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ω-ULTRADISTRIBUTIONS IN THE ABSTRACT CAUCHY PROBLEM*)

BY

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In this work we associate to the generator of a „regular" ω-ultradistribution semi-group on a Banach space X an abstract ω-space which is dense in X and on which the ω-ultradistribution semi-group coincide with a locally-equicontinuous semi-group of operators.

We extens so to ω-ultradistribution semi-groups the result of T. Ushijima from [15] concerning the distribution semi-groups.

The ω-ultradistributional context yields to a generalization of the abstract Gevrey spaces of R. Beals [2] and of the abstract Beurling spaces of I. Ciorănescu [4]. Moreover it enables us to precise and to generalize results from [1] and [2] on the existence of dense subspaces of elements of X for which the abstract Cauchy problem for certain classes of operators is solvable.

§1. Preliminaries. We recall in this paragraph some facts about the ω-ultradistributions considered by the authors in [5], [6], [7] and [8].

Let $\{t_k\}$ $k \geqslant 1$ be a sequence of positive numbers with $t_1 < +\infty$ and $\sum_{k=1}^{\infty} \frac{1}{t_k} < +\infty$; let us define the function

$$\omega_{\{t_k\}}(z) = \prod_{k=1}^{\infty}\left(1 + \frac{iz}{t_k}\right), \quad z \in \mathbf{C}.$$

When the sequence $\{t_k\}$ is fixed, we shall denote simply $\omega_{\{t_k\}}$ by ω. *The function ω is called strogly non-quasianalytic if there exist $L' > 0$, $n' \geqslant 1$ integer and $c' > 0$ such that*

$$\omega(-it) \leqslant c' \mid \omega(L't)^{n'} \mid, \; t > 0. \tag{1.1}$$

If $t_k = k^a$, $a > 1$ (the Gevrey sequence), then one can easily verify that the corresponding function ω satisfies the strong non-quasianaliticity condition.

For $n \geqslant 1$, $k \geqslant 0$ integres, we shall denote by $a_k^{\omega,n}$ the square root of the coefficient of z^{2k} in the power series expansion of $[\omega(z)\omega(-z)]^n$ and by $c_k^{\omega,n}$ the coefficient of z^k in the power series expansion of $\omega(-iz)^n$.

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If ω is strongly non-quasianalytic, then there exist $L_1>0$, $n_1\geqslant 1$ integer and $c_1>0$, such that.

$$c_k^{\omega,n}\leqslant c_1\, L_1^n\, a_k^{\omega,nn_1}, \quad \text{for every } n\geqslant 1,\ k\geqslant 0. \tag{1.2}$$

We define next the following spaces of ω-ultradifferentiable functions ;

$$\mathcal{E}_\omega=\{\varphi\in C^\infty \mid r_{L,n}^{\omega,K}(\varphi)=\sup_{\substack{k\geqslant 0\\ s\in K}} L^k a_k^{\omega,n} \mid \varphi^{(k)}(s)\mid <+\infty,\ L>0,\ n\geqslant 1 \text{ integer},\ K\subset R$$

compact$\}$, endowed with the Fréchet topology defined by the semi-norms $r_{L,n}^{\omega,K}$;

$$\mathcal{D}_\omega(K)=\{\varphi\in\mathcal{E}_\omega \text{ ; supp } \varphi\subset K\},\ K\subset R \textit{ compact},$$

endowed with the topology of the restriction ;

$$\mathcal{D}_\omega=\varinjlim_{K\in R}\mathcal{D}_\omega(K) \quad \text{and}$$

$$\mathcal{D}_\omega^0=\{\varphi\in\mathcal{D}_\omega;\ \text{supp } \varphi\subset(0,+\infty)\}.$$

We remark that when ω is strongly quasi-analytic, then the spaces $\mathcal{E}_\omega$ and $\mathcal{D}_\omega$ can be defined analogously using the sequence $\{c_k^{\omega,n}\}_{k\geqslant 0,\, n\geqslant 1}$ instead of $\{a_k^{\omega,n}\}_{k\geqslant 0,\, n\geqslant 1}$.

Let us underline that our space $\mathcal{E}_\omega$ is the intersection of a decreasing sequence of ultradifferentiable function spaces of Beurling type (defined in [11]) and this assures the stability of $\mathcal{E}_\omega$ under the action of a wide class of differential operators of infinite order. On the other hand our definition of $\mathcal{E}_\omega$ enables to use in handling ω-ultradifferentiable funtions several deep results from the entire function theory, without requiring any additional restrictions like conditions $(M\cdot 2)$ and $(M\cdot 3)$ from [11].

Theorefore the theory of ω-ultradistributions can be efficiently applied in several problems of the operator theory.

Let further X be a Banach space ; then the elements of the vector space $\mathcal{L}(\mathcal{D}_\omega\,;X)$ of all continuous linear mappings of $\mathcal{D}_\omega$ in X, endowed with the topology of the uniform convergence on the bounded subsets of $\mathcal{D}_\omega$ are are called X-valued ω-ultradistributions. The X-valued ω-ultradistributions with compact support can be identified with elements from $\mathcal{L}(\mathcal{E}_\omega\,;X)$.

For $\varphi\in\mathcal{D}_\omega$, we put

$$\tilde{\varphi}(z)=\int_{-\infty}^{+\infty} e^{zt}\varphi(t)\,dt,\quad z\in\mathbf{C}.$$

Then by the Paley-Wiener theorem for $\mathcal{D}_\omega[a,b]$ ([5], [8]), for every $L>0$ and $n\geqslant 1$ integer, we have

$$|\tilde{\varphi}(z)|\leqslant\begin{cases} 2r_{L,n}^{\omega,[a-1,b+1]}(\varphi)\,|\omega(iLz)|^{-n}\,e^{a\operatorname{Re}z}, & \operatorname{Re} z\leqslant 0\\ 2r_{L,n}^{\omega,[a-1,b+1]}(\varphi)\,|\omega(-iLz)|^{-n}\,e^{b\operatorname{Re}z}, & \operatorname{Re} z\geqslant 0.\end{cases} \tag{1.3}$$

If $F\in(\mathcal{E}\,;X)$, supp $F\subset[a,b]$, then denoting by

$$\tilde{F}(z)=F(e^z),\ z\in C.$$

again by the Paley-Wiener theorem from [5], [8], there are $L,c>0$ and $n\geqslant 1$ integer such that

$$\|\tilde{F}(z)\|\leqslant\begin{cases} c\,|\omega(iLz)|^n\,e^{a\operatorname{Re}z}, & \operatorname{Re} z\leqslant 0\\ c\,|\omega(-iLz)|^n\,e^{b\operatorname{Re}z}, & \operatorname{Re} z\geqslant 0.\end{cases} \tag{1.4}$$

The inequalities (1.3) and (1.4) completely characterize those entire functions which are Fourier transforms of functions from $\mathcal{D}_\omega$, respectively of ω-ultradistributions with compact support.

Finally, we define the operator $\omega(-iD)$ from $\mathcal{D}_\omega$ in $\mathcal{D}_\omega$ by (see [6])

$$\omega(\widetilde{-iD)\,\varphi}=\omega(iz)\,\tilde{\varphi},\quad \varphi\in\mathcal{D}_\omega.$$

§2. **ω-Ultradistribution semi-groups.** In all this paragraph A will be a linear closed and densely defined operator in a Banach space X, with the domain $D(A)$ and the function

$$\omega(z)=\prod_{k=1}^{\infty}\left(1+\frac{iz}{t_k}\right),\quad 0<t_1\leqslant t_2\leqslant\ldots,\ t_1<+\infty,\ \sum_{k=1}^{\infty}\frac{1}{t_k}<+\infty.$$

will be fixed.

2.1. Definition. *We say that $A\in C_\omega$ if it exists $\mathcal{E}\in\mathcal{L}(\mathcal{D}_\omega\,;\mathcal{L}(X))$ such that:*

(i) supp $\mathcal{E}\subset[0,+\infty)$;

(ii) $\mathcal{E}(\varphi)X\ D(A)$ for every $\varphi\in\mathcal{D}_\omega$;

(iii) $\mathcal{E}'\ldots A\mathcal{E}=\delta\oplus I$, $\mathcal{E}'-\mathcal{E}A=\delta\oplus I|D(A)$.

$\mathcal{E}$ is called the ω-ultradistribution semi-group generated by A. Using the inequalities (1.3) and (1.4) from §1, by similar reasonings as in [3], we get the :

2.3. Theorem. *The operator $A\in C_\omega$ if and only if there are α, β, $c_0>0$ and $n_0\geqslant 1$ integer, such that the resolvent $R(z;A)=(z-A)^{-1}\in\mathcal{L}(X)$ exists for z in the domain*

$$\Lambda_{\omega,\alpha,\beta}=\{z\in\mathbf{C};\ \operatorname{Re} z\geqslant\alpha\ \ln|\omega(\operatorname{Im} z)|+\beta\}. \tag{2.1}$$

and satisfies

$$\|R(z\,;A)\|\leqslant c_0|\omega(-iz)|^{n_0},\quad z\in\Lambda_{\omega,\alpha,\beta}. \tag{2.2}$$

Moreover, the above conditions (2.1) and (2.2) define a unique $\mathcal{E}$-ultradistribution semi-group $\mathcal{E}\in\mathcal{L}(\mathcal{D}_\omega\,;\mathcal{L}(X))$ given by the formula

$$\mathcal{E}(\varphi)=\frac{1}{2\pi i}\int_{\Gamma\omega,\alpha,\beta}\tilde{\varphi}(z)R(z\,;A)\,dz,\quad \varphi\in\mathcal{D}\omega. \tag{2.3}$$

where $\Gamma_{\omega,\alpha,\beta}$, is the boundary of $\Lambda_{\omega,\alpha,\beta}$, oriented from the lower to the upper half plane.

By standard methods we can prove that an ω-ultradistribution semi-group $\mathcal{E}$ has the following „semi-group" property :

$$\mathcal{E}(\varphi * \psi) = \mathcal{E}(\varphi)\mathcal{E}(\psi) \quad \varphi, \psi \in \mathcal{D}^0_\omega.$$

Let us further define for an $\mathcal{E} \in \mathcal{L}(\mathcal{D}_\omega ; \mathcal{L}(X))$ the subspaces

$$\mathcal{N}_\mathcal{E} = \{x \in X ; \mathcal{E}(\varphi)\, x = 0 \ \textit{for all} \ \varphi \in \mathcal{D}^0_\omega\}.$$

$$\mathcal{R}_\mathcal{E} = \left\{ \sum_{j=1}^{n} \mathcal{E}(\varphi_j)\, x_j \ ; \ n \geqslant 1 \ \textit{integer}, \ \varphi_j \in \mathcal{D}^0_\omega, \ x_j \in X \right\}.$$

Then we put the

2.3. Definition. *We say that an ω-ultradistribution semi-group is regular if $\mathcal{N} = \{0\}$ and $\overline{\mathcal{R}} = X$. We denote by C^0_ω the class of those operators from C_ω which generate regular ω-ultradistribution semigroups.*

Let us remark that there are operators in C_ω which don't belong to C^0_ω, as the following example (due to D. V o i c u l e s c u) shows.

Let us assume that H is an infinite dimensional separable Hilbert space with an orthonormal basis $\{e_n\}_{-\infty < n < +\infty}$. We define the weighted shift $T \in \mathcal{L}(H)$ by

$$T e_n = \frac{1}{t_{|n|+1}}\, e_{n+1}, \ n \text{ integer.}$$

It is easy to verify that $A = T^{-1}$ is a closed and densely defined operator in H such that $R(z ; A) \in \mathcal{L}(H)$ exists for every $z \in \mathbf{C}$ and

$$\| R(z ; A) \| \leqslant \left(\sum_{k=1}^{\infty} \frac{1}{t_k} \right) | \omega(-iz) |, \quad \operatorname{Re} z \geqslant 0.$$

Thus $A \in C_\omega$.

On the other hand, denoting by $\mathcal{E}$ the ω-ultradistribution semi-group generated by A and using the Paley-Wiener theorems (1.2) and (1.3) and the Cauchy integral theorem, we get $\operatorname{supp} \mathcal{E} = \{0\}$, so that

$$\mathcal{N}_\mathcal{E} = H \text{ and } \mathcal{R}_\mathcal{E} = \{0\}, \text{ that is } A \notin C^0_\omega.$$

The following result shows that the class C^0_ω is large enough.

2.4. Proposition. *Let A be a densely defined closed linear operator in X such that for some $\alpha, \beta, c_0 > 0$ and $n_0 \geqslant 1$ integer, $R(z ; A) \in \mathcal{L}(X)$ exists for $z \in \Lambda_{\omega,\alpha,\beta} = \{z \in \mathbf{C};\ \operatorname{Re} z \geqslant \alpha \ln |\omega(\operatorname{Im} z)| + \beta\}$ and satisfies*

$$\|R(z ; A)\| \leqslant c_0(1 + |z|)^{n_0}, \quad z \in \Lambda_{\omega,\alpha,\beta}.$$

Then $A \in C^0_\omega$.

Proof. It is obvious that $A \in C_\omega$.

Let $x \in D(A^{n_0+2})$; we have, for every $z \in \Lambda_{\omega,\alpha,\beta}$

$$R(z ; A)\, x = \sum_{j=1}^{n_0+1} z^{-j-1} A^j x + z^{-n_0-2} R(z ; A) A^{n_0+2}.$$

Let further $\{\varphi_k\}_{k \geqslant 1} \in \mathcal{D}^0_\omega$ such that $\varphi_K \to \delta$ in the space of ω-ultradistributions with compact support. Then we can prove using formula (2.3) and similar arguments as in [1], that, $\mathcal{E}$ beeing the ω-ultradistribution semi-group generated by A, then $\lim \mathcal{E}(\varphi_k)\, x = x$. Thus $\mathcal{R}_\mathcal{E}$ is dense in $D(A^{n_0+2})$. But $D(A^{n_0+2})$ is dense in X, so that finally $\overline{\mathcal{R}}_\mathcal{E} = X$.

Analogously $\mathcal{R}^*_\mathcal{E}$ is X-dense in X^*, where $\mathcal{E}^*$ is the ω-ultradistribution semi-group generated by A^*. Since $\mathcal{N}_\mathcal{E}$ is the annulator of $\mathcal{R}_{\mathcal{E}^*}$ in X, we get $\mathcal{N}_\mathcal{E} = \{0\}$. q.e.d.

Since for $t_2 = t_3 = \cdots = +\infty$, we have $\mathcal{D}_\omega = \mathcal{D}$, by the above proposition, every generator of a distribution semi-group [13] is in C^0_ω.

2.5. *Remark.* If $A \in C_\omega$ and $\mathcal{E}$ is the ω-ultradistribution semi-group generated by A, then the function $[0, +\infty) \ni t \to \mathcal{E}(\tau_t \varphi)\, x$ is a solution for the abstract Cauchy problem for $(A, \mathcal{E}(\varphi)\, x)$, where $x \in X$, $\varphi \in \mathcal{D}^0_\omega$. Hence, if $x_0 \in \mathcal{R}_\mathcal{E}$, then the abstract Cauchy problem has a solution for (A, x_0) and this solution is unique, by the uniqueness result of I. I u. L i u b i c [14]. When $A \in C^0_\omega$, then the abstract Cauchy problem is solvable for a dense subset of "initial conditions", namely for each $x_0 \in \mathcal{R}_\mathcal{E}$.

2.6. Corollary. *Let A be a densely defined closed linear operator in X such that for some increasing function $f : [0, +\infty) \to [1, +\infty)$ with $\int_1^\infty t^{-2} \ln f(t)\, dt < +\infty$, $c_0 > 0$ and $n_0 > 1$ integer, $R(z ; A) \in \mathcal{L}(X)$ exists for z in the region*

$$\Lambda_f = \{z \in \mathbf{C} ;\ \operatorname{Re} z \geqslant \ln f(|\operatorname{Im} z|)\}$$

and satisfies the estimation

$$\| R(z : A) \| \leqslant c_0 (1 + |z|)^{n_0}, \quad z \in \Lambda_f.$$

Then there exists a dense linear subspace $\mathcal{R}$ in X such the abstract Cauchy problem has a unique solution for (A, x_0), whenever $x_0 \in \mathcal{R}$.

Proof. By a result of O. I. I z o n e m c e v and V. A. M a r c e n k o [10], there exist $0 < r_1 \leqslant r_2 \leqslant \ldots < +\infty$, $\sum_{k=1}^{\infty} 1/r_k < +\infty$ and $c > 1$ such that we have

$$f(t) \leqslant c\, | \omega_{\{r_k\}}(t) |, \quad t \in [0, +\infty).$$

Hence $R(z; A)$ exists for $z \in \Lambda_{\omega\{r_k\} 1, \ln c}$ and by the above Propozition, $A \in C^0_\omega$. q.e.d.

As the class of operators considered by R. B e a l s in [1] satisfy the conditions of Corollary 2.6, Proposition 2.4 generalize Theorem 2 from [1]. Using abstract ω-spaces, the asertion of this Corollary will be improved in the last section, in the sense that the space $\mathcal{R}$ is precisely described.

§3. **Abstract ω-spaces.** In all this paragraph $0 < t_1 \leqslant t_2 \leqslant \cdots$, $t_1 < +\infty$, $\sum_{k=1}^{\infty} 1/t_k < +\infty$ will be fixed and $\omega(z) = \prod_{k=1}^{\infty} (1 + iz/t_k)$.

The following technical lemma, essentially proved in [4], is an important tool in our further considerations :

3.1. Lemma. *For every $\alpha > 0$, $0 < \beta < t_1$ and $0 < \varepsilon < 1$, there exists $c > 0$ such that*

$$| \omega(iz) | \geqslant c\, | \omega(|z|) |^\varepsilon$$

for $\operatorname{Re} z \leqslant \alpha \ln |\omega(\operatorname{Im} z)| + \beta$.

We shall define now „abstract ω-spaces" similarly to the abstract Gevrey spaces of R. B e a l s from [2] and to the abstract B e u r l i n g spaces of I. C i o r a n e s c u from [4].

To this purpose, we prove a lemma inspired by the technics used in [4], §3.

For every $\gamma>0$, we define the entire function ω_γ by the formula

$$\omega_\gamma(z)=\omega(z)\prod_{t_k<\gamma}\left(1+\frac{iz}{\gamma}\right)\Big/\left(1+\frac{iz}{t_k}\right),\quad z\in\mathbf{C}.$$

3.2. Lemma. *Let $A\in C^0_\omega$, $\mathcal{E}$ the ω-ultradistribution semi-group generated by A, α, β, $c_0>0$ and $n_0\geqslant 1$ integer, the constants given by Theorem 2.2 and let $\gamma>\beta$. Then for every integer $n\geqslant 2n_0+6$, we have :* (i) *the integral in the formula*

$$B_{\gamma,n}=\frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta}}\omega_\gamma^{-n}(iz)\,R(z\,;A)\,dz$$

converges in the norm-topology of $\mathcal{L}(X)$-and $B_{\gamma,n}$ doesn't depend on the choice of α, β, c_0 and n_0 ;

(ii) *for every $\varphi\in\mathcal{D}^0_\omega$*

$$B_{\gamma,n}\mathcal{E}(\omega^n(-iD)\varphi)=\mathcal{E}(\omega^n(-iD)\varphi)\,B_{\gamma,n}=\mathcal{E}(\varphi)\,;$$

(iii) *$B_{\gamma,n}$ is injective and $B_{\gamma,n}X\supset\mathcal{R}_{\mathcal{E}}$, so that $B^{-1}_{\gamma,n}$ is a densely deficlosed linear operator with*

$$\overline{B^{-1}_{\gamma,n}\mid\mathcal{R}_{\mathcal{E}}}=B^{-1}_{\gamma,n}.$$

(iv) *for every integers $n,m\geqslant 2n_0+6$ holds*

$$B_{\gamma,n}B_{\gamma,m}=B_{\gamma,n+m}.$$

(v) *if $m\geqslant 1$ and $n\geqslant 2n_0+m+6$ are integers, then $B_{\gamma,n}X\subset D(A^m)$ and*

$$A^mB_{\gamma,n}=\frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta}}z^m\omega_\gamma^{-n}(iz)\,R(z\,;A)\,dz$$

where the integral at the right side converges in the norm-toplogy of $\mathcal{L}(X)$.

(vi) *if $\gamma_1,\gamma_2>\beta$, then*

$$B_{\gamma_1,2n}=B_{\gamma_1,n}C_{\gamma_1,\gamma_2,n}=C_{\gamma_1,\gamma_2,n}B_{\gamma_1,n},$$

where

$$C_{\gamma_1,\gamma_2,n}=\frac{1}{2\pi i}\int_{\Gamma_{\omega,\alpha,\beta}}\omega^n_{\gamma_1}(z)\omega^{-2n}_{\gamma_2}(z)\,R(z\,;A)\,dz.$$

Proof. Using the characterisation of $R(z;A)$ of Theorem 2.3 and Lemma 3.1, we get

$$\|\omega_\gamma^{-n}(iz)\,R(z\,;A)\|\leqslant\text{const.}\,|\,\omega\,(|\,z\,|)\,|^{-2},\quad z\in\Gamma_{\omega,\alpha,\beta},$$

so that (i) results.

Taking in account that $\widetilde{(\omega_\gamma^n(-iD)\,\varphi)}\,(z)=\omega_\gamma^n(iz)\,\tilde{\varphi}(z)$, $z\in\mathbf{C}$, that $z\to\omega_\gamma^{-n}(iz)$ is an analytic function on a neighborhood of $\overline{\mathbf{C}\setminus\Lambda_{\omega,\alpha,\beta}}$ and $0\,(|\,z\,|^{-n/2})$ at ∞ on $\hat{\mathbf{C}}\setminus\Lambda_{\omega,\alpha,\beta}$ and using standard arguments from the functional calculus theory, it is easy to verify (ii) and (iv), (see for example the proof of Lemma 2.7 from [4]). (iii) is a direct consequence of (ii). Indeed, if $x\in X$ is such that $B_{\gamma,n}x=0$, then

$$\mathcal{E}(\varphi)\,x=\mathcal{E}(\omega_\gamma^n(-iD)\,\varphi)\,B_{\gamma,n}x=0,\quad\varphi\in\mathcal{D}^0_\omega,$$

so $x=0$. On the other hand,

$$B_{\gamma,n}X\supset\bigcup_{\varphi\in\mathcal{D}^0_\omega}B_{\gamma,n}\mathcal{E}(\omega_\gamma^n(-iD)\,\varphi)\,X=\bigcup_{\varphi\in\mathcal{D}^0_\omega}\mathcal{E}(\varphi)\,X,$$

hence $B^{-1}_{\gamma,n}X\supset\mathcal{R}_{\mathcal{E}}$.

Further, let $\varphi\in\mathcal{D}^0_\omega$; since $B_{\gamma,n}$ and $\mathcal{E}(\varphi)$ commute, we have

$$B^{-1}_{\gamma,n}\mathcal{E}(\varphi)\,B_{\gamma,n}=\mathcal{E}(\varphi)\quad\text{and}\quad\overline{(B^{-1}_{\gamma,n}\mid\mathcal{R}_{\mathcal{E}})}(\mathcal{E}(\varphi)\,B_{\gamma,n})=\mathcal{E}(\varphi)$$

so that

$$B_{\gamma,n}\mathcal{E}(\varphi)\,X\subset D\overline{(B^{-1}_{\gamma,n}\mid\mathcal{R}_{\mathcal{E}})}\quad\text{and}\quad\overline{(B^{-1}_{\gamma,n}\mid\mathcal{R}_{\mathcal{E}})}(B_{\gamma,n}\mathcal{E}(\varphi))=\mathcal{E}(\varphi).$$

Since $\varphi\in\mathcal{D}^0_\omega$ is arbitrary, we conclude that $B_{\gamma,n}\mathcal{R}_{\mathcal{E}}\subset D\overline{(B^{-1}_{\gamma,n}\mid\mathcal{R}_{\mathcal{E}})}$ and $\overline{(B^{-1}_{\gamma,n}\mid\mathcal{R}_{\mathcal{E}})}\,B_{\gamma,n}x=x$, $x\in\mathcal{R}_{\mathcal{E}}$. But $B^{-1}_{\gamma,n}X\subset D\overline{(B^{-1}_{\gamma,n}\mid\mathcal{R}_{\mathcal{E}})}$, that is $D\,(B^{-1}_{\gamma,n})\subset\subset D\,\overline{(B^{-1}_{\gamma,n}\mid\mathcal{R}_{\mathcal{E}})}$, hence (iii) is completely proved.

It is easy to verify that the integral in the formula from (v) converges in the norm-topology of $\mathcal{L}(X)$. Then the assertion from (v) results by the formula

$$A^mR(z\,;A)\,x=z^mR(z\,;A)\,x-\sum_{j=0}^{m-1}z^{m-j-1}A^jx,\quad x\in D(A^m),\ z\in\Gamma_{\omega,\alpha,\beta},$$

by the closedness of A^m, the density of $D(A^m)$ and the continuity of $B_{\gamma,n}$.

Finally (vi) results from standard computations. q.e.d.

The above two Lemmas permit us to conclude :

3.3. Theorem. *let $A\in C^0_\omega$, $\mathcal{E}$ the ω-ultradistribution semigroup generated by A, α, β, $c_0>0$ and $n_0\geqslant 1$ the constants given by Theorem 2.2, $\gamma>\beta$ and $p_0\geqslant 2n_0+6$. Then the space*

$$X_{\omega,A}=\bigcap_{n\geqslant p_0}B_{\gamma,n}X,$$

endowed with the semi-norms

$$\|\,x\,\|_{\gamma,n}=\|\,B^{-1}_{\gamma,n}x\,\|,\quad n\geqslant p_0,$$

is a Fréchet space which doesn't depend on the choice of α, β, c_0, n_0, γ, p_0 and satisfies :

(i) $\quad X_{\omega,A}\subset D(A)$, $AX_{\omega,A}\subset X_{\omega,A}$ *and* $A\mid X_{\omega,A}\in\mathcal{L}(X_{\omega,A})$;

(ii) *$\mathcal{R}_{\mathcal{E}}$ is a dense linear subspace of $X_{\omega,A}$;*

(iii) *the inclusion $X_{\omega,A}\subset X$ is continuous and has dense range.*

Proof. By lemma 3.2. (iv) and (vi), for all $\gamma_1, \gamma_2 > \beta$ and $n_1, n_2 \geq 2n_0 + 6$, $n_2 \geq 3n_1$, we have

$$B_{\gamma_1,n_2}X \subset B_{\gamma_2,2n_1}X \subset B_{\gamma_1,n_1}X$$

and

(3.1) $$\| x \|_{\gamma_1,n_1} \leq \| C_{\gamma_1,\gamma_2,n_1} \| \| B_{\gamma_2,n_2,2n_1} \| \| x \|_{\gamma_2,n_2}, \quad x \in X_{\omega,A}$$

so that $X_{\omega,A}$ doesn't depend on γ and p_0. Again by Lemma 3.2. (i)we conclude that $X_{\omega,A}$ doesn't depend on the choise of $\alpha, \beta, c_0, n_0, \gamma, p_0$. Clearly, it is a Fréchet space.

In order to prove (i), we remark that by Lemma 3.2. (v).$X_{\omega,A} \subset D(A)$. By the same lemma, for $y \in D(A)$, $\gamma > \beta$ and $n \geq 2n_0 + 7$ integer, we get

$$B_{\gamma,n}Ay = \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} \omega_\gamma^{-n}(iz)\, R(z\,;A)\, Ay\,dz = \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} z\omega_\gamma^{-n}(iz)\, R(z\,;A)\, y\,dz - \left[\frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} \omega_\gamma^{-n}(iz)\,dz\right] y = A\,B_{\gamma,n}y$$

and $A B_{\gamma,n} \in \mathcal{L}(X)$. Hence, if $x \in X_{\omega,A}$ then for all $\gamma > \beta$ and $n \geq 2n_0 + 7$ integer, we successively have

$$B_{\gamma,n}^{-1}x \in X_{\omega,A} \subset D(A),$$

$$Ax = A\,B_{\gamma,n}(B_{\gamma,n}^{-1}x) = B_{\gamma,n}A\,(B_{\gamma,n}^{-1}x) \in B_{\gamma,n}X,$$

so that $Ax \in X_{\omega,A}$. Consequently, $AX_{\omega,A} \subset X_{\omega,A}$. Finally, if $x \in X_{\omega,A}$, $\gamma > \beta$ and $n \geq 2n_0 + 7$ is integer, then

$$\| Ax \|_{\gamma,n} = \| B_{\gamma,n}^{-1} A\, B_{\gamma,n} B_{\gamma,2n}^{-1} x \| \leq \| A\, B_{\gamma,n} \| \| x \|_{\gamma,2n},$$

and thus $A \mid X_{\omega,A} \in \mathcal{L}(X_{\omega,A})$.

By Lemma 3.2 (iii), $\mathcal{R}_{\mathcal{E}}$ is a linear subspace of $X_{\omega,A}$ and using (3.1.), it is easy to verify that $\mathcal{R}_{\mathcal{E}}$ is dense in $X_{\omega,A}$. So also (ii) results.

(iii) is an immediate consequence of the inequality

$$\| x \| \leq \| B_{\gamma,n} \| \| x \|_{\gamma,n}, \quad x \in X_{\omega,A}, \quad n \geq 2n_0 + 6,$$

and of the density of $\mathcal{R}_{\mathcal{E}}$ in X. q.e.d.

We define next another class of "abstract ω-spaces "using the coefficient $a_k^{\omega,n}$ as in the definition of the space $\mathcal{E}_\omega$.

3.4. Definition. *Let $A \in C_\omega^0$. We define the Fréchet space $Y_{\omega,A}$ by endowing the linear subspace of X*

$$\left\{x \in \bigcap_{k=0}^{\infty} D(A^k)\,;\ \sup_{k \geq 0} L^k a_k^{\omega,n} \| A^k x \| < +\infty \ \text{for every } L > 0 \text{ and } n \geq 1 \text{ integer}\right\}$$

with the semi-norms

$$x \to \sup_{k \geq 0} L^k a_k^{\omega,n} \| A^k x \|, \quad L > 0, \ n \geq 1 \ \text{integer}.$$

3.5. Lemma. *For each $A \in C_\omega^0$ we have*

$$X_{\omega,A} \subset Y_{\omega,A},$$

where the inclusion is continuous.

Proof. Let $L > 0$ and $n \geq 1$ integer be arbitrary α, β, c_0, n_0 as in Theorem 2.2, $\gamma > \beta$ and $m \geq 2L^2 n + 2n_0 + 6$ integer. Then there exists $c_1 > 0$ such that

(3.2) $$\| R(z\,;A) \| \leq c_1 | \omega(|z|) |^{n_0+1}, \quad z \in \Gamma_{\omega,\alpha,\beta}.$$

and by lemma 3.1, there exists $c_2 > 0$ with

(3.3) $$| \omega_\gamma(iz) | \geq c_2 | \omega(|z|) |^{1/2}, \quad z \in \mathbf{C} \setminus \Lambda_{\omega,\alpha,\beta}.$$

so

(3.4) $$| (L^k a_k^{\omega,n} z^k)\, \omega_\gamma^{-m}(iz) | \geq c_\gamma^{-m} | \omega(|z|) |^{-n_0-3}, \quad k \geq 0, \ z \in \mathbf{C} \setminus \Lambda_{\omega,\alpha,\beta}.$$

Let $x \in X_{\omega,A}$ and $k \geq 0$ integer be arbitrary. By Theorem 3.3. (i) $A \mid X_{\omega,A} \in \mathcal{L}(X_{\omega,A})$ and it commutes obviously with $B_{\gamma,m} \mid X_{\omega,A} \in \mathcal{L}(X_{\omega,A})$, hence

$$L^k a_k^{\omega,n} A^k x = L^k a_k^{\omega,n} B_{\gamma,m} A^k B_{\gamma,m}^{-1} x =$$

$$= \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} L^k a_k^{\omega,n} \omega_\gamma^{-m}(iz)\, R(z\,;A)\, A^k B_{\gamma,m}^{-1} x\,dz =$$

$$= \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} (L^k a_k^{\omega,n} z^k)\, \omega_\gamma^{-m}(iz)\, R(z\,;A)\, B_{\gamma,m}^{-1} x - (L^k a_k^{\omega,n} z^k)\, \omega_\gamma^{-m}(iz) \sum_{j=0}^{k-1} z^{-j-1} A^j B_{\gamma,m}^{-1} x\,dz.$$

Since $z \to (L^k a_k^{\omega,n} z^k) \omega_\gamma^{-m}(iz) \sum_{j=0}^{k-1} z^{-j-1} A^j B_{\gamma,m}^{-1} x$ is analytical on a neighborhood of $\overline{\mathbf{C} \setminus \Lambda_{\omega,\alpha,\beta}}$ and by (3.4) it is $0(|z|^{-2})$ at ∞ on $\overline{\mathbf{C} \setminus \Lambda_{\omega,\alpha,\beta}}$, using the Cauchy integral theorem, we deduce

$$L^k a_k^{\omega,n} A^k x = \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} (L^k a_k^{\omega,n} z^k)\, \omega_\gamma^{-m}(iz)\, R(z\,;A)\, B_{\gamma,m}^{-1} x\,dz.$$

Using now (3.2) and (3.4), it is easy to see that

$$L^k a_k^{\omega,n} \| A^k x \| \leq \frac{c_1 c_2^{-m}}{2\pi} \left(\int_{\Gamma_{\omega,\alpha,\beta}} | \omega(|z|) |^{-2} d|z| \right) \| x \|_{\gamma,m}.$$

We conclude that, denoting

$$c = \frac{c_1 c_2^{-m}}{2} \int_{\Gamma_{\omega,\alpha,\beta}} | \omega(|z|) |^{-2} d|z|,$$

we have

$$\sup_{k \geq 0} L^k a_k^{\omega,n} \| A^k x \| \leq c \| x \|_{\gamma,m}, \quad x \in X_{\omega,A},$$

which proves our statement. q.e.d.

Assuming that ω satisfies the strong non-quasianalyticity condition, we can prove also the converse inclusion, extending Lemma 4, and [4], Theorem 3.8 :

3.6. Theorem. *Let us assume that ω satisfies the strong non-quasianalyticity condition and let $A \in C^0_\omega$. Then the Fréchet spaces $X_{\omega,A}$ and $Y_{\omega,A}$ coincide.*

Proof. By the above lemma we need only to prove that $Y_{\omega,A} \subset X_{\omega,A}$ where the inclusion is continuous. We note that, by (1.2) we have

$$Y_{\omega,A} = \left\{ x \in \bigcap_{k=0}^{\infty} D(A^k) : \sup_{k\geq 0} L^k c_k^{\omega,n} \| A^k x \| < +\infty \text{ for each } L > C,\ n \geq 1 \right\}$$

and the topology of $Y_{\omega,A}$ is given by the semi-norms

$$x \to \sup_{k \geq 0} L^k c_k^{\omega,n} \| A^k x \|,\quad L > 0,\ n \geq 1 \text{ integer.}$$

Let α, β, c_0, n_0 be as in Theorem 2.2, $\gamma > \beta$ and $n \geq 2n_0 + 6$ integer. Denoting by

$$c_\gamma = \sup_{t \in \mathbf{R}} \prod_{t_k < \gamma} \left| \left(1 + \frac{it}{\gamma}\right) \Big/ \left(1 + \frac{it}{t_k}\right) \right| < +\infty,$$

we have

$$| \omega_\gamma(t) | \leq c_\gamma |\omega(t)|,\quad t \in \mathbf{R}.$$

By a theorem of S. N. Bernstein, we have

$$|(\omega_\gamma^n)^{(k)}(t)| \leq c_\gamma^n |(\omega^n)^{(k)}(t)|,\quad t \in \mathbf{R},\ k \geq 0.$$

In particular,

$$\frac{1}{k!} |(\omega^n)^{(k)}(0)| \leq c^n \frac{1}{k!} |(\omega^n)^{(k)}(0)|,\quad k \geq 0,$$

that is

$$c_k^{\omega_\gamma,n} \leq c^n c_k^{\omega,n},\quad k \geq 0. \tag{3.5}$$

Then using (1.1) we get

$$\sum_{k=0}^{\infty} c_{k,\gamma}^{\omega,n} t^k \leq (cc')^n |\omega(t)|^{nn'},\quad t \geq 0$$

and this implies by (3.3)

$$\sum_{k=0}^{\infty} c_k^{\omega_\gamma,n} |z|^k |\omega_\gamma^{-n-2nn'}(iz)| \leq (cc')^n |\omega(|z|)|^{-n_0-3},\quad z \in \mathbf{C} \setminus \Lambda_{\omega,\alpha,\beta}. \tag{3.6}$$

Let now $x \in Y_{\omega,A}$ be arbitrary. Since by (3.5)

$$c_k^{\omega_\gamma,n} \|A^k x\| \leq 2^{-k} c^n \sup_{p \geq 0} 2^p c_p^{\omega,n} \|A^p x\|,\quad k \geq 0,$$

we can define the elements $y_1, y_2, \ldots, y_\infty \in X$ by

$$y_m = \sum_{k=0}^{\infty} (-1)^k c_k^{\omega_\gamma,n} A^k x,\quad m = 1, 2, \ldots, \infty;$$

then $y_m \to y_\infty$ in the norm topology of X. Moreover,

$$\|y_m\| \leq 2c^n \sup_{p \geq 0} 2^p c_p^{\omega,n} \|A^p x\|,\quad m = 1, 2, \ldots, \infty. \tag{3.7}$$

For every integer $1 \leq m \leq +\infty$ we have

$$B_{\gamma,n+2nn'} y_m = \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} \left[\sum_{k=0}^{m} (-1)^k c_k^{\omega_\gamma,n} \omega_\gamma^{-n-nn'}(iz) R(z;A) A^k x \right] dz =$$

$$= \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} \left(\sum_{k=0}^{m} (-1)^k c_k^{\omega_\gamma,n} z^k \right) \omega_\gamma^{-n-nn'}(iz) R(z;A)\, x\, dz.$$

Since $y_m \to y_\infty$, by (3.2), (3.6) and the Lebesgue deminated convergence theorem, we get

$$B_{\gamma,n+2nn'} y_\infty = \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} \omega_\gamma^n(iz) \omega_\gamma^{-n-nn'}(iz) R(z;A)\, x\, dz) = B_{\gamma,2nn'} x.$$

By Lemma 3.2. (iii) and (iv), it follows that $B_{\gamma,n} y_\infty = x$, so that using (3.7), we conclude

$$x \in B_{\gamma,n} X \text{ and } \|B_{\gamma,n}^{-1} x\| = \|y_\infty\| \leq 2c^n \sup_{p \geq 0} 2^p c_p^{\omega,n} \|A^p x\|.$$

Since $x \in Y_{\omega,A}$ and the integer $n \geq 2n_0 + 6$ are arbitrary, the above inequality proves that $Y_{\omega,A} \subset X_{\omega,A}$ and that this inclusion is continuous q.e.d.

§4. Semi-groups and abstract ω-spaces. In this paragraph we establish the connection between the abstract ω-spaces considered in §3 and the abstract Cauchy problem for the generators of regular ω-ultradistributions semi-groups.

4.1. Definition. *Let $-\infty \leq a < b \leq +\infty$ and E a locally convex topological vector space. Then we denote by $\mathcal{E}_\omega((a,b) ; E)$ the vector space of all infinitely strongly differentiable functions $\varphi : (a,b) \to E$ such that for any compact $K \subset (a,b)$, $L > 0$, $n \geq 1$ integer and continuous semi-norm p on E,*

$$r_{L,n}^{\omega,p,K}(\varphi) = \sup_{k \geq 0} [L^k a_k^{\omega,n} \sup_{s \in K} p(\varphi^{(k)}(s))] < +\infty.$$

We endowe $\mathcal{E}_\omega((a,b) ; E)$ with the locally convex topology defined by the semi-norms $r_{L,n}^{\omega,K,p}$, $K \subset (a,b)$ compact, $L > 0$, $n \geq 1$ integer, p continuous semi-norm on E.

We recall next the following terminology from [12] :

4.2. Definition. *Let E be a locally conex topological vector space. We say that a family $\{U_t\}_{t \geq 0} \subset \mathcal{L}(E)$ is a semi-group of class (C_0) if*

$$U_0 = I,\quad U_{t+s} = U_t U_s,\quad t, s \geq 0,$$

and for every $x \in E$, the mapping $[0, +\infty) \ni t \to U_t x \in E$ is continuous.

A semi-group $\{U_t\}_{t \geq 0} \subset \mathcal{L}(E)$ is called locally equi-continuous if for every $K \subset [0, +\infty)$ compact, the family $\{U_t\}_{t \in K}$ is equi-continuous.

We can now give our main result concerning the abstract Cauchy problem.:

4.3. Theorem. *Let $A \in \mathcal{C}_{\omega}^{0}$. Then*
(i) *for each $x \in X_{\omega, A}$ the (ACP) has a unique solution*

$$[0, +\infty) \ni t \to U_t^{\omega, A} x \in X$$

for (A, x);
(ii) *the linear operators $U_t^{\omega,A}$, $t \geq 0$, belong to $\mathcal{L}(X_{\omega,A})$ and commute with $A \mid X_{\omega,A} \in \mathcal{L}(X_{\omega,A})$;*
(iii) *$\{U_t^{\omega,A}\}_{t \geq 0}$ is a locally equi-continuous semi-group of class (C_0) of operators from $\mathcal{L}(X_{\omega,A})$ and its infinitesimal generator is $A \mid X_{\omega,A} \in \mathcal{L}(X_{\omega,A})$.*
(iv) *for each $x \in X_{\omega,A}$ the function*

$$(0, +\infty) \ni t \to U_t^{\omega,A} x \in X_{\omega,A}$$

belongs to $\mathcal{E}_{\omega}((0, +\infty) ; X_{\omega,A})$ and it depends continuously on x;
(v) *denoting by $\mathcal{E}$ the ω-ultradistribution semi-group generated by A, we have*

$$\mathcal{E}(\varphi)x = \int_0^{+\infty} \varphi(t)\, U_t^{\omega,A} x\, dt, \quad \varphi \in \mathcal{D}_{\omega}^{0}, x \in X_{\omega,A}.$$

Proof. Let α, β, c_1, n_0 be as in Theorem 2.2 and $\gamma > \beta$. Let $t \geq 0$ and $n \geq 2\alpha t + 2n_0 + 6$ integer. By (3.2) and (3.3) we have for all $z \in \Gamma_{\omega,\alpha,\beta}$

$$\|e^{tz}\omega_{\gamma}^{-n}(iz)\, R(z\,;A)\| \leq c_1 c_2^{-n} e^{\beta t}\, |\omega(|z|)|^{\alpha t - \frac{n}{2} + n_0 + 1} \leq c_1 c_2^{-n} |\omega(|uz|)|^{-2} e^{\beta t},$$

so the integral in the formula

$$E_{t,n} = \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} e^{tz}\omega_{\gamma}^{-n}(iz)\, R(z\,;A)\, dz$$

converges in the norm tcpology of $\mathcal{L}(X)$ and

$$E_{t,n} \leq \frac{c_1 c_2^{-n}}{2\pi} \left(\int_{\Gamma_{\omega,\alpha,\beta}} |\omega(|z|)|^{-2} dz \right) e^{\beta t}. \tag{4.1}$$

It is clear that $E_{t,n}$ commutes with all operators $B_{\gamma,m}, m \geq 2n_0 + 6$, so that

$$E_{t,n} X_{\omega,A} \subset X_{\omega,A}, \quad E_{t,n} \mid X_{\omega,A} \in \mathcal{L}(X_{\omega,A}). \tag{4.2}$$

On the other hand, using Lemma 3.2. (ii), (2.3) and a reasoning similar to that one used in the proof of Lemma 2.7 in [4], we et gfor every $\varphi \in \mathcal{D}_{\omega}^{0}$

$$E_{t,n} B_{\gamma,n}^{-1} \mathcal{E}(\varphi) = E_{t,n} \mathcal{E}(\omega_{\gamma}^{n}(-iD)\varphi) = \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} e^{tz}\omega_{\gamma}^{-n}(iz)\, R(z\,;A)\, dz\,.$$

$$\cdot \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} \omega_{\gamma}^{n}(iz)\, \tilde{\varphi}(z)\, R(z;A) dz = \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} e^{tz}\, \tilde{\varphi}(z)\, R(z\,;A)\, dz = \mathcal{E}(\varphi(\cdot - t)).$$

that is, with the usual notation $\tau_t\varphi = \varphi(\cdot - t)$,

$$E_{t,n} B_{\gamma,n}^{-1} \mathcal{E}(\varphi) = \mathcal{E}(\tau_t\varphi), \quad \varphi \in \mathcal{D}_{\omega}^{0}. \tag{4.3}$$

By (4.2), (4.3) and Theorem 3.3. (ii), $U_t^{\omega,A} = E_{t,n} B_{\gamma,n}^{-1} \mid X_{\omega,A}$ belongs to $\mathcal{L}(X_{\omega,A})$ and is completely defined by

$$U_t^{\omega,A} \mathcal{E}(\varphi)\, x = \mathcal{E}(\tau_t\varphi)\, x, \quad \varphi \in \mathcal{D}_{\omega}^{0}, \quad x \in X, \tag{4.4}$$

so that it doesn't depend on n. Moreover, by Theorem 2.2. (iii)

$$\mathcal{E}(-\varphi') = A\mathcal{E}(\varphi), \quad \varphi \in \mathcal{D}_{\omega}^{0},$$

thus (4.4) implies for every $\varphi \in \mathcal{D}_{\omega}^{0}$ and $x \in X$

$$A\, U_t^{\omega,A} \mathcal{E}(\varphi)\, x = A\mathcal{E}(\tau_t\varphi)\, x = \mathcal{E}(-\tau_t\varphi')\, x = U_t^{\omega,A} \mathcal{E}(-\varphi')\, x = U_t^{\omega,A} A\mathcal{E}(\varphi)\, x.$$

By Theorem 3.3. (ii) it follows that $U_t^{\omega,A}$ commutes with $A \mid X_{\omega,A} \in \mathcal{L}(X_{\omega,A})$.

Using again (4.4) and Theorem 3.3. (ii), the semi-group property of the family $\{U_t^{\omega,A}\}_{t \geq 0}$ results immediately. An easy computation shows that, choosing for example $n \geq 2\alpha + 2n_0 + 6$,

$$\lim_{0 \leq t \to 0} \| E_{t,n} - B_{\gamma,n} \| = 0.$$

Hence, for every $x \in X_{\omega,A}$ and $m \geq 2n_0 + 6$ integer we have

$$\lim_{0 \leq t \to 0} \| U_t^{\omega,A} x - x \|_{\gamma,m} = \lim_{0 \leq t \to 0} \| B_{\gamma,m}^{-1}(E_{t,n} - B_{\gamma,n})\, B_{\gamma,n}^{-1} x \| \leq$$

$$\leq \overline{\lim_{0 \leq t \to 0}} \| E_{t,n} - B_{\gamma,n} \| \, \| B_{\gamma,m}^{-1} B_{\gamma,n}^{-1} x \| = 0,$$

so the semi-group $\{U_t^{\omega,A}\}_{t \geq 0}$ is of class (C_0).
Let next $K \subset [0, +\infty)$ be compact. We choose $t_0 > 0$ such that $K \subset [0, t_0]$ and $n \geq 2\alpha t_0 + 2n_0 + 6$. Then, using (4.1) and Lemma 3.2 (iii), for every $t \in [0, t_0]$, $m \geq 2n_0 + 6$ integer and $x \in X_{\omega,A}$, we get

$$\| U_t^{\omega,A} x \|_{\gamma,m} = \| B_{\gamma,m}^{-1} E_{t,n} B_{\gamma,n}^{-1} x \| \leq \frac{c_1 c_2^{-n}}{2\pi} \int_{\Gamma_{\omega,\alpha,\beta}} |\omega(|z|)|^{-2} d|z|)\, e^{\beta t_0} \| x \|_{\gamma,n+m},$$

so that the semi-group $\{U_t^{\omega,A}\}_{t \geq 0}$ is locally equi-continuous.

Let further $t \geq 0$ and choose $t_0 > t$ and $n \geq 2\alpha t_0 + 2n_0 + 8$ integer. Using the Cauchy integral theorem and the closedness of A, it is easy to verify that

$$E_{t,n} X \subset D(A) \quad \text{and}$$

$$A E_{t,n} = \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} z e^{tz}\omega_{\gamma}^{-n}(iz)\, R(z\,;A)\, dz.$$

Since for every $0 < s < t_0$, $s \neq t$, we have

$$\left| \frac{1}{t-s} (e^{tz} - e^{sz}) - z e^{tz} \right| \leq |t-s|\, |z^2 e^{t_0 z}|, \quad \operatorname{Re} z \geq 0,$$

denoting by

$$c_3 = \frac{1}{2\pi} \int_{\Gamma_{\omega,\alpha,\beta}} |z^2 e^{t_0 z} \omega_\gamma^{-n}(iz)| \, \| R(z\,;A) \| \, d|z| < +\infty,$$

it follows that

$$\left\| \frac{1}{t-s} (E_{t,n} - E_{s,n}) - A E_{t,n} \right\| \leqslant c_3 |s-t|.$$

Consequently, for each $x \in X_{\omega,A}$ and $m \geqslant 2n_0+6$ integer, we have

$$\left\| \frac{1}{t-s} (U_t^{\omega,A} x - U_s^{\omega,A} x) - A U_t^{\omega,A} x \right\|_{\gamma,m} \leqslant c_3 |s-t| \, \|x\|_{\gamma,n+m}, \quad 0 < s < t_0, \ s \neq t,$$

hence

$$\lim_{0<s\to t} \left\| \frac{1}{t-s} (U_t^{\omega,A} x - U_s^{\omega,A} x) - A U_t^{\omega,A} x \right\|_{\gamma,m} = 0. \tag{4.5}$$

By the above proved facts concerning the operators $\{U_t^{\omega,A}\}_{t\geqslant 0}$ we conclude that (i), (ii), (iii) from the Theorem hold.

In order to prove (iv), let $x \in X'_{\omega,A}$. By (4.5) the function

$$(0, +\infty) \ni t \to U_t^{\omega,A} x \in X_{\omega,A}$$

is strongly differentiable and

$$\frac{d}{dt} U_t^{\omega,A} x = A U_t^{\omega,A} x = U_t^{\omega,A} A x.$$

By induction, we get that the above function is infinitely strongly differentiable and

$$\frac{d^k}{dt^k} U_t^{\omega,A} x = A^k U_t^{\omega,A} x = U_t^{\omega,A} A^k x, \quad k \geqslant 0. \tag{4.6}$$

Let further $K \subset (0, +\infty)$ be compact, $L > 0$ and $n \geqslant 1$, $m \geqslant 2n_0+6$ integers. By the aboveproved locally equi-continuity of $\{U_t^{\omega,A}\}_{t\geqslant 0}$, there exist $m' \geqslant 2n_0+6$ integer and $c_4 > 0$ such that

$$\|U_t^{\omega,A} y\|_{\gamma,m} \leqslant c_4 \|y\|_{\gamma,m'}, \qquad t \in K, \ y \in X_{\omega,A},$$

and by Lemma 3.5 there exist $m'' \geqslant 2n_0+6$ integer and $c_5 > 0$ with

$$\sup_{k\geqslant 0} L^k a_k^{\omega,n} \|A^k y\| \leqslant c_5 \|y\|_{\gamma,m''}, \quad y \in X_{\omega,A}.$$

Hence, using (4.6), we deduce

$$r_{L,n}^{\omega,K,\|\cdot\|_{\gamma,m}} (U_\cdot^{\omega,A}) = \sup_{k\geqslant 0} [L^k a_k^{\omega,n} \sup_{t\in K} \|U_t^{\omega,A} A^k x\|_{\gamma,m}] \leqslant c_4 \sup_{k\geqslant 0} L^k a_k^{\omega,n} \|A^k x\|_{\gamma,m'} =$$

$$= c_4 \sup_{k\geqslant 0} L^k a_k^{\omega,n} \|A^k B_{\gamma,m'}^{-1} x\| \leqslant c_4 c_5 \|B_{\gamma,m'}^{-1} x\|_{\gamma,m''} = c_4 c_5 \|x\|_{\gamma,m'+m''},$$

which proves (iv).

Finally, let $\varphi \in D_\omega^0$ and $x \in X_{\omega,A}$. Let us take $t_0 > 0$ such that supp $\varphi \subset [0, t_0]$ and $n \geqslant 2\alpha t_0 + 2n_0 + 6$ integer, then we get

$$\mathcal{E}(\varphi) B_{\gamma,n} = \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} \tilde{\varphi}(z) R(z\,;A)\, dz \cdot \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} \omega_\gamma^{-n}(iz) R(z\,;A)\, dz =$$

$$= \frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} \tilde{\varphi}(z) \omega_\gamma^{-n}(iz) R(z\,;A)\, dz = \int_0^{t_0} \varphi(t) \left[\frac{1}{2\pi i} \int_{\Gamma_{\omega,\alpha,\beta}} e^{tz} \omega_\gamma^{-n}(iz) R(z\,;A)\, dz \right] dt =$$

$$= \int_0^{t_0} \varphi(t) E_{t,n} dt,$$

so that

$$\mathcal{E}(\varphi) x = \mathcal{E}(\varphi) B_{\gamma,n} B_{\gamma,n}^{-1} x = \int_0^{t_0} \varphi(t) E_{t,n} B_{\gamma,n}^{-1} x dt = \int_0^{+\infty} \varphi(t) U_t^{\omega,A} x dt.$$

Hence (v) is also proved. q.e.d.

Assuming that ω satisfies the strong non quasi-analyticity condition, by Theorem 3.6 we can replace the space $X_{\omega,A}$ in the statements of Theorem 4.3 by the space $Y_{\omega,A}$; we obtain so an extension of some results of T. Ushijima from [15] (see also [9], Theorem 3.1) [2], Theorem 1 and [4], Theorem 4.1.

We remark finally that, using Proposition 2.4, and Theorem 4.3, we can easily obtain the following improvement of Corollary 2.6:

4.4. Corollary. *Let A be a densely defined closed linear operator in X such that for some increasing function $f\colon [0, +\infty) \to [1, +\infty)$ with $\int_1^{+\infty} t^{-2} \ln f(t)\, dt < +\infty$, $c_0 > 0$ and $n_0 \geqslant 1$ integer $R(z\,;A) \in \mathcal{L}(X)$ exists for z in the region*

$$\Lambda_f = \{z \in \mathbf{C}\,;\ \operatorname{Re} z \geqslant \ln f(|\operatorname{Im} z|)\}$$

and satisfies the estimation

$$\|R(z\,;A)\| \leqslant c_0 (1 + |z|)^{n_0}, \quad z \in \Lambda_f.$$

Then there exists a Fréchet $\mathcal{X}$, continuously and densely imbedded in X, and a semi-group $\{U_t\}_{t\geqslant 0}$ of class (C_0) of continuous linear operators on $\mathcal{X}$ such that

$$\mathcal{X} \subset D(A), \ A\mathcal{X} \subset \mathcal{X}, \ A|\mathcal{X} \in \mathcal{L}(\mathcal{X})$$

and $\{U_t\}_{t\geqslant 0}$ is generated by $A|X$. Moreover, for each $x \in \mathcal{X}$

$$[0, +\infty) \ni t \to U_t x \in \mathcal{X} \subset X$$

is the unique solution of the (ACP) for (A, x).

We note that the above result can be proved using Beurling ultradistributions only under additionally assumptions on the function f (see the discussion at the end of [4]). This illustrate the utility of the ω-ultradistributions in the treatment of the abstract Cauchy problem.

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EQUIVALENCE PROBLEMS FOR OPERATORS*)

BY

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We shall be discussing (bounded linear) operators on a complex separable Hilbert space $\mathcal{H}$, usually infinite dimensional. We let $\mathcal{L}(\mathcal{H})$ denote the algebra of such operators on $\mathcal{H}$. A basic problem in any area is that of equivalence; that is, deciding when two objects are the same, where „the same" might have various meanings. The most restrictive notion of equivalence for operators is that of unitary equivalence. Two operators T and $\tilde{T}$ defined on $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively, are said to be unitarily equivalent, denoted $T \cong \tilde{T}$ if there exists a unitary operator $W : \mathcal{H} \to \tilde{\mathcal{H}}$ such that $W^*\tilde{T}W = T$ or they are identical up to a „change of variables" preserving the Hilbert space structure. For a normal operator N on a finite dimensional space, there is orthonormal basis relative to which the matrix for N is diagonal and these entries, which are the eigenvalues of N, taking into account multiplicity, form a complete set of unitary invariants for N. For normal operators on an infinite dimensional space analogous but technically more complicated results hold which provide a complete set of unitary invariants and which involve the spectral theorem and multiplicity theory. We do not elaborate. If we attempt to obtain a complete set of unitary invariants for a general operator T on a finite dimensional $\mathcal{H}$, then we must settle for less conceptual invariants such as the trace of the words in T and T^* of length less than some number depending on the dimension of $\mathcal{H}$. If we weaken the notion of equivalence to that of similarity in which W is replaced by some invertible operator S, then a complete set of similarity invariants in the finite dimensional case is provided by the Jordan canonical form; hence eigenvalues, multiplicities and dimensions of generalized eigenspaces are the similarity invariants for T on finite dimensional $\mathcal{H}$. No analogous results are known for T on infinite dimensional $\mathcal{H}$ except for very special classes of operators.

My aim in this paper is to show that equivalence problems for various classes of operators lead one to introduce ideas from other parts of mathe-

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