We note that the above result can be proved using Beurling ultradistributions only under additionally assumptions on the function f (see the discussion at the end of [4]). This illustrate the utility of the ω-ultradistributions in the treatment of the abstract Cauchy problem.

B I B L I O G R A P H Y

1. Beals R. — *On the abstract Cauchy Problem*, J. Funct. Anal., 10 (1972), 281—300.
2. Beals, R. — *Semi-groups and abstract Gevrey spaces*, J. Funct. Anal., 10 (1972), 300—309.
3. Chazarain, J. — *Problemes de Cauchy abstraits et applications à quelques problèmes mixtes*, J. Funct. Anal., 7 (1971), 386—446.
4. Ciorănescu I. — *Abstract Beurling spaces of class M_p and ultradistribution semi-groups*, Bull. Soc., Math. France, to appear.
5. Ciorănescu, I. and Zsidó, L. — *Les ω-ultradistributions*, C. R. Acad. Sci. Paris, série A, t. 285 (1977), 707—710.
6. Ciorănescu, I. and Zsidó L. — *Opérateurs ω-ultradifférentiels*, C. R. Acad. Sci. Paris, série A, t. 285 (1977), 753—756.
7. Ciorănescu I., and Zsidó L., — *Les ω-ultradistributions comme valeurs au bord des fonctions holomorphes*, C.R. Acad. Sci. Paris, série A, t. 285 (1977), 855—858.
8. Ciorănescu I., and Zsidó L. — *The ω-ultradistributions and their application to operators theory*, to appear.
9. Guillement J. H., and Pham The Lai — *Sur la caractérisation des semigroupes distributions*, J. Fac. Sci., Tokyo, Sec. I. A, 22 (1975), 299—318.
10. Izonemcev O. I. and Marcenko, V. A. — *On majorants of genus zero*, Uspehi Mat. Nauk, 11, no. 2 (1956), 173—178.
11. Komatsu H. — *Ultradistributions*, I, J. Fac. Sci. Univ. Tokyo, Sec. I A, 20 (1973), 25—105.
12. Komura, T. — *Semi-groups of operators in locally convex spaces*, J. Funct. Anal., 2 (1968), 258—296.
13. Lions, J. L. — *Les semi-groupes distributions*, Portugal. Mat., 19 (1960), 141—164.
14. Liubic, I. Iu. — *Conditions for the uniqueness of the solution to the Cauchy's abstract problem*. Dokl. Akad. S.S.S.R. (in Russian), 130 (1960), 969—972.
15. Ushijima T., — *On the generation and smoothness of semi-groups of linear operators*, J. Fac. Sci. Univ. Tokyo, Sect. I. A., 19 (1972), 65—128.

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EQUIVALENCE PROBLEMS FOR OPERATORS*)

BY

R. G. DOUGLAS

We shall be discussing (bounded linear) operators on a complex separable Hilbert space $\mathcal{H}$, usually infinite dimensional. We let $\mathcal{L}(\mathcal{H})$ denote the algebra of such operators on $\mathcal{H}$. A basic problem in any area is that of equivalence; that is, deciding when two objects are the same, where „the same" might have various meanings. The most restrictive notion of equivalence for operators is that of unitary equivalence. Two operators T and $\tilde{T}$ defined on $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively, are said to be unitarily equivalent, denoted $T \cong \tilde{T}$ if there exists a unitary operator $W : \mathcal{H} \to \tilde{\mathcal{H}}$ such that $W^*\tilde{T}W = T$ or they are identical up to a „change of variables" preserving the Hilbert space structure. For a normal operator N on a finite dimensional space, there is orthonormal basis relative to which the matrix for N is diagonal and these entries, which are the eigenvalues of N, taking into account multiplicity, form a complete set of unitary invariants for N. For normal operators on an infinite dimensional space analogous but technically more complicated results hold which provide a complete set of unitary invariants and which involve the spectral theorem and multiplicity theory. We do not elaborate. If we attempt to obtain a complete set of unitary invariants for a general operator T on a finite dimensional $\mathcal{H}$, then we must settle for less conceptual invariants such as the trace of the words in T and T^* of length less than some number depending on the dimension of $\mathcal{H}$. If we weaken the notion of equivalence to that of similarity in which W is replaced by some invertible operator S, then a complete set of similarity invariants in the finite dimensional case is provided by the Jordan canonical form; hence eigenvalues, multiplicities and dimensions of generalized eigenspaces are the similarity invariants for T on finite dimensional $\mathcal{H}$. No analogous results are known for T on infinite dimensional $\mathcal{H}$ except for very special classes of operators.

My aim in this paper is to show that equivalence problems for various classes of operators lead one to introduce ideas from other parts of mathe-

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matics in such a manner that both operator theory and the other areas are enriched. I will be discussing results which I have obtained in collaboration with others over the last few years. A final unifying theme is that all these classes of operators contain one operator, the unilateral shift or its adjoint. For nearly fifty years most attention in operator theory was concentrated on the classes of self-adjoint and compact operators. There were at least two reasons for this. Firstly, the basic results for each class flow in a natural way from the finite dimensional case and secondly, they are undoubtedly the most useful operators for applications. In the last twenty to thirty years increasing attention has been paid to other classes of operators. The simplest and most natural operator which is neither compact nor self-adjoint is the unilateral shift U_+. It can be defined on $l^2=\{(\alpha_0, \alpha_1, \alpha_2 \cdots) : \alpha_i \in \mathbf{C}, \Sigma|\alpha_i|^2<\infty\}$ such that

$$U_+(\alpha_0, \alpha_1, \alpha_2 \cdots)=(0, \alpha_0, \alpha_1, \cdots).$$

Now U_+ can be easily seen to be an isometry, that is, $\|U_+x\|=\|x\|$ for x in l^2, which is not unitary. I believe that it first occurs in a 1929 paper of von Neumann where he obtains it on the Cayley transform of id/dx on the subspace of functions in $L^2(0, \infty)$ with square integrable derivative and which vanish at 0. It's adjoint, the backward shift U_+^*, is defined by

$$U_+^*(\alpha_0, \alpha_1, \alpha_2, \cdots)=(\alpha_1, \alpha_2, \alpha_3, \cdots)$$

for $(\alpha_0, \alpha_1, \alpha_2, \cdots)$ in l^2. Now U_+^* has a feature which sets it apart immediately from the compact or self-adjoint operators; namely, the open unit disk $\mathbf{D}$ consists of eigenvalues for U_+^*. For λ in $\mathbf{C}$, $|\lambda|<1$, the vector $(1, \lambda, \lambda^2, \cdots)$ is in l^2 and

$$U_+^*(1, \lambda, \lambda^2, \cdots)=\lambda(1, \lambda, \lambda^2, \cdots).$$

And although finite sets of the corresponding eigenvectors are linearly independent, the whole family can not be in any strong sense since l^2 is separable.

In [6], [7] Mike Cowen and I introduced a technique involving complex geometry for studying a class of operators each possessing an open set of eigenvalues. Let Ω be a bounded connected open subset of $\mathbf{C}$ and n be a positive integer. We let $\mathcal{B}_n(\Omega)$ denote the operators T in $\mathcal{L}(\mathcal{H})$ which satisfy 1) $\Omega \subset \sigma(T)$, the spectrum of T; 2) $(T-\omega)\mathcal{H}=\mathcal{H}$ for ω in Ω; 3) $\bigvee_{\omega \in \Omega} \ker (T-\omega)=\mathcal{H}$; and 4) $\dim \ker (T-\omega)=n$ for ω in Ω. Properties 1), 2) and 4) guarantee that Ω consists of eigenvalues for T of multiplicity, n, while 3) guarantees that the corresponding eigenspaces span $\mathcal{H}$. Now U_+^* is in $\mathcal{B}_1(\mathbf{D})$ and $\mathcal{B}_n(\Omega)$ contains the adjoints of many subnormal, hyponormal, and weighted shift operators.

We study operators in $\mathcal{B}_n(\Omega)$ by considering the map $\omega \to \ker (T-\omega)$ and showing that it defines a Hermitian holomorphic vector bundle E_T over Ω. We identify E_T as the subbundle of the product bundle $\Omega \times \mathcal{H}$ defined by $E_T=\{(\omega, x) \in \Omega \times \mathcal{H}: Tx=\omega x\}$. Following Subin one can show for each ω_0 in Ω that there is a neighborhood Δ of ω_0 and holomorphic $\mathcal{H}$-valued functions $e_1(\omega), \cdots, e_n(\omega)$ defined on Δ which span $\ker (T-\omega)$ for ω in Δ; hence E_T is a holomorphic vector bundle over Ω. Since each fiber $\ker (T-\omega)$ is a subspace of $\mathcal{H}$, it has a natural Hilbert space structure which provides the Hermitian structure for E_T. The justification for introducing this object into operator theory is contained in

Proposition. *Operators T and $\tilde{T}$ in $\mathcal{B}_n(\Omega)$ are unitarily equivalent if and only if the corresponding bundles E_T and $E_{\tilde{T}}$ are equivalent as Hermitian holomorphic vector bundles.*

This result is a corollary to a rigidity theorem which we prove for bundles obtained as pullbacks of holomorphic maps into a Grassmannian generalizing an earlier result of Calabi.

The proposition enables us to replace the study of operators in $\mathcal{B}_n(\Omega)$ by that of their associated bundles. Now Hermitian holomorphic vector bundles possess a canonical connection which is what one needs to translate vectors in a fiber along an arc in Ω. In particular, the classification problem can be studied in the context of complex geometry and we were able to solve this classification problem for such bundles in terms of local invariants up to some fixed finite order. These invariants include the curvature and its partial derivatives. Fortunately, we don't have to be any more specific, at least not in regard to the application to operator theory, because we were able to define an operator theoretic object which contains the relevant information.

For T in $\mathcal{B}_n(\Omega)$ we define the local operator at ω to be

$$N_\omega=(T-\omega) \mid \ker (T-\omega)^{n+1}.$$

Then N_ω can be seen to be a nilpotent operator of order $n+1$ defined on a finite dimension Hilbert space of dimension $n(n+1)$. Our principal result is:

Theorem: *Operators T and $\tilde{T}$ in $\mathcal{B}_n(\Omega)$ are unitarily equivalent if and only if N_ω is unitarily equivalent to $\tilde{N}_\omega$ for ω in Ω.*

This yields a complete set of unitary invariants for operators in $\mathcal{B}_n(\Omega)$. In case $n=1$, the operator N_ω has a unique matrix of the form

$$\begin{pmatrix} 0_0 & h(\omega) \\ 0 & 0 \end{pmatrix},$$

where $h(\omega)>0$. Thus a complete set of unitary invariants for T in $\mathcal{B}_1(\Omega)$ is given by a positive function on Ω which is real analytic. For U_+^* the function is $(1-|\omega|^2)$. Moreover, the complete invariant $-1/[h(\omega)]^2$ is the curvature of E_T for T in $\mathcal{B}_1(\Omega)$.

We have obtained many more results in our study both in operator theory and complex geometry. A much harder problem than unitary equivalence is that of similarity and we are able to obtain partial results in this case. Extension of our work to a larger class of operators and to commuting n-tuples is currently under way but now we return to the shift operator to describe some other work.

In the paper of von Neumann mentioned earlier he classified all isometries although the result is usually associated with the name of Wold who obtained it over ten years later. von Neumann showed that an isometry has the form $V=W \oplus U_+ \oplus U_+ \oplus \ldots$, where W

is unitary and the number of shifts occurring is equal to $\alpha=\dim\ker V^*$. About 1970 I proved that if V is non unitary and one allows himself a compact perturbation, then W can be dispensed with. More precisely if $\mathcal{K}(\mathcal{H})$ denotes the closed two-sided ideal of compact operators on $\mathcal{K}$ one has

Proposition: *For V a nonunitary isometry on $\mathcal{H}$ there exists K in $\mathcal{K}(\mathcal{H})$ such that $V+K$ is unitarily equivalent to α copies of U_+.*

In particular, this says if we consider the notion of unitary equivalence modulo the compacts, then nonunitary isometries are characterized completely by their multiplicity. If we restrict ourselves to isometries V of finite multiplicity, then one can show that the self-commutator $[V, V^*]= VV^*-V^*V$ is the projection onto $\ker V^*$ and hence compact. In my joint work with L. G. Brown and P. A. Fillmore [3], [4], [5] we extended this classification relative to this weakened notion of equivalence to the class of essentially normal operators, that is, operators T on $\mathcal{H}$ for which $[T, T^*]$ is compact.

Thus one is given T and $\tilde{T}$ in $\mathcal{L}(\mathcal{H})$ and one seeks to determine when there exists a unitary operator W such that $W^*TW-\tilde{T}$ is compact. Since $\mathcal{K}(\mathcal{H})$ is a closed two-sided ideal in $\mathcal{L}(\mathcal{H})$, we can define the quotient $Q(\mathcal{H})=\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ usually called the Calkin algebra and the quotient map $\pi:\mathcal{L}(\mathcal{H})\to Q(\mathcal{H})$. For T in $\mathcal{L}(\mathcal{H})$ the essential spectrum of T is defined by $\sigma_e(T)=\sigma_{Q(\mathcal{H})}(\pi(T))$. Since $W^*TW-\tilde{T}$ compact implies that $\pi(T)$ and $\pi(\tilde{T})$ are unitarily equivalent in $Q(\mathcal{H})$, we have the necessary condition: $\sigma_e(T)=\sigma_e(\tilde{T})$.

For T and $\tilde{T}$ in $n+\mathcal{K}$, where n denotes the normal operators, it is essentially the Weyl-von Neumann-Berg Theorem that this condition is actually sufficient. For general essentially normal operators it is not. For example both U_+ on ℓ^2 and M_z on $L^2(\mathbf{T})$ are essentially normal with essential spectrum equal to the unit circle $\mathbf{T}$ in $\mathbf{C}$, where M_z denotes multiplication by z, but for no unitary operator $W: L^2(\mathbf{T})\to\ell^2$ is it true that $W^*U_+W-M_z$ is compact. This follows since the index of U_+ is -1, while that of M_z is 0. To be more precise we must recall a few more definitions.

An operator T in $\mathcal{L}(\mathcal{H})$ is said to be Fredholm if $\pi(T)$ is invertible in $Q(\mathcal{H})$ or 0 is not in $\sigma_e(T)$ or equivalently, if ran T is closed while both ker T and ker T^* are finite dimensional. The set Fred $(\mathcal{H})$ of Fredholm operators is an open, multiplicative semigroup closed under compact perturbation. The index of a Fredholm operator is defined by $\operatorname{ind}(T)=\dim\ker T-\dim\ker T^*$ and is a continuous, multiplicative homomorphism which is invariant under compact perturbation.

If $W^*TW-\tilde{T}$ is compact, then so is $W^*(T-\lambda)W-(\tilde{T}-\lambda)$ and hence we have another necessary condition: $\operatorname{ind}(\tilde{T}-\lambda)=\operatorname{ind}(T-\lambda)$ for λ not in $\sigma_e(T)=\sigma_e(\tilde{T})$. Our main result is that these two necessary conditions are actually sufficient.

Theorem: *For T and $\tilde{T}$ in $\mathcal{L}(\mathcal{H})$ satisfying $[T, T^*]$ and $[\tilde{T}, \tilde{T}^*]$ compact we have $W^*TW-\tilde{T}$ is compact for some unitary operator W if and only if $\sigma_e(T)=\sigma_e(\tilde{T})$ and* $\operatorname{ind}(T-\lambda)=\operatorname{ind}(\tilde{T}-\lambda)$ *for λ not in $\sigma_e(T)$.*

Corollary: *An operator T is in $h+\mathcal{K}$ if and only if $[T, T^*]$ is in $\mathcal{K}$ and* ind *$(T-\lambda)=0$ for λ not in $\sigma_e(T)$.*

Corollary: *The linear span $h+\mathcal{K}$ is norm-closed.*

This result follows from the characterization of operators in $h+\mathcal{K}$ contained in the previous corollary and no proof not using these results is known. Moreover, the analogous result for commuting pairs of normal operators is false.

The proof of the theorem involves the introduction of ideas from algebraic topology and homological algebra. For T essentially normal we let ε_T be the norm-closed *-subalgebra of $\mathcal{L}(\mathcal{H})$ generated by I, T and $\mathcal{K}$. Then the quotient $\varepsilon_T/\mathcal{K}$ can be identified as the norm-closed *-subalgebra of $Q(\mathcal{H})$ generated by $\pi(I)$ and $\pi(T)$ and since $\pi(T)$ is normal, $\mathcal{E}_T/\mathcal{K}$ is commutative. Therefore by the Gelfand-Naimark Theorem it follows that $\mathcal{E}_T/\mathcal{K}$ is isometrically isomorphic to $C(\sigma_e(T))$ with $T+\mathcal{K}\to z$. Let φ_T be the composite

$$\begin{array}{ccc} \mathcal{E}_T & \to & \mathcal{E}_T(\mathcal{K}) \\ & \searrow & \wr\| \\ \varphi_T & & C(\sigma_e(T)) \end{array}$$

Then we have the extension of $\mathcal{K}(\mathcal{H})$ by $C(\sigma_e(T))$ defined by

$$0\to\mathcal{K}(\mathcal{H})\to\mathcal{E}_T\xrightarrow{\varphi_T}C(\sigma_e(T))\to 0.$$

In our work we consider extensions of $\mathcal{K}(\mathcal{H})$ by $C(X)$ for X a compact metrizable space, introduce an equivalence relation on extensions, and denote the equivalence classes so obtained by Ext (X). Essentially normal operators T and $\tilde{T}$ define equivalent extensions $(\mathcal{E}_T, \varphi_T)\sim(\mathcal{E}_{\tilde{T}}, \varphi_{\tilde{T}})$ if and only if T and $\tilde{T}$ are unitarily equivalent modulo $\mathcal{K}$. Further, we introduce an addition on Ext (X) relative to which Ext (X) is an abelian group. For extensions $(\mathcal{E}_T, \varphi_T)$ and $(\mathcal{E}_{\tilde{T}}, \varphi_{\tilde{T}})$ and $(\mathcal{E}_{\tilde{T}}, \varphi_{\tilde{T}})$ arising from operators (and all do for $X\subset\mathbf{C}$) we have $(\mathcal{E}_T, \varphi_T)+((\mathcal{E}_{\tilde{T}}, \varphi_{\tilde{T}})=(\mathcal{E}_{T\oplus\tilde{T}}, \varphi_{T\oplus\tilde{T}})$ and $(\mathcal{E}_T, \varphi_T)\sim 0$ if and only if T is in $h+\mathcal{K}$.

Next we show that $X\to$ Ext (X)defines a covariant homotopy functor which can be used to define a generalized homology theory of period two on the category of compact metrizable spaces which is dual to the generalized cohomology theory defined by K-theory. I won't pursue this here except to the extent necessary to indicate the proof of our theorem.

Let $\pi^1(X)$ denote the first cohomotopy group of X, that is the group relative to pointwise multiplication of homotopy classes of maps from X to $\mathbf{C}^*=\mathbf{C}\setminus\{0\}$. Important for our proof is the existence of a homomorphism γ_1: Ext $(X)\to$ Hom $(\pi^1(X), \mathbf{Z})$. Actually for general, X, $\pi^1(X)$ should be replaced by $K^1(X)$ but $K^1(X)=\pi^1(X)$ for $X\subset\mathbf{C}$ and we have:

Theorem: *For $X\subset\mathbf{C}$, the homomorphism*

γ_1: Ext $(X)\to$ Hom$(\pi^1(X), \mathbf{Z})$ *is an isomorphism.*

To see how this provides the classification of essentially normal operators, we need to be able to compute $\gamma_1(\mathcal{E}_T, \varphi_T)$. For $X\subset\mathbf{C}$ we can write $\mathbf{C}\setminus X= V_\infty\cup V_1\cup V_2\cup\cdots$, where V_∞ is the unbounded component while $\{V_i\}_{i\geq 1}$ are the bounded components. Then $\pi^1(X)$ can be shown to be the free abelian group with generators $\{[V_i]\}_{i\geq 1}$. And for T essentially normal one can show that $\gamma_1(\mathcal{E}_T, \varphi_T)[V_i] = \mathrm{ind}\,(T-\lambda_i)$ for some λ_i in V_i; therefore the theorem follows since γ_1 is one-to-one. Moreover, since γ_1 is onto, we see that for any assignment of integers, $\{n_i\}$, to the bounded components $\{V_i\}$ of $\mathbf{C}\setminus X$, there exists an essentially normal operator T such that $\sigma_e(T)=X$ and ind $(T-\lambda)=n_i$ for λ in V_i.

The use of algebraic topology allows one to use a sophisticated form of induction for Ext based on the complexity of X. In algebraic topological terminology one uses the Mayer-Vietoris sequence.

There are many further developments of this theory, with applications to operator theory, C^*-algebras, and algebraic topology but we move on.

We return to the shift operator preparing to discuss joint work of M. B. Abrahamse and myself [1], [2]. Recall that one way of define the Hardy space $H^2(\mathbf{D})$ is as the space of homomorphic functions $f:\mathbf{D}\to\mathbf{C}$ with square-summable Taylor coefficients. If we match l^2 with $H^2(\mathbf{D})$ via the Taylor coefficients, then U_+ on l^2 corresponds with the operator T_z defined by multiplication by z on $H^2(\mathbf{D})$. To obtain all pure isometries (isometries with no unitary part), then by von Neumann's Theorem we may use T_z, $T_z\oplus T_z$, etc. Rather than direct sums we can consider Hardy spaces $H^2_{\mathcal{E}}(\mathbf{D})$ of the Hilbert space $\mathcal{E}$-valued homomorphic functions on $\mathbf{D}$ and the operator $T_{\mathcal{E}}$ defined by multiplication by z. For example, we have $T_{\mathbf{C}^n}$ isometrically isomorphic to the direct sum of n-copies of T_z.

Now suppose $\mathcal{M}$ is a closed subspace of $H^2(\mathbf{D})$ for which $T_z\mathcal{M}\subset\mathcal{M}$. Then we can define the restriction operator $T_z|\mathcal{M}$ which is easily seen to be an isometry. Only a little more difficult is the fact $T_z|\mathcal{M}$ is a pure isometry of multiplicity one and hence is unitarily equivalent to T_z on $H^2(\mathbf{D})$. Thus there exists a unitary operator $W: H^2(\mathbf{D})\to\mathcal{M}$ which satisfies $WT_z = (T_z|\mathcal{M})W$. If we view W as an isometry on $H^2(\mathbf{D})$, then the latter identity becomes the fact that W commutes with T_z. Operators which commute with T_z can be shown to be defined by multiplication by some bounded holomorphic function φ on $\mathbf{D}$ and because the operator defined is unitary it follows easily that $|\varphi|=1$ a.e. on $\mathbf{T}=\partial\mathbf{D}$. Thus $\mathcal{M}=\varphi H^2(\mathbf{D})$, where φ is an inner function and we have obtained another proof of Beurling's Theorem characterizing the invariant subspaces of the shift operator. Now my point in describing this proof is not that it is easier than other proofs (it's not), but to make clear the central role played by the fact that a complete set of models for pure isometries is provided by the operators $T_{\mathcal{E}}$ on $H^2_{\mathcal{E}}(\mathbf{D})$.

Since $T_{\mathcal{E}} = M_{\mathcal{E}}|H^2_{\mathcal{E}}(\mathbf{D})$, where $M_{\mathcal{E}}$ denotes multiplication by z on $L^2_{\mathcal{E}}(T)$, these isometries can be seen to be subnormal operators with spectrum contained in $\mathbf{D}$, while the spectrum of its normal extension is contained in $\partial\mathbf{D}=\mathbf{T}$. We study the analogous class of operators associated with more general domains in $\mathbf{C}$. More precisely, let Ω be a bounded, finitely connected, connected open subset of $\mathbf{C}$ with smooth boundary $\partial\Omega$. We let $\mathcal{S}(\Omega)$ denote the pure subnormal operators S which satisfy $\sigma(S)\subset\overline{\Omega}$ and $\sigma(N)\subset\partial\Omega$, where N is the minimal normal extension of S. We seek models for this class.

One can define a Hardy space $H^2(\Omega)$ for Ω and although it's not quite as elementary as for $\mathbf{D}$, it is well understood. And the operator T_z defined

by multiplication by z on $H^2(\Omega)$ does belong to $\mathcal{S}(\Omega)$ and by considering Hilbert-space valued functions we obtain the higher multiplicity examples $T_{\mathcal{E}}$ on $H^2_{\mathcal{E}}(\Omega)$. However, if Ω is not simply connected, then this will not exhaust the possibilities. To obtain a complete set of models for operators in $\mathcal{S}(\Omega)$, we must consider a certain class of vector bundles over Ω with Hilbert space fiber. The particular class is that of flat unitary bundles or bundles such that relative to some open cover of Ω the transition functions are constant unitary maps. Since such bundles are locally a product, they possess both a Hermitian and a holomorphic structure. Then a subspace of the holomorphic cross-sections of E can be used to define a Hardy space $H^2_E(\Omega)$ and multiplication by z defines what we call the bundle shift T_E on $H^2_E(\Omega)$ about which we prove:

Theorem: *An operator S on $\mathcal{H}$ is in $\mathcal{S}(\Omega)$ if and only if S is unitarily equivalent to some bundle shift T_E.*

Moreover, two bundle shifts T_{E_1} and T_{E_2} are unitarily equivalent if and only if the bundles E_1 and E_2 are equivalent as flat unitary vector bundles.

Lastly, two bundle shifts T_{E_1} and T_{E_2} are similar if and only if the ranks of E_1 and E_2 are equal.

The proof involves some intrinsic techniques on Ω as well as pulling back to the universal covering space $\mathbf{D}$ of Ω. An important ingredient in the proof is the theorem of Grauert-Bungart which yields that the holomorphic bundles over Ω are all trivial as holomorphic bundles (not as flat unitary bundles).

Now it is possible to give a reasonable parametrization of the flat unitary bundles over a space. They correspond to the unitary representations of the fundamental group. Since $\pi_1(\Omega)$ is the free group of n-generators where n is the number of holes, then the equivalence classes of flat unitary vector bundles with fiber $\mathcal{E}$ are in natural correspondence with the equivalence classes of unitary representations of $\pi_1(\Omega)$ on $\mathcal{E}$, or equivalently to the n-tuples of unitary operators on $\mathcal{E}$ modulo conjugation.

Now there is considerably more to this theory including an extension of Beurling's Theorem to this context and other people have extended other results about the unilateral shift to the context of bundle shifts but we stop here.

In this note I have tried to indicate how some natural equivalence problems in operator theory lead one to introduce techniques from other areas of mathematics with the resultant enrichment of both operator theory and the other areas. Hilbert spaces are such natural objects that they occur everywhere and natural operators defined on them are readily apparent. It seems likely that the study of classes occuring in nature holds a promise of deep results for which this exposition provides some evidence.

BIBLIOGRAPHY

1. Abrahamse M. B. and Douglas R. G. — *A class of subnormal operators related to multiply-connected domains*, Adv. Math. 19 (1967) 106—148.
2. Abrahamse M. B. and Douglas R. G. — *Operators on multiply-connected domains*, Proc. Royal Irish Acad. 74 A (1974) 135—141.

3. Brown L. G., Douglas R. G. and Fillmore P. A. — *Unitary equivalence modulo the compact operators and extensions of C*-algebras*, in Proc. Conf. on Operator Theory, Lecture Notes in Mathematics, 345, Springer-Verlag, Heidelberg, 1973.
4. Brown, L. G., Douglas R. G., and Fillmore P. A. — *Extensions of C*-algebras, operators with compact self-commutator, and K-homology*, Bull. Amer. Math. Soc. 79 (1973) 973—978.
5. Brown L. G., Douglas R. G. and Fillmore P. A. — *Extensions of C*-algebras and K-homology*, Ann. Math. 105 (1977) 265—324.
6. Cowen M. J. and Douglas R. G. — *Complex geometry and operator theory*, Bull. Amer. Math. Soc. 83 (1977) 131—133.
7. Cowen M. J. and Douglas R. G. — *Complex geometry and operator theory*, Acta Math. 134 (1978).

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SCHWARZ'S LEMMA FOR OPERATORS ON A HILBERT SPACE*)

BY

KY FAN

1. Introduction. In this Seminar report we summarize some of our recent results related to Schwarz's lemma for operators on a Hilbert space. For the proofs of these results, the reader is referred to our papers [2], [3].

Let $\mathcal{H}$ be a Hilbert space which will be always complex. By an operator on $\mathcal{H}$ we shall mean a bounded linear transformation on $\mathcal{H}$. A *proper contraction* is an operator A with $\|A\|<1$. For two Hermitian operators A, B on $\mathcal{H}$, we write $A \geqslant B$ to indicate that $A-B$ is positive, i.e., $((A-B)x, x) \geqslant 0$ for all $x \in \mathcal{H}$. The notation $A > B$ will mean that $A-B$ is positive and invertible.

Throughout this paper, Δ will denote the open unit disk $\Delta=\{z : |z|<1\}$ in the complex plane. $H(\Delta)$ will denote the class of all analytic functions on Δ. For $f \in H(\Delta)$ and a proper contraction A on a Hilbert space we use the usual notation $f(A)$ to denote the operator $\sum_{n=0}^{\infty}[f^{(n)}(0)/n!]\ A^n$ which converges in the norm topology for operators.

2. Schwarz's lemma and related inequalities. We begin with the following result.

Theorem 1. *Let A be a proper contraction on a Hilbert space $\mathcal{H}$. Let $f, g, h \in H(\Delta)$ be such that $f=gh$ and $|h(z)| \leqslant 1$ for $z \in \Delta$. Then*

$$g(A)^*g(A) \geqslant f(A)^*f(A). \tag{1}$$

$$\|g(A)\| < \|f(A)\|. \tag{2}$$

Strict inequality holds in (1) if and only if $g(A)^*g(A)>0$ and h is not a constant function of absolute value 1. Equality in (2) holds if and only if either $g(A)=0$ or h is a constant function of absolute value 1.

A special case of Theorem 1 is

Corollary 1. (Jensen's inequality). *Let A be a proper contraction on a Hilbert space. Let $f \in H(\Delta)$ be such that $f(\Delta) \subset \Delta$. Let $z_1, z_2, \cdots, z_n$ be some*

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