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SCHWARZ'S LEMMA FOR OPERATORS ON A HILBERT SPACE*)

BY

KY FAN

1. Introduction. In this Seminar report we summarize some of our recent results related to Schwarz's lemma for operators on a Hilbert space. For the proofs of these results, the reader is referred to our papers [2], [3].

Let $\mathcal{H}$ be a Hilbert space which will be always complex. By an operator on $\mathcal{H}$ we shall mean a bounded linear transformation on $\mathcal{H}$. A *proper contraction* is an operator A with $\|A\| < 1$. For two Hermitian operators A, B on $\mathcal{H}$, we write $A \geq B$ to indicate that $A - B$ is positive, i.e., $((A-B)x, x) \geq 0$ for all $x \in \mathcal{H}$. The notation $A > B$ will mean that $A - B$ is positive and invertible.

Throughout this paper, Δ will denote the open unit disk $\Delta = \{z : |z| < 1\}$ in the complex plane. $H(\Delta)$ will denote the class of all analytic functions on Δ. For $f \in H(\Delta)$ and a proper contraction A on a Hilbert space we use the usual notation $f(A)$ to denote the operator $\sum_{n=0}^{\infty} [f^{(n)}(0)/n!]\ A^n$ which converges in the norm topology for operators.

2. Schwarz's lemma and related inequalities. We begin with the following result.

Theorem 1. *Let A be a proper contraction on a Hilbert space $\mathcal{H}$. Let $f, g, h \in H(\Delta)$ be such that $f = gh$ and $|h(z)| \leq 1$ for $z \in \Delta$. Then*

$$g(A)^* g(A) \geq f(A)^* f(A), \tag{1}$$

$$\|g(A)\| \leq \|f(A)\|. \tag{2}$$

Strict inequality holds in (1) if and only if $g(A)^* g(A) > 0$ and h is not a constant function of absolute value 1. Equality in (2) holds if and only if either $g(A) = 0$ or h is a constant function of absolute value 1.

A special case of Theorem 1 is

Corollary 1. (Jensen's inequality). *Let A be a proper contraction on a Hilbert space. Let $f \in H(\Delta)$ be such that $f(\Delta) \subset \Delta$. Let $z_1, z_2, \cdots, z_n$ be some*

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(not necessarily all) of the zeros of f in Δ, each of which is allowed to repeat not oftener than its multiplicity. Let

$$g(z)=\prod_{k=1}^{n}\frac{z-z_k}{1-\bar{z}_k z}.$$

Then (1) and (2) hold. Strict inequality in (1) occurs if and only if $g(A)^*g(A)>0$ and f is not of the form $f(z)=\eta g(z)$ for some constant η with $|\eta|=1$. There is equality in (2) if and only if either $g(A)=0$ or $f(z)=\eta g(z)$ with $|\eta|=1$.

In the case $z_1=z_2=\cdots=z_n=0$, Corollary 1 becomes

Corollary 2. (Schwarz's lemma). *Let A be a proper contraction on a Hilbert space. Let $f\in H(\Delta)$ be such that $f(\Delta)\subset\Delta$ and $f(0)=f'(0)=\cdots=f^{(n-1)}(0)=0$ for some integer $n\geqslant 1$. Then*

$$A^{n*}A^n\geqslant f(A)^*f(A), \tag{3}$$

$$\|A^n\|\geqslant\|f(A)\|. \tag{4}$$

Strict inequality in (3) holds if and only if $A^{n*}A^n>0$ and f is not of the form $f(z)=\eta z^n$ for some constant η with $|\eta|=1$. Equality occurs in (4) if and only if either $A^n=0$ or $f(z)=\eta z^n$ with $|\eta|=1$.

The next corollary is a direct consequence of Corollary 2.

Corollary 3. (Harnack's double inequality). *Let A be a proper contraction on a Hilbert space. Let $g\in H(\Delta)$ be such that $g(0)=1$ and $\operatorname{Re} g(z)>0$ for $z\in\Delta$. Then*

$$\frac{1-\|A\|}{1+\|A\|}\leqslant\|g(A)\|\leqslant\frac{1+\|A\|}{1-\|A\|}.$$

3. Pick's theorem and Julia's lemma. The following analog of Pick's theorem generalizes the case n=1 of Corollary 2.

Theorem 2. *Let A be a proper contraction on a Hilbert space. Let $f\in H(\Delta)$ be such that $f(\Delta)\subset\Delta$, and let $z_0\in\Delta$. Then*

$$(I-z_0A^*)^{-1}(A^*-\bar{z}_0I)(A-z_0I)(I-\bar{z}_0A)^{-1}\geqslant\{I-f(z_0)f(A)^*\}^{-1}\{f(A)^*-\overline{f(z_0)}\,I\}\{f(A)-f(z_0)\,I\}\{I-\overline{f(z_0)}\,f(A)\}^{-1} \tag{5}$$

$$\|(A-z_0I)(I-\bar{z}_0A)^{-1}\|\geqslant\|\{f(A)-f(z_0)\,I\}\{I-\overline{f(z_0)}\,f(A)\}^{-1}\|. \tag{6}$$

Strict inequality in (5) occurs if and only if $(A^*-\bar{z}_0I)(A-z_0I)>0$ and f is not of the form

$$f(z)=\varepsilon\frac{z-z_1}{1-\bar{z}_1z}\ \text{with}\ |\varepsilon|=1,\ |z_1|<1. \tag{7}$$

Equality in (6) occurs if and only if either $A=z_0I$ or f is of the form (7).

The classical Julia's lemma ([1], p. 25) has the following generalization.

Theorem 3. *Let $f\in H(\Delta)$ be such that $f(\Delta)\subset\Delta$. Let $\{z_n\}\subset\Delta$ be such that*

$$\lim_{n\to\infty}z_n=\lim_{n\to\infty}f(z_n)=1$$

and

$$\lim_{n\to\infty}\frac{1-|f(z_n)|}{1-|z_n|}=\alpha,$$

where α is finite. Let A be a proper contraction on a Hilbert space. Then

$$\|\{I-f(A)\}\{I-f(A)^*f(A)\}^{-1}\{I-f(A)^*\}\|\leqslant\alpha\|(I-A)(I-A^*A)^{-1}(I-A^*)\|.$$

Furthermore, for any positive number β, inequality

$$\left\|A-\frac{1}{1+\beta}I\right\|<\frac{\beta}{1+\beta}$$

implies

$$\left\|f(A)-\frac{1}{1+\alpha\beta}I\right\|<\frac{\alpha\beta}{1+\alpha\beta}.$$

4. Principle of subordination. A simple consequence of the principle of subordination ([4], p. 421) and Corollary 2 is the following result which has a number of interesting consequences.

Theorem 4. *Let $g\in H(\Delta)$ be univalent on Δ with $g(0)=0$. If $f\in H(\Delta)$, $f(0)=0$ and $f(\Delta)\subset g(\Delta)$, then for every proper contraction A on a Hilbert space $\mathcal{H}$, there is a unique proper contraction B on $\mathcal{H}$ satisfying $f(A)=g(B)$. Furthermore:*

$$A^*A\geqslant B^*B,\quad\|A\|\geqslant\|B\|.$$

We have $A^*A>B^*B$ if and only if $A^*A>0$ and f is not of the form $f(z)=g(\eta z)$ for some constant η with $|\eta|=1$. Equality $\|A\|=\|B\|$ occurs if and only if either $A=0$ or f is of the form $f(z)=g(\eta z)$ with $|\eta|=1$.

In case $g(z)=z$, Theorem 4 becomes the case $n=1$ of Corollary 2. Another special case of Theorem 4 is the following extension of Koebe-Bieberbach's 1/4-theorem for univalent functions ([4], p. 350).

Corollary 4. *Let $g\in H(\Delta)$ be univalent on Δ, and $g(0)=0$, $g'(0)=1$. For every operator C on a Hilbert space $\mathcal{H}$ with $\|C\|<1/4$, there is a unique proper contraction B on $\mathcal{H}$ satisfying $g(B)=C$. Furthermore, $16C^*C\geqslant B^*B$ with strict inequality in case $C^*C>0$; and $\|B\|<4\|C\|$ unless $C=0$.*

Corrollary 5. *Let $g\in H(\Delta)$ be univalent on Δ with $g(0)=0$, $g'(0)=1$. If the image $g(\Delta)$ is a starlike set with respect to the origin, then the set of all $g(A)$, when A runs through all proper constructions on $\mathcal{H}$, is a starlike set of operators. More precisely, for every proper contraction A on $\mathcal{H}$ and any non-negative real number $r<1$, there is a unique proper contraction B on $\mathcal{H}$ such that $g(B)=rg(A)$.*

5. A theorem of von Neumann. An interesting special case of Theorem 1 is the case when g is the constant function 1. This case of Theorem 1 may be restated in the following form.

Theorem 5. *Let A be a proper contraction on a Hilbert space. If $f\in H(\Delta)$ and $f(\Delta)\subset\Delta$, then $\|f(A)\|<1$.*

Theorem 5 is closely related to an important theorem of von Neumann [5] in his theory of spectral sets. Each of the two theorems can be quickly derived from the other. Another closely related result is

Theorem 6. *Let $f\in H(\Delta)$. For $0\leqslant r<1$, let $M(r)=\operatorname{Max}_{|z|=r}\|f(z)\|$. Then*

$$M(r)=\operatorname*{Max}_{\|A\|\leqslant r}\|f(A)\|$$

for $0\leqslant r<1$, where the maximum is taken over all operators A of norm $\leqslant r$ on a Hilbert space $\mathcal{H}$.

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SOME PERTURBATION PROBLEMS FOR VOLTERRA EQUATIONS*)

BY

I. J. LEVIN

We'll consider the behavior of solutions of the nonlinear Volterra equation

$$x(t)+\int_0^t a(t-s)[x(s)+h(s,x(s))]\,ds=q(t)\qquad(0\leqslant t<\infty),\tag{1}$$

where x, h, and q are complex valued N-dimensional vectors and a is an $N\times N$ complex valued matrix. a, h, and q are prescribed ; x is the unknown. Assume that

$$h,\,q\in C,\quad a\in L^1_{loc}\,;\tag{2}$$

thus

$$\int_0^t |a(s)|\,ds<\infty\;\forall\, t\in \mathbf{R}^+=[0,\infty),$$

where $|a|=\sum_{i,j=1}^N |a_{ij}|$. The corresponding norm for vectors is $|x|=\sum_{i=1}^N |x_i|$.

Most of what will be said has a counterpart when the kernel $a(t-s)$ is replaced by $a(t,s)$. However, as that entails many more technicalities, we'll confine ourselves here to the convolution situation.

This is a perturbation study because, first, h, whether it depends on t or not, is assumed, as a function of x to vanish faster than linearly. Thus, h is a nonlinear perturbation of the term x. Second, because q, the forcing term in (1), is the sum of two functions

$$q=\omega+f,$$

where it is assumed that $f(t)\to 0$ as $t\to\infty$. Thus, f is a perturbation of ω, the principal term of q. Typically ω is either constant (possibly zero), or periodic,

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