REFERENCES

1. Carathéodory C. — *Funktionentheorie* II, Birkhäuser, Basel, 1950.
2. Fan K. — *Analytic functions of a proper contraction*, Math. Z. (1978).
3. Fan K. — *Julia's lemma for operators*, (197).
4. Hille E. — *Analytic function theory* II, Ginn, Boston, 1962.
5. Neumann J. von — *Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes*, Math. Nachr. 4 (1950/51), 258—281.

Department of Mathematics
University of California
Santa Barbara, CA 93106 U.S.A.

Analele ştiinţifice ale Universităţii „Al. I. Cuza" Iaşi
Supliment la tomul XXV, s. I a 1979
Cu ocazia centenarului naşterii Acad. Prof. A. Myller

SOME PERTURBATION PROBLEMS FOR VOLTERRA EQUATIONS*)

BY

I. J. LEVIN

We'll consider the behavior of solutions of the nonlinear Volterra equation

$$x(t) + \int_0^t a(t-s)[x(s) + h(s, x(s))]\,ds = q(t) \qquad (0 \leqslant t < \infty), \tag{1}$$

where x, h, and q are complex valued N-dimensional vectors and a is an $N \times N$ complex valued matrix. a, h, and q are prescribed ; x is the unknown. Assume that

$$h, q \in C, \quad a \in L^1_{loc}; \tag{2}$$

thus

$$\int_0^t |a(s)|\,ds < \infty \;\forall t \in \mathbf{R}^+ = [0, \infty),$$

where $|a| = \sum_{i,j=1}^{N} |a_{ij}|$. The corresponding norm for vectors is $|x| = \sum_{i=1}^{N} |x_i|$.

Most of what will be said has a counterpart when the kernel $a(t-s)$ is replaced by $a(t, s)$. However, as that entails many more technicalities, we'll confine ourselves here to the convolution situation.

This is a perturbation study because, first, h, whether it depends on t or not, is assumed, as a function of x to vanish faster than linearly. Thus, h is a nonlinear perturbation of the term x. Second, because q, the forcing term in (1), is the sum of two functions

$$q = \omega + f,$$

where it is assumed that $f(t) \to 0$ as $t \to \infty$. Thus, f is a perturbation of ω, the principal term of q. Typically ω is either constant (possibly zero), or periodic,

---

*) Lecture presented at the Romanian-American Seminar on Operator Theory and Applications, 20—24 March 1978.

**) Visiting Professor, University of British Columbia, 1977—1978. This research was partially supported by the U.S. Army Research Office.

or almost periodic. In which casses q is, respectively, either asymptotically constant, or asyptotically periodic, or asymptotically almost periodic.

The resolvent of a is defined to be the solution, r, of

$$r(t)+\int_0^t a(t-s)\, r(s)\, \mathrm{d}s = a(t) \quad (0 \leqslant t < \infty).$$

In many studies of (1), including this one, it plays a critical role. Some relevant well known properties of the resolvent are, first,

$$a \in L^1_{\text{loo}} \Rightarrow r \in L^1_{\text{loo}}, \quad r \text{ is unique.}$$

Also, the unique solution of the linear equation

$$\theta(t)+\int_0^t a(t-s)\theta(s)\, \mathrm{d}s = q(t) \quad (0 \leqslant t < \infty), \tag{3}$$

which is the linearized form of (1), is given by

$$\theta(t) = q(t) - \int_0^t r(t-s)\, q(s)\, \mathrm{d}s \quad (0 \leqslant t < \infty). \tag{4}$$

This, and the further property that

$$\int_0^t a(t-s)\, r(s)\, \mathrm{d}s = \int_0^t r(t-s)\, a(s)\, \mathrm{d}s (0 \leqslant t < \infty),$$

enables one readily show that a solution of (1) also satifies

$$x(t)+\int_0^t r(t-s)\, h(s, x(s))\, \mathrm{d}s = \theta(t) \quad (0 \leqslant t < \infty). \tag{5}$$

Of course, a solution of (1) needn't exist for all t. In that case, for as long as it does exist, it also a solution of (5). For later purposes it is convenient to write (5) equivalently as

$$x(t)+\int_0^t r(s)\, h(t-s, x(t-s))\, \mathrm{d}s = \theta(t) \quad (0 \leqslant t < \infty). \tag{5}$$

Our key assumption is that

$$r \in L^1. \tag{6}$$

Thus

$$\|r\|_1 = \int_0^\infty |r(t)|\, \mathrm{d}t < \infty.$$

Let x be a continuous solution of (1). It is evident, at least intuitively, that (5), with hypothesis (6), is a better starting point for studying the asymptotic behavior of x than (1) itself. For, a kernel, a, which need not be in L^1 has been replaced by one that is. Also, the small nonlinear term, h, has been isolated. Finally, θ, the solution of the linearized form of (1) is explicitly in the equation.

A few words should be said about (6). When $N=1$ and $a \in L^1$, there is the classical result :

Paley and Wiener [8]. *If $a \in L^1$, then (6) holds if and only if*

$$\int_0^\infty e^{-zt} a(t)\, \mathrm{d}t \neq -1 \qquad (\operatorname{Re} z \geqslant 0).$$

There are some important kernels which are not in L^1 but whose associated resolvent is. The best known of these is $a(t)=t^{-1/2}$, which occurs in many applications, whose resolvent behaves like (const.) $t^{-3/2}$ as $t \to \infty$. Some interesting results of Shea and Wainger [9] give rise to classes of kernels which while not in L^1 have a resolvent which is.

Many papers concerning (1) come uncer the general terms employed so far. Some are: Jordan and Wheeler [1], Kaplan [2], Miller, Nohel, and Wong [4], [5], Nohel [6], [7], and Strauss [10]. Since all ofthese papers deal with the nonconvolution situation, $a=a(t, s)$, any detailed comparison of results, which won't be given here, would have to be made with it. However the two novelties of the present approach also appear in the convolution case which we are discussing. The first is a somewhat sharper a priori bound on the solutions of (1). The second, and more significant one, is in the use of an auxiliary equation to study the asymptotic behavior of the solution of (1) as $t \to \infty$. They lead to more detailed information and much simpler proofs.

We need the

Definition. $\Gamma : \mathbf{R}^+ \to \mathbf{R}^+$, nondecreasing, is called an a priori bound for (1) if

$$|x(t)| \leqslant \Gamma(t) \quad (0 \leqslant t < \hat{t})$$

whenever $x \in C$ is a solution of (1) on $[0, \hat{t})$, where $\hat{t} \in (0, \infty]$.

The first result is

Theorem. 1. *Let (2), (6), $q \in L^\infty$, and*

$$\begin{cases} \forall\, \varepsilon > 0\ \exists\, r(\varepsilon) > 0 \ni \\ |x| \leqslant r(\varepsilon),\ t \in \mathbf{R}^+ \Rightarrow |h(t, x)| \leqslant \varepsilon |x| \end{cases} \tag{7}$$

$$\|q\|_\infty \leqslant \frac{1}{1+\|r\|_1} \frac{\alpha-1}{\alpha} r\left(\frac{1}{\alpha \|r\|_1}\right) \qquad (if\ \|r\|_1 > 0)$$

hold for some $\alpha > 1$. *Then*

$$\Gamma(t) = \begin{cases} (1+\|r\|_1)\dfrac{\alpha}{\alpha-1}\|q\|_\infty & (if \|r\|_1 > 0) \\ \|q\|_\infty & (if \|r\|_1 = 0) \end{cases}$$

is an a priori bound for (1).

Hypothesis (7) says simply that h is $o(x)$ as $x \to 0$ uniformly with respect to t. The restriction on $\|q\|_\infty$ is made, of course, in order to be able to employ (7). The new feature in Theorem 1 is the explicit appearance of $\|q\|_\infty$ in Γ.

As a consequence of the existence of Γ, it is well known that

Corollary 1a. *The hypothesis of Theorem 1 implies that there exists a continuous solution of* (1) *on* $\mathbf{R}^+$.

Another consequence of Theorem 1 is

Corollary 1b. *The hypothesis of Theorem 1 and*

$$\lim_{t\to\infty} \{\max_{|x|<K} |h(t, x)|\} = 0 \quad (\forall K \in \mathbf{R}^+) \tag{8}$$

implies that

$$\lim_{t\to\infty} [x(t) - \theta(t)] = 0 \qquad \forall \textit{ solution } x, \textit{ of } (1).$$

This type of results is usually referred to by saying that (1) and its linearized form, (3), are asymptotically equivalent as $t \to \infty$ if h gets small as $t \to \infty$ (uniformly with respect to x on compact sets). It follows almost trivially from (5)

When (8) isn't assumed, the analysis of x for large t is more interesting. Instead of (7) we now impose the somewhat more stringent condition

$$\begin{cases} \{h \in C,\ h(t, 0) = 0,\ \forall \varepsilon > 0\ \exists \delta(\varepsilon) > 0 \Rightarrow \\ |x_1| \leqslant \delta(\varepsilon),\ |x_2| \leqslant \delta(\varepsilon),\ t \in \mathbf{R}^+ \Rightarrow |h(t, x_1) - h(t, x_2)| \leqslant \varepsilon |x_1 - x_2|. \end{cases} \tag{9}$$

Thus,(9) implies (7) with $\gamma = \delta$. Of course, (9) implies that the solution of(1) guaranteed by Theorem 1 is unique. The key tool is the auxiliary equation

$$y(t) + \int_0^\infty r(s) H(t-s, y(t-s))\, ds = \psi(t) \quad (-\infty < t < \infty). \tag{10}$$

Notice that, unlike (1), (3), and (5), (10) is defined on $\mathbf{R}^1$, not $\mathbf{R}^+$.

Concerning H we assume that

$$\begin{cases} \{H : \mathbf{R}^1 \times \mathbf{C}^N \to \mathbf{C}^N \text{ is an extension of } h \Rightarrow \\ H \text{ satisfies (9) on } \mathbf{R}^1 \times \mathbf{C}^N. \end{cases} \tag{11}$$

For any h satisfying (9), setting

$$\begin{cases} H(t, x) = h(t, x) \quad (t \in \mathbf{R}^+) \text{ and (for example) any of} \\ H(t, x) = 0, \quad H(t, x), = h(t, x) \quad H(t, x) = -h(-t, x) \quad (t \leqslant 0), \end{cases}$$

defines an H such that (11) is satisfied. Thus, (11) is not an additional restriction beyond (9).

In the important special case

$$h(t, x) = h(x) \quad (t \in \mathbf{R}^+),$$

the most „natural" definition of H is

$$H(t, x) = h(x) \quad (t \in \mathbf{R}^1).$$

Other special cases in which there is a natural extension of h are described by

$$(12) \quad (11) \text{ holds and } H(t, x) = H(t+\rho, x) \quad (-\infty < t < \infty),$$

where $\rho > 0$, and by

$$\begin{cases} (11) \text{ holds and } H(t, x) \text{ is almost periodic with respect to } t, \\ \text{uniformly with respect to } x, \text{ for } x \text{ in any compact subset of } \mathbf{C}^N. \end{cases} \tag{13}$$

The forcing term ψ in (10) will, ultimately, be related to θ and, therefore, to q. However, it is convenient to first regard it as an independent function.

Concerning (10) we need the following result which is established by a straightforward succesive approximation argument.

Lemma. *Let* (6), (11), *and*

$$\|\psi\|_\infty \leqslant \frac{\alpha-1}{\alpha}\,\delta\left(\frac{1}{\alpha\|r\|_1}\right) \quad (if\ \|r\|_1 > 0) \tag{14}$$

hold for some $\alpha > 1$. *Then* (10) *has a solution* y_ψ *with*

$$\|y_\psi\|_\infty \leqslant \frac{\alpha}{\alpha-1}\|\psi\|_\infty.$$

Moreover, $y = y_\psi$ *is the unique solution of* (10) *such that*

$$\|y\|_\infty \leqslant \delta\left(\frac{1}{\alpha\|r\|_1}\right).$$

The asymptotic relationship between (1) and (10) is given by

Theorem 2. *Let the hypotheses of Theorem 1 and the Lemma hold for the same* $\alpha > 1$. *Also, let*

$$\lim_{t\to\infty} [\theta(t) - \psi(t)] = 0. \tag{15}$$

Then

$$\lim_{t\to\infty} [x(t) - y_\psi(t)] = 0. \tag{16}$$

Hypothesis (15) is not an additional restriction on the prescribed functions a, h, and q of (1). For if the assumptions of Theorem 2, excluding those relating to ψ, are made, then (first by (4) and then by the hypothesis of Theorem 1)

$$\|\theta\|_\infty \leqslant (1+\|r\|_1)\,\|q\|_\infty \leqslant \frac{\alpha-1}{\alpha}\, r\left(\frac{1}{\alpha\|r\|_1}\right) = \frac{\alpha-1}{\alpha}\,\delta\left(\frac{1}{\alpha\|r\|_1}\right).$$

Hence, taking $\psi=0$, yields a ψ which satisfies the conditions of the Lemma and Theorem 2. The proof of Theorem 2 is quite elementary.

Various known results concerning the asymptotic behavior of x follow immediately from the Lemma and Theorem 2. For example, let

$$\omega(t)=\omega(t+\rho) \quad (-\infty<t<\infty), \tag{17}$$

where $\rho<0$, and let

$$\theta_0(t)=\omega(t)-\int_0^\infty r(s)\omega(t-s)\,ds \quad (-\infty<t<\infty).$$

$$\theta_1(t)=f(t)-\int_0^t r(s)\,f(t-s)\,ds+\int_t^\infty r(s)\,\omega(t-s)\,ds \quad (0\leqslant t<\infty).$$

Then, $q=\omega+f$, (4), (17), and $f(\infty)=0$ imply that

$$\theta(t)=\theta_0(t)+\theta_1(t),\ \ \theta_0(t)=\theta_0(t+\rho),\ \lim_{t\to\infty}\theta_1(t)=0.$$

Corollary 2. *Let the hypotheses of Theorem* 1, *the Lemma*, (12), (17), *and*

$$\|\omega\|_\infty \leqslant \frac{1}{1+\|r\|_1}\,\frac{\alpha-1}{\alpha}\,\delta\left(\frac{1}{\alpha\|r\|_1}\right)$$

$$\lim_{t\to\infty} f(t)=0$$

hold. Then

$$y_{\theta_0}(t)=y_{\theta_0}(t+\rho)$$

and

$$\lim_{t\to\infty}\,[x(t)-y_{\theta_0}(t+\rho)]=0. \tag{18}$$

Thus, x is asymptotically periodic, of period ρ, as $t\to\infty$. The proof consists simply in observing that the successive approximations for (10), with $\psi=\theta_0$, are, in this case, periodic, and that

$$\lim_{t\to\infty}[\theta(t)-\psi(t)]=\lim_{t\to\infty}\theta_1(t)=0.$$

A completely analogous result, with essentially the same proof, holds in the almost periodic situation.

In Corollary 2, x is the unique solution of

$$x(t)+\int_0^t r(s)\,H(t-s,\,x(t-s))\,ds=\theta_0(t)+\theta_1(t) \quad (0\leqslant t<\infty), \tag{$\tilde{5}$}$$

as well as of (1), and y_{θ_0} is the unique solution of

$$y(t)+\int_0^\infty r(s)\,H(t-s,\,y(t-s))\,ds=\theta_0(t) \quad (-\infty<t<\infty). \tag{$\tilde{10}$}$$

The same periodicity of H and θ_0, $\theta_1(\infty)=0$, imply that ($\tilde{10}$) is the limit equation associated with ($\tilde{5}$). Given the uniqueness of y_{θ_0}, from the Lemma, (18) also follows from a general result of Levin and Shea [3]. However, if in ($\tilde{5}$), θ_0 and H are only almost periodic, then ($\tilde{10}$) is only one of a family of limit equations associated with ($\tilde{5}$). Thus, the results of [3] don't quite do the job. In the general situation of Theorem 2, there is, of course, no limit equation possible. Thus, the auxiliary equation, (10), is, at least for the problems studied here, a more adaptable tool than the limit equation. This is particularly true in the nonconvolution situation.

Corollary 1*b* may be regarded as providing sufficient conditions for the asymptotic equivalence of (1) and (3). In contrast, Theorem 2 only gives sufficient conditions for a partial asymptotic equivalence of (1) and (10). That is, for each appropriate q there exist, as has been shown, appropriate ψ's such that (15) and, therefore, (16) hold. However, under the hypothesis of Theorem 2, this procedure can not, in general, be reversed., That is, starting with a ψ which satisfies (14), there does not, in general, exist an appropriate q. This can be shown by simple examples.

REFERENCES

1. Jordan G. S. and Wheeler R. L. — *On the asymptotic behavior of perturbed Volterra integral equations*, SIAM J. Math. Anal. 5 (1974), 273—277.
2. Kaplan J. L. — *On the asymptotic behavior of Volterra integral equations*, SIAM J. Math. Anal. 3 (1972), 148—156.
3. Levin J. J. and Shea D. F. — *On the asymptotic behavior of the bounded solutions of some integral equations*, I, II, III, J. Math. Anal. Appl. 37 (1972), 42—82, 288—326, 537—575.
4. Miller R. K., Nohel J. A., and Wong J. S. W. — *Perturbations of Volterra integral equations*, J. Math. Anal. Appl. 25 (1969), 676—691.
5. Miller R. K., Nohel J. A. and Wong J. S. W. — *A stability theorem for nonlinear mixed integral equations*, J. Math. Anal. Appl. 25 (1969), 446—449.
6. Nohel J. A. — *Asymptotic relationships between systems of Volterra equations*, Ann. di Mat. pura ed appl. (IV) 90 (1971), 149—165.
7. Nohel J. A. — *Asymptotic equivalence of Volterra equations*, Ann. di Mat. pura ed appl. (IV) 96 (1973), 339—347.
8. Paley R. E. A. C. and Wiener N. — *Fourier Transforms in the Complex Domain*, Amer. Math. Soc. Colloq. Publ., vol. 19, Amer. Math. Soc., Providence, R. I., 1934.
9. Shea D. F., and Wainger S. — *Variants of the Wiener-Lévy theorem, with applications to stability problems for some Volterra integral equations*, Amer. J. Math. 97 (1975), 312—343.
10. Strauss A. — *On a perturbed Volterra equation*, J. Math. Anal. Appl. 30 (1970), 564—575.

University of Wisconsin, Madison