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ON SOME NONLINEAR VOLTERRA FUNCTIONAL DIFFERENTIAL EQUATIONS

BY

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1. **Introduction.** In this lecture we use the Banach fixed point theorem to discuss the global existence, uniqueness, dependence on data and regularity of solutions of the abstract Cauchy problem for the functional differential equation

$$\text{(FDE)} \qquad \frac{dy}{dt} + Ay \ni G(y) \quad (0 < t \leqslant T), \ y(0) = y_0,$$

where $T > 0$ is arbitrary, A is a m-accretive (generally unbounded and possibly multivalued) operator on a real Banach space X, $G : C[0, T] ; \overline{D(A)}) \to L^1(0, T ; X)$ is a given mapping, and $y_0 \in \overline{D(A)}$.

The motivation for the study of (FDE) is two fold. First, the Volterra integrodifferential equation

$$(1.1) \qquad \begin{cases} \dfrac{dy}{dt} + mAy + \displaystyle\int_0^t a(t-\tau)\, Ay(\tau)\, d\tau \ni f(t) \quad (0 < t \leqslant T) \\ y(0) = y_0 \end{cases}$$

where $m > 0$ is a constant, and $a : [0, T] \to \mathbf{R}, f : [0, T] \to X$ are given functions can be transformed to (FDE) with Ay and $G(y)$ replaced by mAy and $mG(y)$ respectively under suitable assumptions by introducing the resolvent kernel r associated with the kernel a (see Proposition 3.1 below); the mapping G is defined by

$$(1.2) \qquad G(y)(t) = \frac{f(t)}{m} + r * f(t) + \frac{a(0)}{m^2}\, y(t) + r(t) y_0 - \int_0^t y(t-\tau)\, dr(\tau).$$

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Second, the abstract Volterra equation

$$\text{(V)} \qquad y(t) + \int_0^t b(t-\tau)\, Ay(\tau)\, d\tau \ni F(t) \qquad (0 \leqslant t \leqslant T),$$

where $b : [0, T] \to \mathbf{R}$ and $F : [0, T] \to X$ are given functions, is equivalent to (1.1) (and subsequently to (*FDE*) via (1.2) and Proposition 3.1 below) if b is absolutely continuous on $[0, T]$, $b(0) = m > 0$, $b'(t) = a(t)$a.e., $F \in W^{1,1}(0, T; X)$, $F'(t) = f(t)$ a.e., $y_0 = F(0) \in \overline{D(A)}$.
It should be noted that the mapping G arising from (1.1) or (V) depends on the restriction of y to the interval $[0, t]$, $0 \leqslant t \leqslant T$.

The outline of this lecture is as follows: in Section 2 we recall some facts about evolution equations and use them to deduce the basic results for (*FDE*); in Section 3 we apply the results of Section 2 to obtain a general existence, uniqueness and regularity result for (V) (and for the special case (1.1)), and we discuss how our results for (V) relate to the existing literature for (V).

Our exposition follows closely that of a forthcoming joint paper with M. G. Crandall [7] to which the reader is also referred for further details of proofs and other comments.

2. Discussion of (FDE). We begin by recalling some facts about the evolution equation

$$\text{(E)} \qquad \frac{dv}{dt} + Av \ni g \quad (0 < t \leqslant T), \quad v(0) = v_0,$$

(see Barbu [1], Benilan [4], Brézis [5], Crandall [6]).

Let A be a nonlinear operator on a real Banach space X with norm $\|\cdot\|$. A is *accretive* iff $J_\lambda = (I + \lambda A)^{-1}$ is a contraction on X for $\lambda \geqslant 0$; A is *m-accretive* iff A is accretive and $R(I + \lambda A) = X$ for $\lambda > 0$. Let $g \in L^1(0, T; X)$. A *strong solution* of (E) on $[0, T]$ is a function $v : [0, T] \to X$ such that $v(0) = v_0$, $v \in C([0, T]; X) \cap W^{1,1}(0, T; X)$, $v(t) \in D(A)$ a.e. on $[0, T]$, and there exists $w \in Av$ such that $w(t) \in Av(t)$ and $v'(t) + w(t) = g(t)$a.e. on $[0, T]$.

Definition 2.1. *$v : [0, T] \to X$ is an integral solution of* (E) *on* $[0, T)$ *if $v \in C([0, T]; X)$ and*

$$(2.1) \qquad \|v(t) - v_0\| - \|v(s) - v_0\| \leqslant \int_s^t [v(\alpha) - v_0,\ g(\alpha) - w]_+\, d\alpha$$

for $t > s$, $(s, t) \in [0, T)$, $v_0 \in D(A)$, and $w \in Av_0$, where

$$[p, q]_+ = \lim_{\lambda \downarrow 0} \frac{1}{\lambda}(\|p + \lambda q\| - \|p\|) = \inf_{\lambda > 0} \frac{1}{\lambda}(\|p + \lambda q\| - \|p\|).$$

for $p, q \in x$.

Note that $\|p + \lambda q\| \geqslant \|p\|$ for $\lambda \geqslant 0$ iff $[p, q]_+ \geqslant 0$; thus A is accretive iff $[p_1 - p_2,\ q_1 - q_2]_+ \geqslant 0$ for $q_i \in Ap_i$, $i = 1,2$. Since $|[p, q]_+| \leqslant \|q\|$, and since $g \in L^1(0, T; X)$ the integral in (2.1) is well defined. Benilan [4] has proved the following powerful result about integral solutions of (E).

Theorem. *Let A be m-accretive, $v_0 \in \overline{D(A)}$ and $g \in L^1(0, T; X)$. Then* (*E*) *has a unique integral solution $v \in C([0, T]; \overline{D(A)})$ on $[0, T]$. If v, $\hat{v}$ are integral solutions on $[0, T]$ of* (*E*) *corresponding respectively to initial data v_0, $\hat{v}_0$ and forcing terms g, $\hat{g}$, then*

$$(2.2) \qquad \|\hat{v}(t) - v(t)\| \leqslant \|v_0 - \hat{v}_0\| + \int_0^t \|\hat{g}(\sigma) - g(\sigma)\|\, d\sigma \quad (0 \leqslant t \leqslant T).$$

Moreover, if $g \in BV([0, T]; X)$ and $v_0 \in D(A)$, then

$$(2.3) \qquad \|v(\xi) - v(\eta)\| \leqslant |\xi - \eta| \{\|g(0^+) - w\| + \operatorname{var}(g : [0, t])\}$$

for $w \in Av_0$ and $0 \leqslant \xi, \eta \leqslant t \leqslant T$. In particular, v is Lipschitz continuous on $[0, T]$. If, in addition, X is reflexive, then v is a strong solution of (*E*) *on* $[0, T]$.

We shall use Benilan's theorem together with Banach's fixed point theorem to establish the existence, uniqueness, and regularity of solutions of (*FDE*) on $[0, T]$. Let $v = H(g)$ denote the unique integral solution of (E) on $[0, T]$.

Definition 2.2. *A solution of* (*FDE*) *is a function $y \in C([0, T]; \overline{D(A)})$; such that $y = H(G(y))$.*

Theorem 1. *Assume that A is m-accretive, $y_0 \in \overline{D(A)}$, and let $G : C([0, T]; \overline{D(A)}) \to L^1(0, T; X)$ satisfy*

$$(2.4) \qquad \begin{cases} \|G(u) - G(v)\|_{L^1(0,t;X)} \leqslant \displaystyle\int_0^t \gamma(s)\, \|u - v\|_{L^\infty(0,s;X)}\, ds \\ \text{for some } \gamma \in L^1(0, T; \mathbf{R}^+),\ 0 \leqslant t \leqslant T, \text{ and } u, v \in C([0, T]; D(A)). \end{cases}$$

Then (*FDE*) *has a unique solution $y \in C([0, T]; D(A))$ on* $[0, T]$.

We remark that assumption (2.4) implies that the value of $G(y)$ at $t \in [0, T]$ depends only on the restriction of y to $[0, t]$. The idea of the proof is very simple. Let $v = H(g)$ denote the unique integral solution of (E) on $[0, T]$, $g \in L^1(0, T; X)$. We seek a fixed point of the map $K : C([0, T]; \overline{D(A)}) \to C([0, T]; \overline{D(A)})$ defined by $K(y) = H(G(y))$. By property (2.2) of integral solutions

$$\|K(u)(t) - K(v)(t)\| \leqslant \int_0^t \|G(u)(s) - G(v)(s)\|\, ds \quad (0 \leqslant t \leqslant T),$$

for $u, v \in C([0, T]; \overline{D(A)})$, $u(0) = v(0) = y_0$. Applying assumption (2.4) it is now an easy matter to show that some iterate of the map K, say K^J, is strict contraction on $C([0, T]; \overline{D(A)})$ for J sufficiently large; thus the map K has a unique fixed point. For details see [7].

Under further assumptions one can apply the second part of Benilan's theorem to obtain greater regularity of solutions of (*FDE*).

Theorem 2. *In addition to the assumptions of Theorem 1 assume that there is a function $k : [0, \infty) \to [0, \infty)$ such that*

$$(2.5)\qquad \begin{cases} \operatorname{var}(G(y) : [0, t]) \leqslant k(R)(1 + \operatorname{var}(y : [0, t])) \\ \text{and } \|G(y)(0^+)\| \leqslant k(R) \quad (0 \leqslant t \leqslant T), \end{cases}$$

whenever $y \in C([0, T] ; \overline{D(A)})$ is of bounded variation and $\|y\|_{L^\infty(0, T; X)} \leqslant R$. If $y_0 \in D(A)$, then the solution y of (FDE) is Lipschitz continuous on $[0, T]$. If X is also reflexive, then the solution y is a strong solution of (FDE) on $[0, T]$.

A strong solution of (FDE) on $[0, T]$ is a function $y : [0, T] \to X$ such that $y(0) = y_0$, $y \in C([0, T] ; X) \cap W^{1,1}(0, T ; X)$, such that $y(t) \in D(A)$ a.e. and there exists $w \in Ay$ such that $w(t) \in Ay(t)$ and $y'(t) + w(t) = G(y(t))$ a.e. on $[0, T]$.

For the proof of Theorem 2 one defines $y_0 : [0, T] \to X$ by $y_0(t) = y_0$ and $y_{n+1} = K(y_n) = H(G(y_n))$, $n = 0, 1, \cdots$ These iterates converge uniformly and are uniformly bounded on $[0, T]$. By Benilan's theorem and assumption (2.5) one shows that there exists a constant $c > 0$ such that

$$\operatorname{var}(y_{n+1} : [0, t]) \leqslant c(1 + \int_0^t \operatorname{var}(y_n : [0, s])\, ds)$$

for $0 \leqslant t \leqslant T$ so that $\operatorname{var}(y_{n+1} : [0, t]) \leqslant c \exp(ct)$. Thus $\{\operatorname{var}(y_n : [0, T])\}$ and by (2.5) $\{\operatorname{var}(G(y_n)) : [0, T]\}$ are both bounded, and $\{y_n\}$, and hence also $y = \operatorname{unif} \lim y_n$, is Lipschitz continuous on $[0, T]$. For more details see [7].

Finally, tee solution y of (FDE) depends on the data A, G, x in the following sense:

Theorem 3. *Let the assumptions of Theorem 1 be satisfied. Let m-accretive operators A_n in X, mappings $G_n : C([0, T] ; X) \to L^1(0, T ; X)$, and $y_{0n} \in \overline{D(A)}$ be given for $n = 1, 2, \cdots$ Assume that the inequality (2.4) holds for G replaced G_n, with the same γ, for $n = 1, 2, \cdots$, and $u, v \in C([0, T] ; \overline{D(A)})$. For $u \in C([0, T] ; \overline{D(A)})$ assume that $\lim_{n\to\infty} G_n(u) = G(u)$ in $L^1(0, T ; X)$, $\lim_{n\to\infty} y_{0n} = y_0 \in \overline{D(A)}$, and*

$$(2.6)\qquad \lim_{n\to\infty} (I + \lambda A_n)^{-1} z = (I + \lambda A)^{-1} z \quad (z \in X,\ \lambda > 0).$$

Let $y_n \in C([0, T] ; \overline{D(A_n)})$ be solutions of (FDE) on $[0, T]$ with A replaced by A_n, G replaced by G_n, y_0 replaced by y_{0n}, and let $y \in C([0, T] ; \overline{D(A)})$ be the solution of (FDE) on $[0, T]$. Then $\lim_{n\to\infty} y_n = y$ in $C([0, T] ; X)$.

The proof of Theorem 3 follows from the observation that under our assumptions the mapping $K(A, x, G)(y) = H(A, x, G(y))$ of Theorem 1 has the property that in the iterate K^j, which is a strict contraction for some j, both j and the contraction constant depend only on the function γ of (2.4), and the latter is assumed to be uniform in n; for details see [7].

3. The Abstract Volterra Equation. We shall next apply Theorems 1, 2, 3 to study the nonlinear Volterra equation (V). Let $b \in (L^1(0, T; \mathbf{R})$, $F \in L^1(0, T ; X)$. *We shall say that y is a strong solution of the Volterra equation (V) on $[0, T]$ if $y \in L^1(0, T ; X)$, and if there exists $w \in L^1(0, T ; X)$ such that $w(t) \in Ay(t)$ and $y(t) + b * w(t) = F(t)$* a.e. *on $[0, T]$.* One can establish the following equivalence between strong solutions of (FDE) with a particular G and strong solutions of (V):

Proposition 3.1. *Let $b \in AC([0, T] : \mathbf{R})$, $b' \in BV([0, T] ; \mathbf{R})$, $F \in W^{1,1}(0, T : X)$ and $b(0) > 0$. Let y be a strong solution of (V) on $[0, T]$. Then y is a strong solution of the FDE*

$$\text{(I)}\qquad \frac{dy}{dt} + b(0)Ay \ni b(0)G(y), \quad y(0) = y_0, \quad (0 < t \leqslant T),$$

with the identifications:

$$\text{(i) } G(y)(t) = \frac{f(t)}{b(0)} + r * f(t) + (a(0)/b^2(0))y(t) + r(t)x - \int_0^t y(t-s)\,dr(s)$$

(2.7) (ii) $f(t) = F'(t)$
(iii) $y_0 = F(0)$
(iv) $a = b'$
(v) $r \in L^1(0, T; \mathbf{R})$ *is defined by* $b(0)\, r + a * r = - a/b(0)$.

Conversely, let $r \in BV([0, T) ; \mathbf{R})$, $f \in L^1(0, T ; X)$, $y_0 \in \overline{D(A)}$ and G be given by (2.7) (i). Let y be a strong solution of FDE (I) on $[0, T]$. Then y is a strong solution of (V) on $[0, T]$, where

(2.8)

$$\text{(i) } F(t) = y_0 + \int_0^t f(s)\, ds$$

$$\text{(ii) } b(t) = b(0) + \int_0^t a(s)\, ds$$

$$\text{(iii) } \frac{a}{b(0)} + a * r = -\, b(0) r$$

The proof of Proposition 3.1 is straightforward. The assumptions on b and F permit defferentiation a.e. on $[0, T]$ of a strong solution y of (V). The differentiated equation is then „solved" for Ay by means of the resolvent kernel r associated with $a = b'$ and the „variation of constants" formula. A known result about r is that $a \in BV([0, T] ; \mathbf{R})$ implies that $r \in BV([0, T)$; $\mathbf{R})$, a fact which together with an integration by parts is used in arriving at the formula (2.7) (i) for $G(y)$. The converse is proved by reversing the steps (for details see [7]). A part of Proposition 3.1 which motivates our approach is contained in MacCamy [17] who, however, then studied (FDE) by an entirely different approach.

If b and F in (V) satisfy the assumptions of Proposition 3.1 it follows from the definition of G in (2.7) (i) that $u, v \in C([0, T]; \overline{D(A)})$, then

$$\|G(u)(t) - G(v)(t)\| \leqslant (|r(0^+)| + \operatorname{var}(r : [0, t]))\|u - v\|_{L^\infty(0,t;X)}, \quad (0 \leqslant t \leqslant T),$$

where r is the resolvent kernel corresponding to $b' = a$. Thus assumption (2.4) of Theorem 1 is satisfied with $\gamma(s) = |r(0)| + \operatorname{var}(r : [0, s])$. Moreover, if $f = F' \in BV([0, T]; X)$, (2.7) (i), (ii), (iii) imply

$$\operatorname{var}(G(y) : [0, t]) \leqslant C(1 + \operatorname{var}(y : [0, t])) \quad (0 \leqslant t \leqslant T),$$

and $\|G(y)(0^+)\| \leqslant C$, where C is a constant depending on $F(0)$, $F'(0^+)$, var $(F' : [0, T])$, $r(0^+)$, and var $(r : [0, T])$; therefore, assumption (2.5) of Theorem 2 is satisfied. Thus applying Theorems 1 and 2 and Proposition 3.1 we obtain the following result for the Volterra equation (V) (for details see [7]; there the result is stated somewhat differently).

Theorem 4. *Let A be an m-accretive operator on X; let $b \in AC([0, T]; \mathbf{R})$, $b' \in BV([0, T]; \mathbf{R})$, $F \in W^{1,1}(0, T; X)$ and let $b(0) > 0$. If $F(0) \in \overline{D(A)}$ then (V) has a unique solution y on $[0, T]$ in the sense that $y \in C([0, T]; \overline{D(A)})$ and y is a unique solution of the FDE (1) in the sense of Theorem 1 with $y_0 = F(0)$ and $G(y)$ identified in Proposition 3.1. If in addition, $F' \in BV([0, T]; X)$ and $F(0) \in D(A)$ then the solution y is Lipschitz continuous on $[0, T]$. If X is also reflexive then y is a strong solution of (V) on $[0, T]$.*

It is clear that Theorem 4 can be applied to obtain a global existence and uniqueness result for the Cauchy problem (1.1) with

$$b(t) = m + \int_0^t a(\tau)d\tau, \quad m > 0, \text{ and } F(t) = y_0 + \int_0^t f(\tau)d\tau, \; y_0 \in \overline{D(A)} \text{ (resp. } D(A)).$$

Equation (1.1) which represents a mathematical model for heat flow in a material with memory, was studied by M a c C a m y [17] by a G a l e r k i n method for a particular case of the operator $A = \partial\varphi$, where φ is a particular convex lower semicontinuous proper function on real Hilbert space H. The more general equation (V) was studied in a real Hilbert space setting by V. B a r b u [2], [3] and by S. O. L o n d e n [15] for the case $A = \partial\varphi$ and by G. G r i p e n b e r g [9] for the case of a maximal monotone operator on a real Hilbert space H; these studies preceeded ours and in some cases included the case of operator valued kernels b which our method can also handle easily. In each of these papers (V) is studied by the following approach: (i) replace (V) by a set of approximating equations (V_λ), $\lambda > 0$, resulting from replacing the operator A by the Y o s i d a approximation A_λ; (ii) show that the solutions y_λ of (V_λ) satisfy useful apriori bounds (in particular they form a Cauchy sequence); (iii) show that as $\lambda \downarrow 0^+$ the solutions y_λ of (V_λ) converge to a solution y of (V); (iv) establish uniqueness. Our approach is simpler and yields more general results.

We may, however, establish a close connection between Theorem 4 and the alternative method used in earlier papers as outlined above by appealing to our Theorem 3. Let $\lambda > 0$ and define the Yosida approximation A_λ of the m-accretive operator A on X by

$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda), \quad J_\lambda = (I + \lambda A)^{-1}.$$

$A_\lambda : X \to X$ is Lipschitz continuous, so a simple contraction argument shows that the approximating problem

$$(V_\lambda) \qquad y_\lambda + b * A_\lambda y_\lambda = F$$

has a unique strong solution y_λ on $[0, T]$, under the assumptions: $b \in L^1(0, T; \mathbf{R})$, and $F \in L^1(0, T; X)$. By Proposition 3.1 y_λ is a strong solution of

$$(FDE_\lambda) \qquad \frac{dy_\lambda}{dt} + b(0)\, A_\lambda\, y_\lambda = b(0)G(y_\lambda), \quad y_\lambda(0) = F(0).$$

One also has $\lim_{\lambda \downarrow 0} (I + \mu A_\lambda)^{-1} z = (I + \mu A)^{-1} z$ for $\mu > 0$, $z \in X$.

Let y_λ be this strong solution of (V_λ) on $[0, T]$; by Theorem 3 and Proposition 3.1 the y_λ converge to y as $\lambda \downarrow 0$ in $C([0, T]; X)$, where y is the solution of (V) in Theorem 4.

We also remark that if the Volterra equation (V) has a strong solution y on $[0, T]$ under the assumptions of Theorem 2, then from Theorem 2 and Proposition 3.1, $\lim_{\lambda \downarrow 0} y_\lambda = y$ in $C([0, T]; X)$ exists, where y_λ is the strong solution of the approximating equation (V_λ). However, under our assumptions the solutions y_λ of (V_λ) converge to a limit y as $\lambda \downarrow 0$, whether or not (V) has a strong solution. For this reason we refer to the solution y of (V) has a strong solution. For this reason we refer to the solution y of (V) of Theorem 4 as the *generalized solution* of V on $[0, T]$. This generalized solution is a continuous function of the data b, A, F via Theorem 3; an explicit estimate of the continuous dependence of the solution of (V) on F is given in [7, Theorem 5].

We should mention how the case $A = \partial\varphi$, where $\varphi : H \to \mathbf{R}$ is convex, l.s.c, proper, is treated in the present context. It follows from Theorem 4 and [5, Theorem 3.6] that the solution y of (V) satisfies $y' \in L^2(0, T; H)$ if only $F(0) \in D(\varphi)$ and $F' \in L^2(0, T; H)$. If, however, one only has $F(0) \in \overline{D(\varphi)}$ and $F' \in L^2(0, T; H)$, then $ty' \in L^2(0, T; H)$. This remark easily gives the result of G r i p e n b e r g [12, Theorem 2] and similarly a variant of it [12, Theorem 3].

We refer to the remarks in [7] on how the case $A(t)$ an m-accretive operator on X for almost all $t \in [0, T]$, $\overline{D(A(t))} = D$ constant a.e. is handled in the present context, and with the same proofs. These remarks give easy generalizations of G r i p e n b e r g [9, Theorem 2]; he subsequently obtained such results independently by the alternative method in [10]. We note also that G r i p e n b e r g [11] has obtained some existence results for (V) in the case of some special operators $A(t)$ with $D(A(t))$ nonconstant; these results are apparently not motivated by physical examples and appear rather artificial; see also G r i p e n b e r g [13] for a discussion of evolution equations related to such operators by the method of E v a n s [8].

There are several known results for (V) in the case $A = \partial\varphi$ where φ is a proper convex lsc function : $H \to (-\infty, \infty]$ which are closely connected to Therorem 4, and which weaken the assumption $b' \in BV\,[0, T]$ concerning the kernel b. S. O. Londen and O. J. Staffans [16] show that this assum tion may be replaced by the frequency domain condition: there exist constants α, T_0 such that

$$\omega \int_0^T \sin(\omega t)\, b'(t)\, dt \leqslant \alpha, \quad \omega \in \mathbf{R}^+, \quad T \leqslant T_0,$$

without altering the conclusions. The kernel

$$b(t) = 1 + \int_0^t \tau \sin \frac{1}{\tau}\, d\tau$$

satisfies this frequency domain condition, and clearly $b'(t) = t \sin \frac{1}{t} \notin BV\,[0, T]$ for any $T > 0$. Earlier, Barbu [2, Theorem 3] used the assumption that b is a kernel of positive type on $[0, T]$ for $T > 0$ sufficiently small to deduce existence of solutions of (V) on $[0, \infty)$. In a different direction Gripenberg [12, Theorem 1] shows that the assumption $b' \in BV[0, T]$ in Theorem 4 (also only for the case $A = \partial\varphi$) can be replaced by: there exists $T_0 > 0$ such that $b' \in C((0, T_0); \mathbf{R})$ and b' is nonnegative and nonincreasing on $(0, T_0)$ with the growth of b' restricted by the inequality $b'(t) \leqslant c_0 \log \frac{1}{t}$ on $(0, T_0)$ for some constant $c_0 > 0$. If one takes $b(0) = 1$, $b'(t) = -\log t$ $(0 < t < 1)$, $b'(t) = 0$ $(t \geqslant 1)$, one sees that $\lim_{t \to 0^+} b'(t) = +\infty$ and it is also easy to show that the frequency domain condition of [16] is not satisfied. The results of [12], [16] are obtained by the alternative approach.

Finally we remark that our method can be used to study the nonconvolution version of (V):

$$y(t) + \int_0^t b(t, s)\, Ay(s)\, ds \ni F(t) \quad (0 \leqslant t \leqslant T),$$

where A, F are as in Theorem 4, and the kernel b defined on $\{(t, s) : 0 \leqslant s \leqslant t \leqslant T\}$, with $b(t, t) > 0$ and b sufficiently smooth. For details see [7] and the thesis of C. Rennolet [18]; similar results were obtained G. Gripenberg [13] independently by the alternative approach.

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