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FLOW-INVARIANCE FOR DIFFERENTIAL EQUATIONS ASSOCIATED TO NONLINEAR OPERATORS*)

BY

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1. Introduction. The main purpose of this paper is to present some existence results obtained by the authors in the area of semi-linear and nonlinear evolution equations, using flow-invariance techniques, results extending those of M a r t i n [5], P a v e l [6] and P a z y [9]. Our paper is very closely related to the works of A t t o u c h [1], C r a n d a l l [4] and S c h i a f f i n o [10]. As we shall give a short sketch of proof only for one of our theorems, we refer the reader to [7] and [8] for further details, examples and applications.

We begin with the basic notations and definitions we shall use in the sequel. Let R be the set of real numbers, $a, b, c, d, T \in R$, X a real Banach space with the norm denoted by $\|\cdot\|$ and X^* its dual with the corresponding norm $\| \|_*$. If $x \in X$ and $r>0$, then $S(x,r)$ is the open ball with center x and radius r and $B(x, r)$ is the closed ball with center x and radius r. If $B \subset X$ denote by $\bar{B}$ the closure of B and by $d(x, B)$ the usual distance between x and B. If $B \subset X$ and $C \subset X$ are two nonempty and bounded sets, denote by $\rho(B, C)$ the Hausdorff distance between B and C, i.e.: $\rho(B, C)=\inf\{r>0;\ B \subset \bigcup_{y \in C} S(y, r);\ C \subset \bigcup_{x \in B} S(x, r)\}$. It is well known that $\rho(B, C)=0$ iff $\bar{B}=\bar{C}$. Let $\Phi: X \to X^*$ be the duality mapping of X, i.e.: $\Phi(x)=\{x^* \in X^*;\ x^*(x)=\|x\|^2=\|x^*\|_*^2\}$ for all $x \in X$. For each pair $(x, y) \in X \times X$ let us define:

$$<y, x>_s = \sup\{<y, w^*>;\ w^* \in \Phi(y)\}.$$

In the definition above $<y, w^*>$ denotes the value of w^* at y, i.e., $<y, w^*> = w^*(y)$ for $y \in X$ and $w^* \in X^*$.

Definition 1. *A set $D \subset X$ is called locally closed, if for each $x \in D$ there exists $r>0$ such that $B(x, r) \cap D$ is closed.*

Definition 2. *A set $G \subset X$ is called semi-locally closed if there exist $c, d \in R$, $c<d$ such that:*

(i) $G=\bigcup_{z \in [c, d[} D_z$,

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(ii) *For each* $x \in G$, *there exists* $r > 0$ *such that* $B(x, r) \cap D_\varepsilon$ *is closed for all* $\varepsilon \in [c, d[$.

(iii) *For each* $r > 0$, $\varepsilon_0 \in [c, d[$ *and* $x \in D_{\varepsilon_0}$, *there exists* $\delta > 0$ *such that the mapping* $\varepsilon \to B(x, r) \cap D_\varepsilon$ *is continuous in the Hausdorff metric on* $[\varepsilon_0, \varepsilon_0 + \delta] \cap [c, d[$.

It is obvious that each locally closed set is semi-locally closed but the converse is not true. Moreover, all open sets and all closed sets are semi-locally closed.

Let G be a semi-locally closed set, $G = \bigcup_{\varepsilon \in [a, b[} D_\varepsilon$ and suppose that for each $t \in [a, b[$, $F(t, \cdot) : \tilde{D}_t \to 2^X$ is a nonempty, convex and weakly compact valued mapping, while $D_t \subset \tilde{D}_t \subset X$ and $\bigcup_{\varepsilon \in [a, b[} \tilde{D}_t = \tilde{G}$.

Definition 3. *The mapping* F *is called demi-closed if for each sequence* $\{(t_n, x_n)\}_{n=1}^{\infty}$ *from* $\tilde{\mathcal{D}} = \{(t, x) \in [a, b[\times X \,;\, x \in \tilde{D}_t\}$, *strongly convergent to* $(t_0, x_0) \in \tilde{\mathcal{D}}$ *and for each sequence* $\{y_n\}_{n=1}^{\infty}$, *with* $y_n \in F(t_n, x_n)$, *weakly convergent to* y_0, *it follows* $y_0 \in F(t_0, x_0)$.

Definition 4. *The mapping* F *is called locally bounded, if for each* $(t_0, x_0) \in \tilde{\mathcal{D}}$ *there exist* $\delta_1 > 0$, $\delta_2 > 0$ *and* $M > 0$ *such that* $\|F(t, x)\| \leqslant M$ *for all* $(t, x) \in \tilde{\mathcal{D}}$ *with* $|t - t_0| \leqslant \delta_1$ *and* $\|x - x_0\| \leqslant \delta_2$.

Here we have used the notation:

$$\|F(t, x)\| = \sup \{\|y\| \,;\, y \in F(t, x)\}. \tag{1.1}$$

Let us consider the following problems:

$$\dot{x}(t) \in Ax(t) + F(t, x(t)), \quad a \leqslant t < T \,;\; x(a) = x_0. \tag{1.2}$$

where $A : X \to X$ generates a strongly continuous semigroup of linear operators $S(t)$ and:

$$\dot{x}(t) = A(t)\, x(t), \quad a \leqslant t_0 \leqslant t < t_0 + T \,;\; x(t_0) = x_0, \quad x_0 \in D(t_0), \tag{1.3}$$

where for each $t \in [a, b[$, $A(t)$ is a nonlinear operator from $D(t)$ to X.

Definition 5. *The continuous mapping* $x : [a, T[\to \tilde{G}$ *is a mild solution for Eq. (1.2), if there exists a strongly measurable function* $f : [a, T[\to X$, *such that* $f(s) \in F(s, x(s))$ *a.e. on* $[a, T[$ *and in addition,*

$$x(t) = S(t - a)\, x_0 + \int_a^t S(t - s) f(s)\, ds, \quad a \leqslant t < T. \tag{1.4}$$

It is obvious that when F is single-valued, our last definition yields to the classical definition due to Browder [3].

Definition 6. *The function* $x : [a, T[\to X$ *is a strong solution for Eq. (1.3) if* x *is continuously differentiable on* $[a, T[$, $x(t) \in D(t)$ *for all* $t \in [a, T]$ *and* x *satisfies (1.3) on* $[a, T[$.

The next Section is concerned with the hypothesis and the formulation of the main results.

2. Statement of the main results. We suppose that the reader is familiar with the basic properties of linear semigroups in Banach spaces and we refer to [2] for the background material relevant to this subject. We begin with the hypotheses we need throughout this paper.

(H_1) *X is a reflexive real Banach space.*

(H_2) *$A : X \to X$ generates a strongly continuous semigroup of linear operators $S(t)$, with $S(t)$ compact for all $t > 0$.*

(H_3) *$G = \bigcup_{\varepsilon \in [0,1[} D_\varepsilon$ is a semi-locally closed set in X,*

(H_3') *$G = \bigcup_{t \in [a,b[} D_t$ is a semi-locally closed set in X,*

(H_4) *$F : [a, b[\times G \to 2^X$ is a nonempty, convex and weakly compact valued mapping. Moreover, F is demi-closed and locally bounded.*

(H_4') *For each $t \in [a, b[$, $F(t, \cdot) : D_t \to 2^X$ is a nonempty, convex and weakly compact valued mapping. Moreover, F is demi-closed and locally bounded.*

(H_5) *For each $x \in G$, $t \in [a, b[$ and $\varepsilon \in [0, 1[$, with $x \in D_\varepsilon$, it follows:*

$$\lim_{h \downarrow 0} \left[d\left(S(h)\, x + \int_t^{t+h} S(t + h - s) f(t, x)\, ds,\; D_{\varepsilon + SMh}\right) \right] / h = 0, \tag{2.1}$$

where M denotes an upper bound of F in a fixed neighbourhood of (t, x), while $S = \sup \{\|S(t)\| \,;\, t \in [0, b - a[\}$ and $f(t, x)$ is an arbitrary element in $F(t, x)$.

(H_5') *For each $x \in G$, $t \in [a, b[$ for which $x \in D_t$ it follows:*

$$\lim_{h \downarrow 0} d\left(S(h)\, x + \int_t^{t+h} S(t + h - s) f(t, x)\, ds,\; D_{t+h}\right) / h = 0 \tag{2.2}$$

where $f(t, x)$ is an arbitrary element in $F(t, x)$.

We need also the following conditions:

(I_1) *X is a real Banach space and X^* is its dual,*

(I_2) *For each $t \in [a, b[$, $A(t) : \mathcal{D}(t) \to X$ and the function $(t, x) \to A(t) \cdot x$ is continuous from $\mathcal{D} = \{(t, x) \in [a, b[\times X \,;\, x \in D(t)\}$ to X.*

(I_3) *For each $(t_0, x_0) \in \mathcal{D}$, there exist $r > 0$ and $T > 0$ such that the mapping $t \to D(t) \cap B(x, r)$ is nonempty, closed valued and continuous in the Hausdorff metric on $[t_0, t_0 + T]$.*

(I_4) *For each $(t, x) \in \mathcal{D}$ and $(t, y) \in \mathcal{D}$ the following condition:*

$$< A(t)x - A(t)\, y,\; x - y >_s \leqslant L \|x - y\|^2 \tag{2.3}$$

holds.

(I_4') *The mapping $(t, x) \to A(t)x$ is compact from $\mathcal{D}$ to X.*

(I_5') *For each $(t, x) \in \mathcal{D}$ the following condition:*

$$\lim_{h \downarrow 0} d(x + hA(t)x,\; D(t + h)) / h = 0 \tag{2.4}$$

holds.

Now, we are able to formulate our main results.

Theorem 1 — *Assume that* (H_1), (H_2), (H_3), (H_4) *and* (H_5) *hold.*

Then, for each $x_0 \in G$, there exists $T \in]a, b[$ such that the problem (1.2) has at least a mild solution on $[a, T[$.

Theorem 2. — *Assume that (H_1), (H_2), (H_3'), (H_4') and (H_5') hold. Then, for each $x_0 \in D_a$, there exists $T \in [a, b[$ such that the problem (1.2) has at least a mild solution on $[a, T[$.*

Theorem 3. — *Assume that (I_1), (I_2), (I_3) and (I_4) are satisfied.*

Then, (I_5) holds if and only if for each $t_0 \in [a, b[$ and $x_0 \in D(t_0)$ there exists $T > 0$ such that the problem (1.3) has an unique strong solution on $[t_0, t_0 + T]$.

Theorem 4. — *Assume that (I_1), (I_2), (I_3) and (I_4') are satisfied.*

Then, (I_5) holds if and only if for each $t_0 \in [a, b[$ and $x_0 \in D(t_0)$ there exists $T > 0$ such that the problem (1.3) has at least a strong solution on $[t_0, t_0 + T_0]$.

Remarks. (1) Let us remark that if G is open and (H_1), (H_2), (H_3) and (H_4) are satisfied, then (H_n) is also satisfied.

(2) It should be noted that if X is a Hilbert space and D_t in (H_3') are open convex and bounded for each $t \in [a, b[$, then (H_1), (H_2), (H_3') and (H_4') implies (H_5'). A simple proof of this fact may be obtained by using Lemma 2.1. and Lemma 3.1. from [11].

(3) If $D(t)$ in (I_2) are open, convex and bounded for each $t \in [a, b[$ and X is a Hilbert space, then, using the lemmas cited before, one may easily deduce that (I_1), (I_2) and (I_3) implies (I_5).

As the proofs of the theorems 1, 3, 4 may be found in [7] and [8] we shall indicate in what follows only a short sketch of proof for theorem 2.

3. Proof of Theorem 2. — Let $x_0 \in D_a$. From (H_3') (see also definition 2 (ii)), it follows that there exists $r > 0$ such that $B(x_0, r) \cap D_a$ is closed. Using (H_4'), namely the locally closedness asumption on F, one may assert that we can choose $r > 0$ small enough and $a < \bar{t} \leq b$ such that the restriction of F to $\mathcal{D}_{\bar{t}} = \{(t, x) \in [a, \bar{t}[\times X;\ x \in D_t\}$ is bounded by a constant denoted by $M - 1$. More precisely, one has $\|F(t, x)\| \leq M - 1$ for all $(t, x) \in \mathcal{D}_{\bar{t}}$. Choose $\widetilde{T} \in]a, \bar{t}[$ such that:

$$a + MT_0 \leq b \text{ and } \max_{t \in [0, T_0]} \|S(t)x_0 - x_0\| + MT_0 \leq r \tag{3.1}$$

where $T_0 = \widetilde{T} - a$. Let us remark that $\widetilde{T}$ may be choosen small enough such that in addition for each $t \in [a, \widetilde{T}]$, $B(x_0, r) \cap D_t$ is closed and the mapping $t \to B(x_0, r) \cap D_t$ is continuous in the Hausdorff metric on $[a, \widetilde{T}]$. One may suppose without loss of generality that:

$$\sup_{t \in [0, T_0]} \| S(t) \| \leq 1 \tag{3.2}$$

Fix a natural number n and define the approximation of the solution x_n, as follows:. Put $t_0^n = a$, $x_0^n = x_0$ and suppose that we have constructed x_n on $[a, t_i^n]$ where $x_n(t_j^n) = x_j^n$, with $x_j^n \in D_{t_j^n}$ for all $j = 0, 1, \ldots, i$. *If* $t_i^n < \widetilde{T}$, then choose the largest number $d_i^n \in]0, 1/n]$ which for:

$$d\Big(S(d_i^n)x_i^n + \int_{t_i^n}^{t_i^n + d_i^n} S(t_i^n + d_i^n - s) f(t_i^n, x_i^n)\, ds,\ D_{t_i^n + d_i^n}\Big) \leq d_i^n/2n. \tag{3.3}$$

and $t_i^n + d_i^n = t_{i+1}^n \leq T$.

In (3.3) $f(t_i^n, x_i^n)$ is an arbitrary element of $F(t_i^n, x_i^n)$. The existence of such d_i^n is guaranteed by (H_5'). Now, we are able to define x_{i+1}^n as a fixed element of $D_{t_i^n + d_i^n}$ (whose existence follows from (3.3)) which for:

$$\Big\| S(d_i^n)x_i^n + \int_{t_i^n}^{t_i^n + d_i^n} S(t_i^n + d_i^n - s) f(t_i^n, x_i^n)\, ds - x_{i+1}^n \Big\| \leq d_i^n/n. \tag{3.4}$$

It is easy to verify that for each $n \in N$ and $i \in N$, $x_i^n \in B(x_0, r) \cap D_{t_i^n}$.

Denote by $p_i^n = \dfrac{1}{d_i^n}\Big(x_{i+1}^n - S(d_i^n)x_i^n - \displaystyle\int_{t_i^n}^{t_i^n + d_i^n} S(t_i^n + d_i^n - s) f(t_i^n, x_i^n)\, ds\Big)$ and define:

$$x_n(t) = S(t - t_i^n)x_i^n + \int_{t_i^n}^{t} S(t - s) f(t_i^n, x_i^n) ds + (t - t_i^n) p_i^n \tag{3.5}$$

for $t \in [t_i^n, t_{i+1}^n]$. Let us remark that $x_n(t_i^n) = x_i^n$ and therefore x_n is continuous on its domain. Moreover, x_n may be rewritten under the following form:

$$\begin{aligned} x_n(t) = S(t - a)\, x_0 &+ \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} S(t - s) f(t_j^n, x_j^n)\, ds + \int_{t_i^n}^{t} S(t - s) f(t_i^n, x_i^n)\, ds + \\ &+ \sum_{j=0}^{i-1} (t_{j+1}^n - t_j^n) S(t - t_{j+1}^n) p_j^n + (t - t_i^n)\, p_i^n \end{aligned} \tag{3.6}$$

for $t \in [t_i^n, t_{i+1}^n]$. Define the step functions $a_n(s) = t_i^n$ for $s \in [t_i^n, t_{i+1}^n[$, $a_n(\widetilde{T}) = \widetilde{T}$ and let us observe that from (3.6) we get:

$$x_n(t) = S(t - a) a x_0 + \int_a^t S(t - s) f_n(s)\, ds + g_n(t), \tag{3.7}$$

where $f_n(s) = f(t_i^n, x_i^n)$ for all $s \in [t_i^n, t_{i+1}^n[$ and the function g_n is defined by: $g_n(t) = \sum_{j=0}^{i-1} (t_{j+1}^n - t_j^n) S(t - t_{j+1}^n) p_j^n + (t - t_i^n) p_i^n$. It is easy to verify that $\| p_i^n \| \leq 1/n$ and consequently $\|g_n(t)\| \leq (t - a)/n$ for all $t \in [a, \widetilde{T}]$.

Thus, we have constructed the approximation of the solution x_n given by (3.7) and we shall prove that for sufficiently large n, x_n is defined on the whole interval $[a, \tilde{T}]$. First of all one has to prove that for $n \in N$, large enough, the sequence $\{x_i^n\}_{i=1}^{\infty}$ is strongly convergent to an element $x^n \in X$.

To this end, let us observe that the sequence $\{x_i^n\}_{i=1}^{\infty}$ is a Cauchy sequence. This simple fact, the proof of which we left to the reader, follows directly from the boundedness assumptions and from the fact that $t \to S(t)$ is continuous in the uniform convergence topology for $t > 0$ (as $S(t)$ is compact for $t > 0$).

Now, we have to verify that for $n \in N$, large enough, the sequence $\{t_i^n\}_{i=1}^{\infty}$ converges to $\tilde{T}$, i.e., that by the construction above one may cover the whole interval $[a, \tilde{T}]$. Suppose, by contradiction, that for a fixed $n \in N$ for which $\lim\limits_{i \to \infty} x_i^n = x^n$ exists, one has:

$$\sum_{i=1}^{\infty} d_i^n = d < T_0, \text{ i.e., } a + \sum_{i=0} d < \tilde{T}.$$

Let us consider the set W of all weak cluster points of $\{f(t_i^n, x_i^n)\}_{i=1}^{\infty}$ which is nonempty as X is reflexive and $\{f(t_i^n, x_i^n)\}_{i=1}^{\infty}$ is bounded. Moreover, from (H_4') one has $W \subset F(a+d, x^n)$. Let us choose $f(a+d, x^n)$ in W and consider h small enough in $]0, 1/n]$ such that:

$$d\Big(S(h)\, x^n + \int_{a+d}^{a+d+h} S(a+d,+h-s)\, f(a+d, x^n)\, ds,\ D_{a+d+h}\Big) \leqslant h/4n, \tag{3.8}$$

and $a + d + h < \tilde{T}$. Let us consider also the sequences:

$$v_i^n = S(h)\, x_i^n + \int_{t_i^n}^{t_i^n+h} S(t_i^n + h - s)\, f(t_i^n, x_i^n) ds,$$

$$v_i^n(\varepsilon) = S(h) x_i^n + \int_{t_i^n}^{t_i^n+h-\varepsilon} S(t_i^n + h - s)\, f(t_i^n, x_i^n)\, ds.$$

Then, one may easily deduce that

$$v_i^n(\varepsilon) = S(h)\, x_i^n + S(\varepsilon) \int_{t_i^n}^{t_i^n+h-\varepsilon} S(t_i^n + h - s - \varepsilon)\, f(t_i^n, x_i^n) ds$$

and therefore, as $S(\varepsilon)$ is compact and $\{f(t_i^n, x_i^n)\}_{i=1}^{\infty}$ is bounded, we conclude that $\{v_i^n(\varepsilon)\}_{i=1}^{\infty}$ is precompact in X. By a simple evaluation, we get also $\|v_i^n - v_i^n(\varepsilon)\| \leqslant (M - l)\, \varepsilon$, relation that, in virtue of the last remark, implies the precompactness of $\{v_i^n\}_{i=1}^{\infty}$.

Then, there exists a convergent subsequence of $\{v_i^n\}_{i=1}^{\infty}$ designed by $\{v_{i_k}^n\}_{k=1}^{\infty}$, such that in addition $\{f(t_{i_k}^n, x_{i_k}^n)\}_{k=1}^{\infty}$ converges weakly to $f(a+d, x^n)$. Taking into account the uniqueness of the weak limit we get:

$$\lim_{k \to \infty} v_{i_k}^n = S(h)\, x^n + \int_{a+d}^{a+d+h} S(a + d + h - s)\, f(a+d, x^n)\, ds.$$

On the other hand, as $t \to B(x_0, r) \cap D_t$ is continuous in the Hausdorff metric, one may assert that (3.8) and the relation above imply:

$$d\Big(S(h)\, x_i^n + \int_{t_i^n}^{t_i^n+h} S(t_i^n + h - s)\, f(t_i^n, x_i^n)\, ds,\ D_{t_i^n+h}\Big) \leqslant h/2n. \tag{3.9}$$

for infinite many $i \in N$.

But let us remark that (3.9) contradicts the maximality of d_i^n and therefore $d < \tilde{T}$ cannot hold. Then, we may conclude that for each $n \in N$ which for $\{x_i^n\}_{i=1}^{\infty}$ is convergent, $\lim\limits_{i = \infty} t_i^n = \tilde{T}$.

Now, we shall prove that $\{x_n\}_{n=1}^{\infty}$ given by (3.7) is relatively compact in $C(a, T; X)$ for $a < T < \tilde{T}$. As $\{g_n\}_{n=1}^{\infty}$ is strongly convergent to 0 in $C(a, T; X)$, it suffices to show that one may apply the infinite dimensional version of Arzelà and Ascoli's Theorem for the sequence $\{y_n\}_{n=1}^{\infty}$ where:

$$y_n(t) = \int_a^t S(t - s)\, f_n(s)\, ds.$$

It is easy to verify that $\{y_n\}_{n=1}^{\infty}$ is uniformely bounded and equicontinuous on $[a, T]$. We have only to check out the precompactness of $\{y_n(t)\}_{n=1}^{\infty}$ for each $t \in]a, T]$. Let $\varepsilon \in]0, t-a[$ and define:

$$y_n^{\varepsilon}(t) = \int_a^{t-\varepsilon} S(t - s)\, f_n(s)\, ds = S(\varepsilon) \int_a^{t-\varepsilon} S(t - s - \varepsilon)\, f_n(s)\, ds. \tag{3.10}$$

As $S(\varepsilon)$ is compact for $\varepsilon > 0$ and $\{f_n\}_{n=1}^{\infty}$ is uniformely bounded, (3.10) shows that $\{y_n^{\varepsilon}(t)\}_{n=1}^{\infty}$ is precompact in X. Let us observe now that $\|y_n(t) - y_n^{\varepsilon}(t)\| \leqslant (M - 1)\varepsilon$ and therefore $\{y_n(t)\}_{n=1}^{\infty}$ is precompact in X.

Finally, we shall prove that one may choose a subsequence of $\{x_n\}_{n=1}^{\infty}$ strongly convergent in $C(a, T; X)$ to a mild solution of Eq. (1.2).

To this end let us consider a strong cluster point x of $\{x_n\}_{n=1}^{\infty}$ in $C(a, T; X)$ and let us observe that $\{f_n\}_{n=1}^{\infty}$ is relatively weakly compact in $L^2(a, T; X)$. Let $\{x_{n_k}\}_{k=1}^{\infty}$ be the subsequence of $\{x_n\}_{n=1}^{\infty}$ that converges to x and consider $W(s)$ the set of all weak cluster points of $\{f_n(s)\}_{n=1}^{\infty}$. Therefore, $W(s)$ is nonempty and contained in $F(s, x(s))$ for all $s \in [a, T]$. Now, choose a subsequence of $\{f_n\}_{k=1}^{\infty}$ denoted again by $\{f_{n_k}\}_{k=1}^{\infty}$, weakly convergent in $L^2(a, T; X)$ to some function f.

Then, as F is convex valued, $W(s) \subset F(s, x(s))$ for all $s \in [a, T]$, it follows from a technical result due to Kato (see [7], Lemma 2.7.) that $f(s) \in F(s, x(s))$ a.e. on $[a, T]$.

To complete the proof we have only to remark that:

$$x(t) = S(t-a)\, x_0 + \int_a^t S(t-s)\, f(s)\, ds$$

and also that for each $t \in [a, T]$ $x(t) \in D_t$. These last two relations follow from standard arguments involving the properties of the weak limit and from the construction of the sequence $\{x_n\}_{n=1}^{\infty}$.

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SOLVABILITY OF SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS AT RESONANCE VIA THE A-PROPER TYPE OPERATOR THEORY*)

BY

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In this talk, part of which is based on [1], we show how the theory of operators of A-proper type *** is used to establish the solvability of abstract and differential equations

$$Lx + Nx = f \quad (x \in X,\ f \in Y), \tag{1}$$

for a given f in Y, where X and Y are real Banach spaces, $L \in L(X, Y)$ is A-proper with the null space $N(L) \neq \{0\}$ and ind $(L) = 0$ and $N: X \to Y$ is a nonlinear map such that $L + N$ is weakly A-proper, $\|Nx\|/\|x\| \to 0$ as $\|x\| \to \infty$ and N satisfies various asymptotic positivity conditions which have their origin in the existence result of Landesman and Lazer [2] and the subsequence papers of [3—11] and others. It will be indicated in §1 (for more details see [1]) that the class of weakly A-proper maps $T = L + N$ is not only well suited for the problem at hand, but it is also large enough so as to allow us to *extend* and *unify* the abstract results of [3] (including some in [4, 5, 6]) with some of [7, 8] (and [9] if the linear part is bounded).

Using our abstract results obtained in [1], some of which are stated in §1, we show how one establishes the existence of strong or weak solutions for elliptic equations when the nonlinear part N depends on the highest order derivatives (without being Lipschitzian with respect to them as in [7, 8]) or the underlying domain in R^n is unbounded so that N is neither compact nor set-contractive. In this talk we are primarily concerned with applying our results to those Boundary Value (BV) Problems to which the abstract results of other authors need not apply (see [1] for other examples).

In §2 we discuss briefly the existence of strong solutions for two elliptic Boundary Value Problems which have been treated in detail in [1]. Problem A, which is of the form

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*** See § 1 for the definitions of various notions, Examples, the precise statement s of the results mentioned here and their relation to results in this area obtained by other author s when N is of a particular type.