and contained in $F(s, x(s))$ for all $s \in [a, T]$. Now, choose a subsequence of $\{f_n\}_{k=1}^{\infty}$ denoted again by $\{f_{n_k}\}_{k=1}^{\infty}$, weakly convergent in $L^2(a, T; X)$ to some function f.

Then, as F is convex valued, $W(s) \subset F(s, x(s))$ for all $s \in [a, T]$, it follows from a technical result due to Kato (see [7], Lemma 2.7.) that $f(s) \in F(s, x(s))$ a.e. on $[a, T]$.

To complete the proof we have only to remark that :

$$x(t) = S(t-a)\, x_0 + \int_a^t S(t-s) f(s)\, ds$$

and also that for each $t \in [a, T]$ $x(t) \in D_t$. These last two relations follow from standard arguments involving the properties of the weak limit and from the construction of the sequence $\{x_n\}_{n=1}^{\infty}$.

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SOLVABILITY OF SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS AT RESONANCE VIA THE A-PROPER TYPE OPERATOR THEORY*)

BY

W. V. PETRYSHYN**)

In this talk, part of which is based on [1], we show how the theory of operators of A-proper type *** is used to establish the solvability of abstract and differential equations

$$Lx + Nx = f \quad (x \in X,\ f \in Y), \tag{1}$$

for a given f in Y, where X and Y are real Banach spaces, $L \in L(X, Y)$ is A-proper with the null space $N(L) \neq \{0\}$ and ind $(L) = 0$ and $N : X \to Y$ is a nonlinear map such that $L + N$ is weakly A-proper, $\|Nx\|/\|x\| \to 0$ as $\|x\| \to \infty$ and N satisfies various asymptotic positivity conditions which have their origin in the existence result of Landesmar and Lazer [2] and the subsequence papers of [3—11] and others. It will be indicated in §1 (for more details see [1]) that the class of weakly A-proper maps $T = L + N$ is not only well suited for the problem at hand, but it is also large enough so as to allow us to *extend* and *unify* the abstract results of [3] (including some in [4, 5, 6]) with some of [7, 8] (and [9] if the linear part is bounded).

Using our abstract results obtained in [1], some of which are stated in §1, we show how one establishes the existence of strong or weak solutions for elliptic equations when the nonlinear part N depends on the highest order derivatives (without being Lipschitzian with respect to them as in [7, 8]) or the underlying domain in R^n is unbounded so that N is neither compact nor set-contractive. In this talk we are primarily concerned with applyng our results to those Boundary Value (BV) Problems to which the abstract results of other authors need not apply (see [1] for other examples).

In §2 we discuss briefly the existence of strong solutions for two elliptic Boundary Value Problems which have been treated in detail in [1]. Problem A, which is of the form

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*** See § 1 for the definitions of various notions, Examples, the precise statements of the results mentioned here and their relation to results in this area obtained by other authors when N is of a particular type.

$$\text{(A)} \quad \begin{cases} -\Delta u - g(x, \nabla u, \Delta u) + h(u) = f(x) & (x \in Q) \\ \dfrac{\partial u}{\partial \eta} = 0 & (x \in \partial Q) \end{cases}$$

was first studied in [8] by means of the degree theory for condensing mappings We include this example here to indicate the differences in assumptions on the function $g(x, \nabla u, \Delta u)$ for the respective abstract existence results in [8] and in [1] to be applicable. In Problem B the abstract result is modelled on situations which arize when one attempts to treat semilinear elliptic problems on unbounded domains in R^n which are of the form

$$\text{(B)} \qquad Au - \lambda u + Nu = f \qquad (u \in D(A),\ f \in H,\ \lambda \in R),$$

where H is a Hilbert space, A a densely defined, positive definite, self-adjoint operator whose essential spectrum $\sigma_l(A)$ is bounded below, and

$N : D(A^{1/2}) \to H$ is such that $N = N_1 + N_2$ with N_2

compact and N_1 monotone. Detailed proofs of the results summarized in §1 and §2 can be found in [1]. Theorem 1.5 in §1 is new.

Section 3 contains new results in the sense that using the theory of weakly A-proper mappings we are able to establish the variational solvability w.r.t. a given subspace V of $W_2^m(Q)$ $(V \geqslant \overset{\circ}{W}{}_2^m(Q))$ of the elliptic BV Problem

$$\text{(C)} \quad \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x) D^\beta u) + \sum_{|\alpha| \leq m} (-1)^\alpha D^\alpha b_\alpha(x, u, \ldots, D^m u) = f(x)$$

under conditions on the functions $b_\alpha : Q \times R^{S_m} \to R$ for $|\alpha| \leqslant m$ which are different and, even in special cases, are weaker than those imposed in [7] and [10]. We also show that the main result for Equation (C) obtained in [7, Theorem 3] and in some the result in [10, Theorem 2] follow as special cases of our Theorem 3.1 but the theories used in [7, 10] for the more restrictive maps N determined by the nonlinear part in Equation (C) cannot be applied to the more general situation treated in Theorems 3.1 and 3.3. See the second half of §3 for more details. Let us finally add that our results for Equation (C) generalize those in [2, 3, 6, 11] and others to the case when the nonlinear part is also of order 2m. This is shown in *Special Cases* of this section.

Finally let me make a general observation concerning the abstract results for weakly A-proper mappings stated in Section 1 which we believe provide sufficient motivation why mappings of A-proper type should be studied. We do not assume that the generalized inverse of L or the resolvent of L is compact nor do we assume that N is compact, monotone or condensing. Consequently, from the point of view of applications where X and Y are usually separable spaces and a set-contractive map $N : X \to Y$ is also ball-contractive, our theorems in Section 1 are proper extensions of the corresponding results of the authors mentioned above and, as will be indicated in Sections 2 and 3, in some cases they are applicable to PDE's under conditions on N which preclude the applicability of the abstract results of the other

authors. Moreover, their arguments cannot be applied to the solvability of Equation (1) when $L+N$ is weakly A-proper.

§1. Let $\{X, Y\}$ be real Banach spaces with a projectionally complete scheme $\Gamma \equiv \{X_n, P_n; Y_n, Q_n\}$, i.e.,$X\{_n\} \subset X$ and $\{Y_n\} \subset Y$ are sequences of oriented finite dimensional spaces and $P_n : X \to X_n$ and $Q_n : Y \to Y_n$ are projections such that dim $X_n = \dim Y_n$ for each $n \in N$, $P_n x \to x$ for each $x \in X$ and $Q_n y \to y$ for each y in Y. When X and Y are Hilbert spaces the projections $\{P_n\}$ and $\{Q_n\}$ are assumed to be orthogonal.

It was shown by the author in [1] that the following class of maps is not only well suited for the study of the existence of solutions of semilinear equations of the form (1) but it also large enough so as to allow us to extend and unify the abstract existece results of [3, 4, 5, 6] with some of those of [7, 8, 10] obtained by these authors for special classes of maps satisfying different conditions and using various types of arguments. We also indicate the extensions of some results in [9] when the linear part is bounded but its generalized inverse is not necessarily compact.

Definition 1.1. *$T : X \to Y$ is said to be weakly A-proper (resp. A-proper) w.r.t. Γ if $T_n \equiv Q_n T|X_n : X_n \to Y_n$ is continuous for each $n \in N$ and if $\{x_{n_j} | x_{n_j} \in X_{n_j}\}$ is any bounded sequence such that $T_{n_j}(x_{n_j}) \to g$ for some g in Y, then there exist a subsequence $\{x_{n_{j(k)}}\}$ and x in X such that $x_{n_{j(k)}} \rightharpoonup x \in X$ (resp. $x_{n_{j(k)}} \to x$ in X) and $Tx = g$, where „$\rightharpoonup$" and „$\to$" denote the weak and strong convergence respectively.*

We restricted ourselves to projectional schemes since these are the ones which we shall use in sections 2 and 3. However,the results obtained in [1] are valid for the more general admissible schemes (see [1]). The generality of the class of weakly A-proper maps is illustrated by the following examples (others and the definitions of some of the notions used here will be given later).

Example 1. All A-proper maps. This class includes compact and ball-condensing vector fields, P-compact maps, maps of type (S) and (KS) as well as of strongly K-monotone or accretive type (see [12, 13]). Extending a recent result in [14], it was shown independently in [15] and in [16] that the class of A-proper mappings also includes the maps $T : X \to X$ of the form $T = I - F + M + C$, where F is ball-condensing, M is accretive and continuous, and C is compact. The interesting feature of the latter class is that neither the theory of accretive nor condensing type maps is applicable to equations involving the above class of maps. We shall see in §2 that Problem B leads to the equation involving such maps.

To state further examples of weakly A-proper (but not A-proper) mappings, we need the following notion which is due to [17] when $Y = X^*$. We say that $T : X \to Y$ is of *type* (KM) provided the following conditions hold :

(i) If $x_j \rightharpoonup x$ in X, $Tx_j \rightharpoonup g$ in Y and lim $(Tx_j, Kx_j) \leqslant (g, Kx)$, then $Tx = g$, where for the sake of simplicity we assume that $K \in L(X, Y^*)$ and $R(K)$ is dense in Y^*.

(ii) T is continuous from finite dimensional subspaces of X into Y equipped with weak* topology.

Example 2. Let X be reflexive and Γ such that $Q_n^*Kx=Kx$ for all $x\in X_n$ and each $n\in N$. If $T:X\to Y$ is either weakly continuous or bounded and of type (KM), then T is weakly A-proper w.r.t. Γ and $T(\bar{B}(0,\rho))$ is closed for each $\rho>0$. As a special case we see that when $Y=X^*$ and $K=I$, then $T:X\to X^*$ is weakly A-proper whenver T is of type (M) and, in particular, when T is either monotone, pseudomonotone, semimonotone, or weakly continuous (see [1]).

Our next example indicates the class of semilinear weakly A-proper mappings which appear in sections 2 and 3 in our applications of abstract results from section 1 to elliptic BV Problems.

Example 3. Suppose X is reflexive, Γ such that $Q_n^*Kx=Kx$ for $x\in X_n$, $L\in L(X,Y)$ is A-proper w.r.t. Γ, $N:X\to Y$ bounded and of type (KM) or weakly continuous and $C:X\to Y$ completely continuous. Then $L+N+C$ is weakly A-proper w.r.t. Γ provided $\varphi(x)=(Lx,Kx)$ weakly lower semicontinuous on X. The latter holds, for example, when $K=A_1+A_2$ with $A_1,A_2\in L(X,Y^*)$ such that $(Lx,A_1x)\geq 0$ for $x\in X$ and A_2 compact (see [1]).

To state our results we recall that $A\in L(X,Y)$ is called Fredholm if $d=\dim N(A)<\infty$, $R(A)$ is closed and its codim $d^*=\dim N(A^*)<\infty$. The index of A is defined by ind $(A)=d-d^*$. A nonlinear map $N:X\to Y$ is *asymptotically zero* if $\|Nx\|/\|x\|\to 0$ as $\|x\|\to\infty$ or, equivalently, $N(x)=o(\|x\|)$ as $\|x\|\to\infty$.

The proof of the basic existence Theorem 1.2 below for Equation (1) when $N(L)\neq\{0\}$ is based on the following result which has an independent interest (see [1]).

Theorem 1.1. *Suppose $A\in L(X,Y)$ is A-proper w.r.t. Γ and injective. If $N:X\to Y$ is asymtptotically zero and $T=A+N$ is weakly A-proper w.r.t. Γ, then the equation*

$$Ax+Nx=f \tag{1.1}$$

has a solution in X for each f in Y.

Employing a perturbation method used in similar situations by other authors (e.g. [3, 9, 21]) and Theorem 1.1 we first proved in [1] the following general existence Theorem 1.2 below for (1) with $N(L)\neq\{0\}$ and then used it as a basis for obtaining existence results for the general class of weakly A-proper maps $T=L+N$ with N satisfying certain positivity conditions.

Theorem 1.2. *Suppose $L\in L(X,Y)$ is A-proper w.r.t. Γ with* ind $(L)=0$ *and $N:X\to Y$ is such that $T=L+N$ is weakly A-proper w.r.t. Γ, $N(x)=o(\|x\|)$ as $\|x\|\to\infty$ and $T(\bar{B}(0,\rho))$ is closed for each $\rho>0$. Then we can construct a compact map $C\in L(X,Y)$ such that*

$$Lx+Nx=f \qquad (x\in X, f\in Y) \tag{1.2}$$

is solvable for a given f in Y provided that

(A) *Either the set $S_k^+=\{x_k\in X\mid Lx_k+\lambda_kCx_k+Nx_k-f=0,\ \lambda_k\to 0^+\}$ or the set $S_k^-=\{x_k\in X\mid Lx_k+\lambda_kCx_k-f=0,\ \lambda_k\to 0^-\}$ is bounded by a constant independent of k.*

The map $C\in L(X,Y)$ mentioned above is constructed as follows. Since ind $(L)=0$, i.e., $d=d^*$, there exists closed subspaces X_1 of X and Y_2 of Y with $\dim Y_2=d$ such that $X=N(L)\oplus X_1$ and $Y=Y_2\oplus R(L)$. Let M

be an isomorphism of $N(L)$ onto Y_2, P a projection of X onto $N(L)$ and set $C=MP$. Note that since L is A-proper and $C\in L(X,Y)$ compact, $L_\lambda=L+\lambda C$ is A-proper and injective for each $\lambda\neq 0$ while $L+\lambda C+N$ remains weakly A-proper for each $\lambda\neq 0$. Thus, for each $k\in N$, the equation

$$Lx_k+\lambda_kCx_k+Nx_k=f \qquad (\lambda_k\to 0^+ \text{ or } \lambda_k\to 0^-) \tag{1.3}$$

has a solution for each f in Y by Theorem 1.1 and so the rest of Theorem 1.2 follows.

It was shown in [1] that the boundedness condition (A) is satisfied if in addition to the growth condition $N(x)=o(\|x\|)$ as $\|x\|\to\infty$ one also assumes that N satisfies certain asymptotic positivity conditions. Such conditions have their origin in the existence result of [2] for PDE's, and have since been considered by many authors including those in [3—11]. For numerous other contributions, where the solvability of Equation (1) is studied when $N(L)\neq\{0\}$ via the Lyapunov-Schmidt or the Alternate Method, see the supplementary bibliography.

The basic consequence of Theorem 1.2 obtained in [1] is the following result which, as was shown in [1], extends abstract results of [3, 4, 5, 6] and some results of [8] and of [9]. In preparing this talk we also found that the recent result of [10] is included in [1] and the relation exists between those in [7] and [1].

Theorem 1.3. *Suppose Y is a Hilbert space, $L\in L(X,Y)$ is A-proper w.r.t. Γ with* ind $(L)=0$, *$N:X\to Y$ is such that $T=L+N$ is weakly A-proper w.r.t. Γ and $T(\bar{B}(0,\rho))$ is closed for each $\rho>0$ and M is as above.*

Then, for a given $f\in Y$, Equation (1.2) *is solvable provided one of the following conditions holds:*

B (1) $N(x)=o(\|x\|)$ as $\|x\|\to\infty$ *and*

(2) *Either* $\overline{\lim}\,(Nx_k;My)>(f,My)$ *or* $\overline{\lim}\,(Nx_k;My)<(f,My)$ *whenever $\{x_k\}\subset X$ is such that $\|x_k\|\to\infty$ and $x_k/\|x_k\|\to y\in N(L)$.*

C(1) $\|Nx\|\leq a\|x\|^\alpha+b$ *for $x\in X$ and some $\alpha\in[0,1)$ and $a,b\in R^+$.*

(2) *Either* $\overline{\lim}\,(N(t_k\bar{y}_k+t_k^\alpha z_k),My)>(f,My)$ *or* $\lim\,(N(t_k\bar{y}_k+t_k^\alpha z_k),My)<(f,My)$ *whenever $y\in N(L)\cap\partial B(0,1)$ and sequences $\{t_k\}\subset R^+$, $\{\bar{y}_k\}\subset N(L)$ and $\{z_k\}\subset X_1$ are such that $y_k\to y$ in X, $t_k\to\infty$ and $\|z_k\|\leq c$ for* all $k\in N$ and some $c\in R^+$.

D (1) *Condition C(1) holds and*

(2) *Either* $\overline{\lim}\,(Nx_k,My_k)>(f,My)$ *or* $\lim\,(Nx_k,My_k)<(f,My)$ *whenever $\{x_k\}\subset X$ is such that $\|Px_k\|\to\infty$ and $Px_k/\|Px_k\|\to y\in N(L)$.*

It is shown [1] that either the set S_k^+ or the set S_k^- is bounded by a constant independent of k if one of the above positivity conditions holds.

Remark 1.1. If N is also continuous and $T=L+N$ is A-proper, then Theorems 1.2 and 1.3 hold without the assumption that $T(\bar{B}(0,\rho))$ is closed since, as was shown by the author (see [12]), every continuous A-proper map is proper and, in particular, $T(\bar{B}(0,\rho))$ is closed in Y.

Remark 1.2. Assuming that $X\subset Y$ and $Y=N(L)\oplus R(L)$ the conditions $B(1)-B(2)$ (with B(2) in the equivalent form B(2)$'$: $\lim\,(Nx_k,y)>(f,y)$ or $\overline{\lim}\,(Nx_k,y)<(f,y)$ whenever $\|x_k\|\to\infty$ and $x_k/\|x_k\|\to y\in N(L)$) were used in [8] to establish the solvability of (1.2) when $N: X \to Y$ is such that $N(L+Q)^{-1}: Y \to Y$ is condensing.

When X is a Hilbert space and $Y=X$, conditions $C(1)-C(2)$ were used earlier in [3] as extensions of the hypotheses used in [4, 6] with N compact. We add that when N satisfies (C1), then the positivity conditions imposed in [6, Theorem 2.3] and in [8, Proposition 2.5] imply (D2) of Theorem 1.3. The positivity condition in [10, Theorem 1] when $\alpha=0$ in (C1) also implies (D2).

To indicate the generality of Theorem 1.2 we state an extension to weakly A-proper maps $T=L+N: H \to H$ one (for others see [1]) of the numerous results obtained in [9] for a different but related class of maps. We assume that H has a projectional scheme $\Gamma=\{X_n, P_n\}$ and following [9] we denote by $\alpha>0$ the largest number such that $(Lx, x) \geq -\alpha^{-1}\|Lx\|^2$ for $x \in H$.

Theorem 1.4. *Suppose $L \in L(H, H)$ is A-proper with $N(L)=N(L^*)$, $T=L+N$ is weakly A-proper, $Nx=0(\|x\|)$ as $\|x\| \to \infty$ and $T(\bar{B}(0, \rho))$ is closed for each $\rho>0$. Then, for a given $f \in H$, Equation* (1.2) *is solvable provided that*

E (1) $(Nx-Ny,\ x) \geq \frac{1}{\gamma}\|Nx\|^2 - C(y)$ $\forall x, y \in H$ *and some positive* $\gamma<\alpha$,

where $C(y)$ is independent of x.

(2) $\liminf_{t\to\infty} (N(tx), x) > (f, x)$ *for* $x \in B_1^L \equiv N(L) \cap \partial B(0, 1)$.

Theorem 1.4 is related to [9, Corollary II. 7] where it is assumed that $L: D(L) \subset H \to H$ is unbounded but $(\lambda I+L)^{-1}: H \to H$ is compact and $N: H \to H$ is monotone.

As the first consequence of Theorems 1.2, 1.3 and 1.4 for special maps N discussed in Example 3 we state the following corollary which includes a number of special cases mentioned in the Introduction.

Corollary 1.1. *Suppose X, K and Γ are as in Example 3, $L \in L(X, Y)$ is A-proper w.r.t. Γ with* ind $(L)=0$, *$N: X \to Y$ is such that $Nx=0(\|x\|)$ as $\|x\| \to \infty$ and one of the following three conditions holds:*

(H1) *N is compact.*

(H2) *N is weakly continuous.*

(H3) *N is bounded and of type* (KM).

Then Equation (1.2) *has a solution for a given $f \in Y$ provided condition* (A) *of Theorem 1.2 holds or Y is a Hilbert space and either* (B), (C) *or* (D) *of Theorem 1.3 holds. If $X=Y=H$, the assertion is also true when* (E) *of Theorem 1.4 holds.*

Corollary 1.1 is proved by showing that in each of the above cases the map $T=L+N$ is weakly A-proper and $T(\bar{B}(0, \rho))$ is closed for each $\rho>0$.

Special cases. If $Y=X^*$ and X is reflexive, then the natural choices for K and Γ are $K=I$ and $\Gamma=\{X_n, P_n; P_n^*(X_n), P_n^*\}$ (or Γ is an injective scheme $\{X_n, V_n; X_n^*, V_n^*\}$ as in [1]). In this case N in (H3) is of type (M) in the sense of [17]. This class includes all monotone, semimonotone, pseudomonotone and weakly continuous maps. Hence Corollary 1.1 is vaid for these maps $N: X \to X^*$.

In case $Y=X=H$, Corollary 1.1 extends essentially Theorems 1, 2 and 3 of [3] as well as the corresponding earlier results of [4, 6] where N is compact.

If Q is any bounded set in X, then th *ball-measure of noncompactness* of Q, $x(Q)$, is defined by $\chi(Q)=\inf\{r>0 \mid Q$ can be covered by a finite number of balls of radii $\leq r\}$. A continuous map $F: X \to X$ is *k-ball-contractive* (resp. *ball-condensing*) if $\chi(F(Q) \leq k\chi(Q)$ for all bounded $Q \subset X$ and some $k \geq 0$ (resp. $\chi(F(Q))<\chi(Q)$ whenever $\chi(Q) \neq 0$). For the survey of the theory of these classes of mappings see [24].

It was shown in [1] that if $\{X_n, P_n\}$ is a projectional scheme in X, $L \in L(X, \tilde{Y})$ is Fredholm of inf $(L)=0$, $\tilde{Y}_n=(L+C)(X_n)$ for each $n \in N$, where $C \in L(X, Y)$ is compact map appearing in Theorem 1.2, and $Q_n: Y \to \tilde{Y}_n$ a projection with $\|Q_n\|=1$ for each n, then L is A-proper w.r.t. projectionally complete scheme $\Gamma_L = =\{X_n, P_n; \tilde{Y}_n, Q_n\}$ for (X, Y). Moreover, if N is such that $N(L+C)^{-1}: Y \to Y$ is ball-condensing, then $L+N$ is A-proper w.r.t. Γ_L. Consequently, Theorem 1.3 includes [8, Proposition 2.4 and 2.5].

We note that Theorems 1.2 to 1.4 are also applicable to Equation (1.2) studied in [7] when $F: X \to Y$ is k-ball-contractive with $k \in [0, e[0, (L))$, where $e(L) \equiv \sup\{r>0 \mid r\chi(Q) \leq \chi(L(Q))$ for each bounded $Q \subset X\}$ since, as was shown by Hetzer (see [7, 22]), $e(L)>0$ and, as we shall see in §3, $L+N$ is A-proper w.r.t. Γ_L under these conditions.

Finally let us remark that [10, Theorem 1] follows from Theorem 1.3 when $Y=X=H$, with H separable, since the assumptions in [10] that L and $T=L+N$ are of type (S) (i.e. if $x_n \rightharpoonup x$ in H and $(Tx_n-Tx, x_n-x) \to 0 \Rightarrow \Rightarrow x_n \to x$ in H) imply that they are A-proper (see Example 1), while our condition (D) with $\alpha=0$ is implied by the conditions in [10] that $R(N)$ is bounded and either $e(w)>(f, w)$ (or $e(w)<(f, w)$) for $w \in B_1^L$, where $e(w)= =\lim_{t\to\infty}(N(u+tw), w)$ uniformly for $w \in B_1^L$ and $\|u\| \leq d$ for each given $d>0$.

In preparing this talk the author noted that Theorem 1.1 admits the following nontrivial and useful extension. To state it, first we recall that a continuous map $N: X \to Y$ is called *quasibounded* with the *quasinorm* $|N|$ if $|N|=\lim_{|x|\to\infty}\sup\{\|Nx\|/\|x\|\}<\infty$.

Theorem 1.5. *Suppose $A \in L(X, Y)$ is A-proper w.r.t. Γ and injective. Then there exists a constant $\mu>0$, such that if $T=A+N: X \to Y$ is weakly A-proper and $|N|<\mu$, then the equation*

$$Ax+Nx=f \tag{1.4}$$

has a solution in X for each f in Y.

The proof of Theorem 1.5, its application, and its possible use in the study of the solvability of (1.2) when $N(L) \neq \{0\}$ and N is quasibounded will be given elsewhere.

§2. In this section we indicate how the results of §1 are used to establish the existence of strong solutions for two types of BV Problems studied in [1].

Let $Q \subset R^n$ be a bounded domain with sufficiently smooth boundary ∂Q so that the Sobolev Imbedding Theorem holds on Q. Let $W_2^m(Q) \equiv W_2^m$ be the Sobolev space of all real functions $u(x)$ such that u and its generalized derivatives $D^\alpha u \in L^2(Q)$ for $|\alpha| \leq m$, where $D^\alpha = D^{|\alpha|}/\partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n}$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ is the multiindex with $|\alpha| = \alpha_1 + \cdots + \alpha_n$. W_2^m is a separable Hilbert space with the inner product and norm

$$<u, v> = \sum_{|\alpha| \leq m} \int_Q D^\alpha u D^\beta v dx, \quad \|u\|^2 = <u, u> \quad (u, v \in W_2^m).$$

Let $\overset{\circ}{W}{}_2^m$ be the subspace of W_2^m, which is the completion in the W_2^m-norm of $C_0^\infty(Q)$, the set of infinitely differentiable functions with compact support in Q.

Problem A. To compare our existence result [1, Theorem 3.1] with that of [8, Theorem 3.7] we first discuss the solvability of

$$(2.1) \qquad \begin{cases} -\Delta u(x) - g(x, \nabla u(x), \Delta u(x)) + h(u(x)) = f(x) & (x \in Q) \\ \dfrac{\partial u}{\partial \eta}(x) = 0 & (x \in \partial Q), \end{cases}$$

which was studied in [8] under rather restrictive condition on g and the Sobolev spaces. We present it here in order to contrast the applicability of our Theorem 1.3 for A-proper maps with the results in [8] for k-set-contractive perturbations. To state the conditions on g and h under which (2.1) has a solution in W_2^m for each f in $L^2(Q)$ we let $X = \left\{ u \in W_2^m : \dfrac{\partial u}{\partial \eta} = 0 \text{ for } x \in \partial Q \right\}$ and $Y = L^2(Q)$. Let $L : X \to Y$ be defined by $Lu = -\Delta u$ for $u \in X$.

It is known that $N(L)$ consists of constant functions, $R(L)$ consists of those functions whose mean value is zero, ind $(L) = 0$, $Y = N(L) \oplus R(L)$ and $K \equiv L + I : X \to Y$ is a linear homeomorphism, where $I : X \to Y$ is the inclusion map which is compact by Sobolev Imbedding Theorem. In what follows we use $\|\cdot\|_0$ to denote the equivalent norm in X given by $\|u\|_0 = \|Ku\|_{L^2}$.

Let $\{X_n\} \subset X$ be such that dist $(u, X_n) \to 0$ for each $u \in X$ and let $Y_n = K(X_n)$ for $n \in N$. If $P_n : X \to X_n$ and $Q_n : Y \to Y_n$ are projections, then $\Gamma_L = \{X_n, P_n ; Y_n, Q_n\}$ is complete for (X, Y). Since

$$(2.2) \qquad (Lu, Ku) \geq \|u\|_0^2 - \|u\|^2_{L^2} \quad \text{for all } u \in X$$

and X is compactly imbedded in L^2, it follows from (2.2) and the assertion (J3) of Theorem 3.1J in [12] that $L : X \to Y$ is A-proper w.r.t. Γ_L. Assume that

(a1) $g : Q \times R^n \times R \to R$ is continuous and there are $\mu \in [0, 1)$ and $\varphi \in L^2$ such that $|g(x, r, s)| \leq \varphi(x)$ for $(x, r, s) \in Q \times R^n \times R$ and $(g(x, r, s_1) - g(x, r, s_2))(s_1 - s_2) \geq -\mu |s_1 - s_2|^2$ for $(x, r) \in Q \times R^n$ and $s_1, s_2 \in R$.

(a2) $h : R \to R$ is continuous and there are $a, b \in R^+$ and $\delta \in (0, 1)$ such that $|h(s)| \leq a + b|s|^\delta$ for $s \in R$ and $\lim_{s \to \infty} h(s) = \infty$, while $\lim_{s \to -\infty} h(s) = -\infty$.

If we define $C, G : X \to Y$ by $Cu = h(u(x))$ and $Gu = g(x, \nabla u, \Delta u)$ for $u \in X$, then (a2) implies that C is compact and $\|Cu\|_{L^2}/\|u\|_0 \to 0$ as $\|u\|_0 \to \infty$, while (a1) implies that G is continuous, $R(G)$ is bounded and for $u, v \in X$

$$(2.3) \qquad \begin{aligned} &(Gu - Gv, Ku - Kv) \geq -\mu \|\Delta(u - v\|^2_{L^2} - 2p\|u - v\|_{L^2} + \\ &\qquad (g(x, \nabla u, \Delta v) - g(x, \nabla v, \Delta v), \quad \Delta u - \Delta v), \end{aligned}$$

where $p = \|\varphi\|_{L^2}$. From (2.2)–(2.3) we get

$$(2.4) \qquad \begin{aligned} &((L + G)u - (L + G)v, Ku - Kv) \geq (l - \mu)\|u - v\|_0^2 - \|u - v\|^2_{L^2} - \\ &-2p\|u - v\|_{L^2} + (g(x, \nabla u, \Delta v) - g(x, \nabla v, \Delta v), \Delta u - \Delta v) \text{ for } u, v \in X. \end{aligned}$$

In view of (2.4), (J3) of Theorem 3.1J [12] implies that $L + G$ is A-proper w.r.t. Γ_L and so is the map $T = L + N$ with $N = G + C$ because C is compact. Since $(Nu_n, w) \to \infty$ as $n \to \infty$ whenever $\{u_n\} \subset X$ is such that $\|u_n\| \to \infty$ and $w_n = u_n/\|u_n\| \to w \in N(L)$, Theorem 1.3 implies the validity of the following improvement of [8, Theorem 3.7] established in [1].

Theorem 2.1. *Suppose g and h satisfy (a1) and (a2) respectively. Then Equation (2.1) has a solution in X for each $f \in L^2$.*

Remark 2.1. In [8] the solvability of (2.1) for each $f \in L^p$ was proved by means of the degree theory of condensing vector fields under the more restrictive and essential conditions that $p > n$, $|g(x, r, s_1) - g(x, r, s_2)| \leq \geq \mu|s_1 - s_2|$ for some $\mu \in (0,1)$ and all $(x, r) \in Q \times R^n$, $s_1, s_2 \in R$, and that $g(x, r, s)$ is continuous in $r \in R^n$ uniformly w.r.t. $(x, s) \in Q \times R$.

Remark 2.2. If we assume that g is Lipschitzian with respect to the derivatives of order 2, then instead of BV Problem (2.1) we could similarly treat the problem

$$(2.1)' \qquad \begin{cases} -\Delta u - g'(x, u, \nabla u, D^2 u) + h(u(x)) = f(x), & x \in Q, \\ \dfrac{\partial u}{\partial \eta}(x) = 0, & x \in \partial Q, \end{cases}$$

if instead of (a1) we assume that g' satisfies

(a1)′ $g' : Q \times R \times R^n \times R^{s_2} \to R$ is continuous and there exist $\mu > 0$ and $\varphi \in L^2$ such that $|g(x, t, r, s)| \leq \varphi(x)$ for $(x, t, r, s) \in Q \times R \times R^n \times R^{s_2}$ and for $(x, t, r) \in Q \times R \times R^n$

$$(2.5) \qquad |g'(x, t, r, s_1) - g'(x, t, r, s_2)| \leq \mu|s_1 - s_2|^p, \quad \forall s_1, s_2 \in R^{S_2'}$$

where the vector space $R^{S_2} = \{\zeta_\alpha \,|\, |\alpha| = 2\}$ and $D^2 u = \{D^\alpha u \,|\, |\alpha| = 2\}$ and μ is such that $\mu c < 1$ with $c > 0$ being the smallest positive number such that

$$(2.6) \qquad \Big(\sum_{|\alpha| = 2} \|D^\alpha u\|^2_{L^2}\Big)^{1/2} \leq c\|Ku\|_{L^2} \qquad \forall u \in X.$$

If we define $G' : X \to Y$ by $G'u = g'(x, u, \nabla u, D^2 u)$ for $u \in X$, then essentially the same arguments and the Hölder inequality imply (2.4) with G' replacing G and g' replacing g. Consequently $T = L + N' : X \to Y$ is A-proper w.r.t. Γ_L with $N' = G' + C$, $\|N'u\|_{L^2}/\|u\|_0 \to 0$ as $\|u\|_0 \to \infty$ and $(Nu_n, w) \to \infty$ whenever $\{u_n\} \subset X$ is such that $\|u_n\|_0 \to \infty$ and $u_n/\|u_n\|_0 \to w \in N(L)$. Thus we have the following

Theorem 2.1.′ *Supose g' and h satisfy (a1)′ and (a2) respectively. Then Equation (2.1)′ has a solution in X for each f in L^2.*

Problem B. As a second application we consider the abstract problem that is modelled on situations which arise when one attempts to treat the semilinear elliptic problems on unbounded domains Q in R^n, i.e., for a given f in a separable real Hilbert space H we consider the solvability of

$$(2.7)\qquad Au-\lambda u+Nu=f \qquad (u\in D(A),\ f\in H)$$

under the following hypotheses:

(b1) A is densely defined, positive definite, self-adjoint operator whose essential spectrum $\sigma_e(A)$ is bounded below, i.e., there is $\gamma>0$ such that, for each $\varepsilon>0$, $\sigma(A)\cap(-\infty,\gamma-\varepsilon)$ consists of a nonempty set of isolated eigenvalues, each of finite multiplicity, with $\lambda_0<\lambda_1<\cdots<\gamma$.

(b2) λ is an eigenvalue of A such that $|\lambda|\gamma^{-1}<1$

(b3) $N:H_0\to H$ is such that $N=N_1+N_2$ with $N_2:H_0\to H$ compact and $N_1:H_0\to H$ continuous and monotone (i.e., $(N_1u-N_1v,u-v)\geqslant 0$ $\forall u, v\in H_0$) and $\|Nu\|/|u|_0\to 0$ as $|u|_0\to\infty$, *where* H_0 is the completion of $D(A)$ in $|u|_0=(Au,u)^{1/2}$ with $[u,v]=(Au,v)$ $\forall u,v\in D(A)$.

It is known that H_0 is continuously imbedded into H, $A^{-1}:H\to H$ is bounded, self-adjoint and positive, its square root $A^{-1/2}:H\to H$ is also self-adjoint, positive and bounded, and $A^{1/2}$, considered as a mapping from H_0 to H, is a linear homeomorphism. Moreover, it follows from (b1) that $A^{-1}:H\to H$ is γ^{-1}-ball-contractive and $S\equiv A^{-1/2}$: $H\to H$ is $\gamma^{-1/2}$-ball-contractive (see [19]). This and (b2) show that λA^{-1} is k-ball-contractive with $k=|\lambda|\gamma^{-1}<1$, the operator $L\equiv I-\lambda A^{-1}$ is self-adjoint with $N(L)\neq\{0\}$ and $u\in N(L)$ if and only if $Au-\lambda u=0$. Furthermore, $F\equiv SNS$: $H\to H$ is such that $F=F_1+F_2$ with $F_2=SN_2S$ compact and $F_1\equiv SN_1S$ monotone and continuous by (b3). Note also that $u\in D(A)$ is a solution of (2.7) if and only if $v=A^{1/2}$ and v is a solution of

$$(2.8)\qquad Lv+Fv=S(f)\qquad (v\in H,\ S(f)\in H).$$

We are now in the situation where we can apply to the equivalent Equation (2.8) the results of §1. To do this we set $Y=X=H$, $\Gamma_1=\{X_n, P_n\}$ and $T=L+F$. Example 1 shows that L is A-proper w.r.t. Γ_1 and therefore Fredholm of ind $(L)=0$ (see [12, 20]), while $T=L+F$ is also A-proper w.r.t. Γ_1 (see [14]). Moreover, the last part of (b3) implies that $\|Fv\|/\|v\|\to 0$ as $\|v\|\to\infty$.

The above discussion and the A-proper mapping version of Theorem 1.3 imply the validity of the following result in [1].

Theorem 2.2. *Suppose that* (b1), (b2) *and* (b3) *hold. Then* (2.7) *has a solution* $u\in D(A)$ *for a given* f *in* H *if the following condition holds:*

(h1) *Either* $\lim\sup_n(Nu_n,u_0)>(f-u_0)$ *or* $\lim\inf_n(Nu_n,u_0)<(f,u_0)$ *whenever* $\{u_n\}\subset H_0$ *is such that* $|u_n|_0\to\infty$ *and* $u_n/|u_n|_0\to u_0$ *in* H_0 *with* $u_0\in N(A-\lambda I)$.

§3. To discuss the weak solvability of the BV Problem (C) we must introduce some further notation. In what follows we let R^{S_m} be the vector space whose elements are $\zeta=\{\xi_\alpha:|\alpha|\leqslant m\}$, $\eta=\{\eta_\beta:|\beta|\leqslant m-1\}\in R^{S_{m-1}}$ and each ζ in R^{S_m} we write as a pair (η,ζ) with $\eta\in R^{S_{m-1}}$ and $\zeta=\{\zeta_\alpha:|\alpha|=m\}\in R^{S'_m}(\equiv R^{S_m-S_{m-1}})$. We also set $\zeta(u)=\{D^\alpha u:|\alpha|\leqslant m\}$, $\eta(u)=\{D^\alpha u:|\alpha|\leqslant m-1\}$ and $\zeta(u)=\{D^\alpha:|\alpha|=m\}$.

Let V be a closed subspace of $\mathring{W}_2^m$ such that $V\supseteq W_2^m$. For a given f in L^2, the *BV Problem corresponding to* V for the elliptic equation

$$(3.1)\qquad \sum_{|\alpha|,|\beta|<m}(-1)^{|\alpha|}D^\alpha(a_{\alpha\beta}(x)\,D^\beta u)+\sum_{|\alpha|<p}(-1)^{|\alpha|}D^\alpha b_\alpha(x,u,\cdots,D^p u)=f(x)$$

is the problem of establishing the existence of an element $u\in V$, called the *weak solution* of (3.1), such that

$$(3.2)\qquad \sum_{|\alpha|,|\beta|<m}(a_{\alpha\beta}D^\beta u, D^\alpha V)+\sum_{|\alpha|<p}(b_\alpha(x,u,\cdots,D^p u), D^\alpha v)=(f,v)\quad \forall v\in V,$$

assuming that $p\leqslant m$ and the integrals in (3.2) make sense for each $u\in V$ with (.) denoting the inner product in L^2.

To formulate the above BV Problem, as an equivalent operator equation in V, to which Theorems 1.2 and 1.3 are applicable, we impose the following conditions on the functions $a_{\alpha\beta}(x)$ and b_α's

(H1) $a_{\alpha\beta}(x)\in L^\infty(Q)$ *with* $a_{\alpha\beta}=a_{\beta\alpha}$ *for all* $|\alpha|,|\beta|\leqslant m$ *and there exists a constant* $\mu_0>0$ *such that*

$$(3.3)\qquad \sum_{|\alpha|=|\beta|=m}a_{\alpha\beta}(x)\zeta_\alpha\zeta_\beta\geqslant\mu_0\sum_{|\alpha|=m}|\zeta_\alpha|^2\quad \forall x\in Q,\quad \zeta\in R^{S'_m}.$$

(H2) *For each* $|\alpha|\leqslant p$, $b_\alpha:Q\times R^{S_p}\to R$ *satisfies the Carathéodory conditions and there are constants* $c_0>0$ *and* $\sigma\in[0,1)$ *and* $h\in L^2$ *such that*

$$(3.4)\qquad |b_\alpha(x,\zeta)|\leqslant c_0\Big(\sum_{|\beta|<p}|\zeta_\beta|^\sigma\Big)+h(x)\qquad (\zeta\in R^{S_p},\ x\in Q(a,e)).$$

To define the „variational BV Problem w.r.t. V", we note first that by virtue of (H1) and (H2) and the results on Nemytskii operators (see[18]), the generalized forms

$$a(u,v)=\sum_{|\alpha|,|\beta|<m}(a_{\alpha\beta}(x)D^\beta u, D^\alpha v),$$

$$b(u,v)=\sum_{|\alpha|,<p}(b_\alpha(x,u,\cdots,D^p u), D^\alpha v)$$

are well defined on V and there exist bounded continuous maps, L, $N:V\to V$, with L linear, such that

$$(3.5)\qquad \langle Lu,v\rangle=a(u,v),\quad \langle Nu,v\rangle=b(u,v)\quad \forall u,v\in V.$$

Further, to each $f\in L^2$ there exists a unique $w_f\in V$ such that $(f,v)=\langle w_f,v\rangle$ for all $v\in V$. Thus, $u\in V$ is a weak solution of (3.1) if and only if u is a solution of the operator equation

$$(3.6)\qquad Lu+Nu=w_f\qquad (u\in V,\ w_f\in V).$$

In order to apply Theorems 2 and 1.3 to Equation (3.6) we first note that since V is a separable Hilbert space there exists a projectionally complete scheme $\Gamma=\{X_n, P_n\}$ with respect to which the following holds.

Lemma 3.1. *If* (H1) *holds, then* $L\in L(V,V)$ *is self-adjoint and* A*-proper w.r.t.* Γ. *Furthermore, there exist* $c_1>0$ *and* $c_2>0$ *such that*

$$\|Nu\| \leqslant c_1\|u\|^{\sigma} + c_2 \quad for \quad u \in V. \tag{3.7}$$

Proof. The fact that L is self-adjoint follows trivially from (H1). In virtue of Example 1, to show that L is A-proper, it suffices to prove that L is of type (S), i.e., if $\{u_n\} \subset V$ is such that $u_n \rightharpoonup u$ in V and $\lim_n <Lu_n - Lu, u_n - u> = 0$, then $u_n \to u$ in V. Now, L can be written in the form $L = L_1 + L_1'$, where

$$<L_1 u, u> = \sum_{|\alpha|=|\beta|=m} (a_{\alpha\beta}(x) D^{\beta}u, D^{\alpha}v) \quad (u, v \in V) \tag{3.8}$$

and L_1' is compact in view of the complete continuity of the imbedding of W_2^m into W_2^{m-1} (see [18]). Now, let $\{u_n\} \subset V$ be such that $u_n \rightharpoonup u$ in V and

$$\lim_n <Lu_n - Lu, u_n - u> = 0. \tag{3.9}$$

Since $D^{\alpha}u_n \to D^{\alpha}u$ in L^2 for $|\alpha| < m$, to prove that $u_n \to u$ in V, it suffices to show that $D^{\alpha}u_n \to D^{\alpha}u$ in L^2 for $|\alpha| = m$. It follows from (3.8)–(3.9) and the compactness of $L_1' \in L(V, V)$ that $\lim_n <L_1u_n - L_1u, u_n - u> = 0$. This, (3.8) and (3.3) imply that $D^{\alpha}u_n \to D^{\alpha}u$ in L^2 for $|\alpha| = m$.

Finally, using the Holder's inequality, we obtain (3.7) from (3.4). Q.E.D.

Now, when $p = m - 1$ then $N : V \to V$ is completely continuous and in this case the solvability of (3.2) or (3.6) for a given f in L^2, under the assumption that $N(L) \neq \{0\}$ has been studied in [6] (see also [3] and the references cited there), under various growth conditions some of which will be discussed later, by using the L e r a y - S c h a u d e r fixed point theorem or the degree theory for compact vector fields. Since in this case $T = L + N$ is trivially A-proper and, in particular, weakly A-proper with $T(B)$ closed in V for each ball $B \subset V$, it is obvious that our Theorems. 1.2 and 1.3 apply in this case.

When $p = m$, then $N : V \to V$ is not compact and in this case the solvability of (3.2) or (3.6) has been studied by H e t z e r [7], by the coincidence degree theory for set-contraction maps, under the basic assumption that the functions $b_{\alpha}(x, \eta(u), \zeta(u))$ are Lipschitzian w.r.t. the highest order terms $\zeta(u)$. A slightly different version of (3.2) when $p = m$ was studied, under certain monotonicity conditions, in a paper [10] which the author received from Nečas while preparing this talk. We shall discuss more fully the conditions and results in [7, 10] at the end of this section.

In this paper we study the solvability of (3.2) or (3.6) when $p = m$ under conditions on the functions b_{α}'s which are different and weaker than those used in [7, 10]. In view of our Theorems 1.2 and 1.3, the basic problem is solving (3.2) or (3.6), when $N(L) \neq \{0\}$ and f is given, consists in providing concrete (and practically meaningful and variable) analytical assumptions on the functions $b_{\alpha}(x, \zeta(u)) \equiv b_{\alpha}(x, u, \cdots D^m u)$ so that the following holds:

(H3) *$T = L + N : V \to V$ is weakly A-proper w.r.t. Γ and $T(\overline{B}(0, \rho))$ is closed in V for each $\rho > 0$.*

We shall first prove the basic existence results, Theorem 3.1 below, for the weak solvability of (3.1) in V under the assumption (H3) and then

indicate a number of sufficient analytic conditions on b_{α}'s which will imply the validity of (H3). In addition to new results for (3.1) as a special case of Theorem 3.1, we deduce [7, Theorem 3] as well as indicate the generalizations of the results in [2, 3, 6, 11, 25] and others to the case when $p = m$.

To state Theorem 3.1 and its consequences concerning the variational solvability of (3.1) when $N(L) \neq \{0\}$, as in [7] we assume the following additional condition.

(H4) *For each $|\alpha| \leqslant m$, there is a measurable function $h_{\alpha} : Q \times S^1 \to R$, where $S^1 = \{\xi \in R^{s_m} : |\xi| = 1\}$, and $k_{\alpha} \in L^{1/1-\sigma}(Q)$ with $|h_{\alpha}(x, \xi)| \leqslant k_{\alpha}(x)$ for $x \in Q$ (a.e) and $\xi \in S^1$ such that if $\{\xi^{(n)}\} \subset S^1$ and $\{r_n\} \subset R$ are any sequences with $\xi^{(n)} \to \xi \in S^1$ and $r_n \to \infty$, then $\lim_n b_{\alpha}(x, r_n\xi^{(n)})/r_n^{\sigma} = h_{\alpha}(x, \xi)$ for $x \in Q$ (a.e.).*

Assuming that $N(L) \neq \{0\}$ we can, in view of (H4), define the functional $c : B_1^L \equiv \{w \in N(L) : \|w\| = 1\} \to R$ by

$$c(w) = \sum_{|\alpha| \leqslant m} \int_{Q_0(D^{\alpha}w)} h_{\alpha}(x, \xi(w)(x)/|\xi(w)(x)|) |\xi(w)|^{\sigma} D^{\alpha}w(x)\, dx. \tag{3.10}$$

where for any $v : Q \to R$ we set $Q_0(v) = \{x \in Q : v(x) \neq 0\}$. We are now in a position to deduce from Theorem 1.3 the following existence results for (3.2) or Equation (3.6).

Theorem 3.1. *Suppose (H1)–(H4) hold, $N(L) \neq \{0\}$ and $c(w)$ is given by (3.10) for $w \in B_1^L$.*

(A1) *If $\sigma \in [0, 1)$ and $c(w) > 0$ for $w \in B_1^L$, then Equation (3.6) with $w_f = 0$ has a solution in V.*

(A2) *If $\sigma \in (0, 1)$ and $c(w) > 0$ for $w \in B_1^L$, then Equation (3.6) has a solution in V for each f in L^2.*

(A3) *If $\sigma = 0$ and $c(w) > (f, w)$ for $w \in B_1^L$, then Equation (3.6) has a solution for such f's in L^2.*

Proof. (A1) In virtue of Lemma 3.1 and Theorem 1.3, to prove (A1) it suffices to show that

$$\begin{cases} \overline{\lim} <Nu_n, w> > 0 \text{ whenever } \{u_n\} \subset V \text{ is such that} \\ \|u_n\| \to \infty \text{ and } w_n = u_n/\|u_n\| \to w \in B_1^L. \end{cases} \tag{3.11}$$

Now, if (3.11) were not true, then we could find a sequence $\{u_n\} \subset V$ such that $t_n \equiv \|u_n\| \to \infty$ and $w_n = u_n/t_n \to w$ in V with $w \in B_1^L$ but $\overline{\lim}_n <Nu_n, w> \leqslant 0$. If this is the case, then to any given $\varepsilon > 0$ there corresponds an integer $n_{\varepsilon} \geqslant 1$ such that

$$<Nu_n, w> = \sum_{|\alpha| \leqslant m} \int_Q b_{\alpha}(x, \xi(u_n))\, D^{\alpha}w(x)\, dx \leqslant \varepsilon \quad \text{for} \quad n \geqslant n_{\varepsilon},$$

or, equivalently,

$$\sum_{|\alpha| \leqslant m} t_n^{-\sigma} \int_Q b_{\alpha}(x, \xi(u_n))\, D^{\alpha}w(x)\, dx \leqslant t_n^{-\sigma}\varepsilon \text{ for } n \geqslant n_{\varepsilon}. \tag{3.12}$$

By going if necessary to subsequences, we can assume that $D^{\alpha}w_n(x) \to D^{\alpha}w(x)$

a.e. on Q for all $|\alpha|\leqslant m$ and so we find for almost each fixed $x\in Q_0(D^\alpha w)$ an integer $n_0(x)\in N$ such that $D^\alpha w_n(x)\neq 0$ for $n\geqslant n_0(x)$. Hence $|\xi(w_n)(x)|>0$ for almost every $x\in Q_0(D^\alpha w)$ and all $n\geqslant n_0(x)$. Now, since $|\xi(t_n w_n)(x)|\to\infty$ for $x\in Q_0(D^\alpha w)$ (a.e.)

$$\xi^{(n)}=\frac{\xi(u_n)(x)}{|\xi(u_n)(x)|}=\frac{\xi(w_n)(x)}{|\xi(w_n)(x)|}\to\frac{\xi(w)(x)}{|\xi(w)(x)|}\equiv\xi\in S^1$$

for each fixed $x\in Q_0(D^\alpha w)$ (a.e.) and $r_n^\sigma=t_n^\sigma|\xi(w_n)|^\sigma$, we use (H4) to conclude that for each such x and all $|\alpha|\leqslant m$

$$\lim_n\frac{b_\alpha(x,\xi(u_n)(x))}{t_n^\sigma}=\lim_n\frac{b_\alpha(x,r_n\xi^{(n)})}{r_n^\sigma}|\xi(w_n)(x)|^\sigma= \tag{3.13}$$

$$=h_\alpha(x,\xi(w)/|\xi(w)|)|\xi(w)(x)|^\sigma.$$

On the other hand, by virtue of (H2), we have

$$\left|\frac{b_\alpha(x,\xi(u_n)(x))}{t_n^\sigma}\right|\leqslant\frac{c_0}{t_n^\sigma}\,(t_n^\sigma|\xi(w_n)|^\sigma+h)\quad\text{for } |\alpha|\leqslant m$$

with the right-hand side being bounded by an L^2-function. This and (3.13) imply that $\{t_n^{-\sigma}b_\alpha(x,\xi(u_n))\}$ converges weakly to $h_\alpha(x,\xi(w)/|\xi(w)|)|\xi(w)|^\sigma$ in $L^2(Q_0(D^\alpha w))$. Hence

$$t_n^{-\sigma}\int_{Q_0(D^\alpha w)} b_\alpha(x,\xi(u_n))\,D^\alpha w(x)\,dx \text{ converges to} \int_Q h_\alpha(x,\ \xi(w)/|\xi(w)|)\,|\xi(w)|^\sigma D^\alpha w(x)dx$$

for each $|\alpha|\leqslant m$. It follows from this the passage to the limit in (3.12) that

$$\sum_{|\alpha|\leqslant m}\int_Q h_\alpha(x,\xi(w)/|\xi(w)|)|\xi(w)|^\sigma D^\alpha w(x)\,dx\leqslant 0,$$

in contradiction to the assumption that $c(w)>0$ for all $w\in B_1^L$. This establishes the validity of (A1).

(A2) We claim that if $\sigma\in(0,1)$ and $c(w)>0$ for $w\in B_1^L$, then for any given $f\in L^2$ we have $\overline{\lim}\langle Nu_n,w\rangle>\langle w_f,w\rangle$ whenever $\{u_n\}\subset V$ is such that $t_n\equiv\|u_n\|\to\infty$ and $w_n=u_n/t_n\to w\in B_1^L$.

Indeed, if for some f in L^2 this were not the case, then we would find a sequence $\{u_n\}\subset V$ such that $t_n=\|u_n\|\to\infty$ and $w_n=u_n/t_n\to w\in B_1^L$ but $\overline{\lim}\langle Nu_n,w\rangle\leqslant(f,w)=\langle w_f,w\rangle$. As above, this implies that to any given $\varepsilon>0$ there exists $n_\varepsilon\in N$ such that for $n\geqslant n_\varepsilon$

$$\sum_{|\alpha|\leqslant m}\int_Q t_n^{-\sigma}b_\alpha(x,\xi(u_n))\,D^\alpha w(x)\,dx\leqslant t_n^{-\sigma}(\varepsilon+(f,w)). \tag{3.14}$$

Since $\sigma>0$ and $t_n^\sigma\to\infty$, as in (A1) we obtain from (3.14) the relation $c(w)\leqslant 0$ with $w\in B_1^L$, contradicting our assumption that $c(w)>0$ for all $w\in B_1^L$.

(A3) When $\sigma=0$ and $c(w)>(f,w)$ for a given f in L^2 and all $w\in B_1^L$, then by the same arguments as in (A2), one obtains from the contradictory assumption that $\overline{\lim}\,(Nu_n,w)\leqslant(f,w)$ for some $\{u_n\}\subset V$ with $\|u_n\|\to\infty$ and $u_n/\|u_n\|\to w\in B_1^L$ the relation

$$\sum_{|\alpha|\leqslant m}\int_Q h_\alpha(x,\xi(w)/|\xi(w)|)\,D^\alpha w(x)\,dx\leqslant\varepsilon+(f,w)$$

with $w\in B_1^L$. Since the above inequality is true for any given $\varepsilon>0$, we have the contradiction to the assumption that f in L^2 is such that $c(w)>(f,w)$ for all w in B_1^L. Q.E.D.

Special Cases. We shall now indicate some sufficient conditions on $b_\alpha(x,\xi(u))$'s which imply the validity of (H3).

Theorem 3.2. *Assume in addition to (H1), (H2) and (H4) that*

(H3a) *There exists $\mu\in[0,\mu_0]$ and a continuous function c of $R^+\times R^+$ to R^+ such that for each ball $B(0,r)\subset V$ and $u,v\in B(0,r)$ $\sum_{|\alpha|\leqslant m}(b_\alpha(x,\xi(u))-b_\alpha(x,\xi(v)),\ D^\alpha u-D^\alpha v)\geqslant-\mu\sum_{|\alpha|=m}\|D^\alpha(u-v)\|^2-c(r,\|u-\|')$, where $\|\cdot\|'=\|\cdot\|_{W_2^k}$ for some $k<m$ and $c(r,t\rho)/t\to 0$ as $t\to 0^+$ for any fixed r and ρ.*

Then the assertions (A1), (A2) and (A3) of Theorem 3.1 hold.

Proof. In view of Theorem 3.1, to prove Theorem 3.2, it suffices to show that (H3a) implies (H3). The latter fact follows from

Lemma 3.2. *If* (H1), (H2) *and* (H3a) *hold, then $T=L+N: V\to V$ satisfies the hypothesis* (H3).

Proof of Lemma 3.2. It will be shown in Lemma 3.3 below that when $0\leqslant\mu<\mu_0$, then under condition which is weaker than (H3a) the map $T=L+N$ is A-proper and thus satisfies (H3). Thus, it suffices to treat the case when $\mu=\mu_0$.

Now, if $\mu=\mu_0$ then, since $L=L_1+L_1'$ with $L_1'\in L(X,Y)$ compact, (H1) and (H3a) imply that for $u,v\in B(0,r)\subset V$

$$\langle Tu-Tv,\ u-v\rangle\geqslant\langle L_1'u-L_1',\ u-v\rangle-c(r,\|u-v\|').$$

In view of Example 2, to establish (H3), it suffices to show that T is of type (M). So let $\{u_n\}\subset V$ be such that $u_j\to u_0$ in V, $Tu_j\to g$ in V and $\lim\sup_j\langle Tu_j,u_j\rangle\leqslant\langle g,u_0\rangle$. We claim that $Tu_0=g$. Let $r>0$ be such that u_0 and $\{u_j\}$ lie in $B(0,r)$, and note that for all u in $B(0,r)$

$$\langle Tu_j-Tu,\ u_j-u\rangle\geqslant\langle L_1'u_j-L_1'u,\ u_j-u\rangle-c(r,\|u_j-u\|').$$

It follows from this that

$$\langle Tu_j,u\rangle+\langle Tu,u_j\rangle\leqslant\langle Tu_j,u_j\rangle+c(r,\|u_j-u\|')+$$

$$+\langle Tu,u\rangle-\langle L_1'u_j-L_1'u,\ u_j-u\rangle.$$

Since $u_j\to u_0$ in V, $Tu_j\to g$ in V and $\lim\sup_j\langle Tu_j,u_j\rangle\leqslant\langle g,u_0\rangle$, it follows that $\|u_j-u\|'\to 0$, $\langle L_1'u_j-L_1'u,\ u_j-u\rangle\to\langle L_1'u_0-L_1'u,\ u_0-u\rangle$ and thus taking the limit superior in the last inequality we get for all $u\in B(0,r)$ the relation: $\langle g,u\rangle+\langle Tu,u_0\rangle\leqslant\langle g,u_0\rangle+c(r,\|u_0-u\|')+\langle Tu,u\rangle-$

$-<L_1'u_0-L_1'u, u_0-u>$. It follows from this that $<Tu-g, u-u_0>\geq <L_1'u_0-L_1'u, u_0-u-c(r, \|u_0-u\|')$. This inequality implies that $Tu_0=g$. If not, then there exists $w_0\in V$ such that $<Tu_0-g, w_0><0$. Let $\{t_j\}\subset R^+$ be such that $t_j\to 0$ and $u_0-t_jw_0\in B(0, r)$. Choosing $u_0-t_jw_0$ as u in the last inequality we obtain

$$<T(u_0-t_jw_0)-g, w_0>\geq t_j<L_1'(w_0, w_0>-t_j^{-1}c(r, t_j\|w_0\|') \qquad \forall j\in N.$$

Since T is continuous, passing to the limit as $j\to\infty$ we get the relation $<Tu_0-g, w_0>\geq 0$, which contradicts our choice of w_0. Q.E.D.

As our next consequence of Theorem 3.1 we have

Theorem 3.3. *Let* (*H*1), (*H*2) *and* (*H*4) *hold and assume that*

(H3b) *There is* $\mu\in[0, \mu_0)$ *so that for* $x\in Q$ *and* $\eta\in R^{S_{m-1}}$

$$\sum_{|\alpha|=m}(b_\alpha(x, \eta, \zeta)-b_\alpha(x, \eta, \zeta'))(\zeta_\alpha-\zeta_\alpha')\geq-\mu\sum_{|\alpha|=m}|\zeta_\alpha-\zeta_\alpha'|^2\,\forall\zeta, \zeta'\in R^{S_m}.$$

Then the assertions (*A*1), (*A*2) *and* (*A*3) *of Theorem* 3.1 *hold.*

Proof. Since, by Proposition 1.1c in [12], $T(B)$ is closed in V whenever $T: V\to V$ is continuous and A-proper, Theorem 3.3 follows from Theorem 3.1 and

Lemma 3.3. *If* (*H*1), (*H*2) *and* (*H3b*) *hold, then* $T=L+N$ *is A-proper w.r.t.* Γ.

Proof of Lemma 3.3. In virtue of Example 1, to prove the A-properness of $T=L+N$ w.r.t. Γ, it suffices to show that T satisfies condition (S). So, let $\{u_n\}\subset V$ be any sequence such that $u_n\rightharpoonup u$ in V and

$$\lim_n<Tu_n-Tu, u_n-u>=0. \tag{3.15}$$

Since $D^\alpha u_n\to D^\alpha u$ in L^2 for $|\alpha|<m$ and $b_\alpha(x, \xi(u_n))$'s are bounded in L^2, it follows that

$$\lim_n\,[\sum_{|\alpha|<m}(b_\alpha(x, \xi(u_n))-b_\alpha(x, \xi(u)), D^\alpha u_n-D^\alpha u)]=0.$$

In view of this and the fact that $L_1(u_n)\to L_1'(u)$ in V, it follows from (3.15) that

$$\lim\,[\sum_{|\alpha|=|\beta|=m}(a_{\alpha b}D^\beta(u_n-u), D^\alpha u_n-D^\alpha u)+$$

$$\sum_{|\alpha|=m}(b_\alpha(x, \eta((u_n), \zeta(u_n))-b_\alpha(x, \eta(u_n), \zeta(u)), D^\alpha u_n-D^\alpha u)]=0. \tag{3.16}$$

Since $\eta(u_n)\to\eta(u)$ in L^2, $b_\alpha(x, \eta(u_n), \zeta(u))\to b_\alpha(x, \eta(u), \zeta(u))$ in L^2 and $D_\alpha u_n\rightharpoonup D^\alpha u$ in L^2 for $|\alpha|=m$ (see [18]), it follows that

$$\sum_{|\alpha|=m}(b_\alpha(x, n(u_n), \zeta(u))-b_\alpha(x, \eta(u), \zeta(u)), D^\alpha u_n-D^\alpha u)\to 0.$$

This and (3.16) imply that

$$\lim_n\,[\sum_{|\alpha|=|\beta|=m}(a_{\alpha\beta}D^\beta(u_n-u), D^\alpha u_n-D^\alpha u)+\sum_{|\alpha|=m}(b_\alpha(x, \eta(u_n), \zeta(u_n))-$$

$$-b_\alpha(x, \eta(u_n), \zeta(u)), D^\alpha u_n-D^\alpha u)]=0.$$

Since, in view of (H1) and (H3b), the expression in the brackets is $\geq(\mu_0-\mu)\sum_{|\alpha|=m}|D^\alpha u-D^\alpha u|^2$, it follows that $D^\alpha u_n\to D^\alpha u$ in L^2 for $|\alpha|=m$, i.e. $T=L+N$ is of type (S) and so A-proper. Q.E.D.

Remark 3.1. Note that (H3b) holds if, for example, $b_\alpha(x, \eta, \zeta)$ is Lipschitzian in ζ for $|\alpha|=m$ with small Lipschitz constant. Indeed, for a given n and m, let $c_{nm}\geq 1$ be an integer such that

$$(\sum_{|\alpha|=m}|\zeta_\alpha|)^2\leq c_{nm}(\sum_{|\alpha|=m}|\zeta_\alpha|^2) \tag{3.17}$$

and suppose that the following condition holds:

(H5) There is $k>0$ such that for $x\in Q$(a.e.) and $\eta\in R^{S_{m-1}}$

$$|b_\alpha(x, \eta, \zeta)-b_\alpha(x, \eta, \zeta')|\leq k(\sum_{|\beta|=m}|\zeta_\beta-\zeta_\beta'|^2)^{1/2}\,\forall\zeta, \zeta'\in R^{S_m}. \tag{3.18}$$

If $\mu_0-kc_{nm}^{1/2}>0$, where μ_0 appears in (H1), then (H3b) holds with $\mu=kc_{nm}^{1/2}$. Note that one can easily determine c_{nm} for given n and m. Thus, for example, if $n=3$ and $m=1$ then $c_{31}=3$ and if $n=3$ and $m=2$, then $c_{32}=6$.

Remark 3.2. The condition that $c(w)>0$ for all $w\in B_1^L$ certainly holds if the following assumption used in [7] holds:

$$\text{(H6)}\qquad\int_{Q_0(D^\alpha w)}h_\alpha(x, \xi(w)/|\xi(w)|)|\xi(w)|^\sigma D^\alpha w(x)\,dx\geq 0\quad\text{for } w\in B_1^L \text{ and } |\alpha|\leq m$$

and for at least one α the integral is strictly greater than 0.

Assuming the more restrictive conditions (H5) and (H6) and using the coincidence degree theory for set-contractions, it was shown by Hetzer [7] that (A1) and (A2) of Theorem 3.1 hold provided

(H7) *The constant* k *in* (*H5*) *is such that* $k<\inf\{|\lambda|:\lambda\in\sigma_e(L)\}$, *where* $\sigma_e(L)$ *denotes the essential spectrum of the self adjoint operator* L.

Let us now show that Theorem 3.1. includes as a special case, the following slight generalization of [7, Theorem 3].

Theorem 3.4. *Suppose* (*H*1), (*H*2), (*H*4), (*H*5), (*H*6) *and* (*H*7) *hold. Then the assertions* (*A*1), (*A*2) *and* (*A*3) *of Theorem* 3.1 *hold.*

Proof. In view of the preceding discussion, to deduce Theorem 3.4 from Theorem 3.1, it suffices to show that L and $L+N$ are A-proper w.r.t. a suitable projectionally complete scheme for (V, V) when instead of (H3.) one assumes (H5) and (H7).

To prove the latter, we first note that the arguments as those used in [7] show that, in view of (H5), N is k-ball-contractive and, as was shown in [7], $e(L)=\inf\{|\lambda|: \lambda\in\sigma_e(L)\}$, where $eL)=\sup\{r: r\in R^+, r\chi(M)\leq\chi(L(M))$ for each bounded $M\subset V\}$; further, $e(L)>0$ since L is Fredholm (see also [22]).

Now, it was shown by the author in [1] that if $\{X_n, P_n\}$ is projectionally complete for (V, V), $L\in L(V, V)$ is Fredholm with ind $(L)=0$, $\tilde{Y}_n=(L+C)(X_n)$ for each $n\in N$, Q_n is the orthogonal projection of V onto $\tilde{Y}_n$, where $C\in L(V, V)$ is any compact map such that $L+C$ is bijective, then L is A-proper w.r.t. $\Gamma_L=\{X_n, P_n; \tilde{Y}_n, Q_n\}$. We add that at least one such C exists since ind $(L)=0$. We claim now that, since N is k-ball-contractive with, $k<c(L)$ the map $L+N$ is A-proper w.r.t. Γ_L. Indeed, let $\{x_{n_j}|x_{n_j}\in X_{n_j}\}$ be any bounded sequence such that $g_{n_j}\equiv Q_{n_j}(L+N)(x_{n_j})\to g$ for some g in V. Since $Q_{n_j}(L+C)(x_{n_j})=g_{n_j}-Q_{n_j}N(x_{n_j})+Q_{n_j}Cx_{n_j}$ and $Q_{n_j}(L+C)(x_{n_j})=(L+C)$ (x_{n_j}) for each $j\in N$, it follows that $Lx_{n_j}=-Cx_{n_j}+g_{n_j}-Q_{n_j}N(x_{n_j})+Q_{n_j}Cx_{n_j}$ for each j. The last equality, the precompactness of the sets $\{Cx_{n_j}|j\geqslant 1\}$, $\{g_{n_j}|j\geqslant 1\}$ and $\{Q_{n_j}Cx_{n_j}|j\geqslant 1\}$ and the fact that $X(\{Q_{n_j}Nx_{n_j}\})\leqslant X(\{Nx_{n_j}\})\leqslant$ $\leqslant kX(\{x_{n_j}\})$ imply that $X(\{Lx_{n_j}\})\leqslant kX(\{x_{n_j}\})$. This and the assumption $k<c(L)$ imply that $X(\{x_{n_j}\})=0$ and so $\{xn_{n_j}\}$ has a convergent subsequence. It now follows from the continuity of L and N and the projectional completeness of Γ_L that $L+N$ is A-proper w.r.t. Γ_L. Q.E.D.

Remark 3.3. For the case when $\sigma=0$ in (H2), $\zeta=\zeta_\alpha$ and $\zeta'=\zeta'_\alpha$ and $\mu=0$ in (H3b), and $h_\alpha(p,\zeta)\equiv k_\alpha(x)$ in (H4), the assertion (A3) of Theorem 3.3 is related to [10, Theorem 2] which is proved in [10] under the limit conditions on $b_\alpha(x,\eta,\zeta)$ that are somewhat different than those used in (H4). At any rate, as was noted in Remark 1.2, Theorem 2 in [10] can be deduced from Theorem 1.3 since in this case (D) holds.

Remark 3.4. If $V=\mathring{W}_2^m(Q)$ and $\sigma=0$, the assertion $(A3)$ of Theorem 3.1 provides a generalization of Landesman-Lazer result (e.g. [2, 11]) and others (see [7, 9, 8] for further references) to the case when N is neither compact nor set-contractive.

Remark 3.5. If b_α depends only on x and the α-th derivative $D^\alpha u$ (we write $b_\alpha(x,D^\alpha u)$, the hypothesis (H4) reduces to:

(H4)′ There exist functions $h_\alpha^+(x), h_\alpha^-(x)\in L^{2/1-\sigma}$ such that $\lim\limits_{\xi_\alpha\to\pm\infty} b_\alpha(x,\xi_\alpha)/|\xi_\alpha|^\sigma=h_\alpha^\pm(x)$ for $x\in Q$ (a.e.) and all $|\alpha|\leqslant m$. In this case the functional e: $B_1\to R$ is given by

$$e(w)=\sum_{|\alpha|<m}\left[\int\limits_{Q_+(D^\alpha w)} h_\alpha^+(x)|Dw(x)|^{1+\sigma}dx-\int\limits_{Q_-(D^\alpha w)} h_\alpha^-(x)|D^\alpha w(x)|^{1+\sigma}dx\right],$$

where $\underset{(\pm)}{Q}(D^\alpha w)\equiv\{x\in Q: D^\alpha w(x)\underset{(<)}{>}0\}$ for all $|\alpha|\leqslant m$.

It should be added that in this case condition (H3b) takes the form

$$\text{(H3b)}'\qquad\begin{cases}\sum\limits_{|\alpha|=m}(b_\alpha(x,\zeta_\alpha)-b_\alpha(x,\zeta'_\alpha))(\zeta_\alpha-\zeta'_\alpha)\geqslant-\mu\sum\limits_{|\alpha|=m}|\zeta_\alpha-\zeta'_\alpha|^2 \text{ for}\\ x\in Q \text{ and } \zeta_\alpha,\zeta'_\alpha\in R^{S'_m} \text{ and some } \mu\in[0,{}_0).\end{cases}$$

It is clearly implied by the assumption

$$\text{(H5)}'\qquad |b_\alpha(x,\zeta_\alpha)-(b(x,\zeta'_\alpha)|\leqslant\mu|\zeta_\alpha-\zeta'_\alpha|\forall\zeta_\alpha,\zeta'_\alpha\in R^{S'_m}$$

imposed in [7]. The condition $e(w)>0$, is certainly implied by the assumption (H6)′ in [7].

Note also that if $\sigma=0$ in (H4)′ (and in (H2) with $b_\alpha=(x,\xi)=b(x,\xi_\alpha)$) and if (H3b)′ holds, then it follows from (A3) of Theorem 3.3 that Equation (3.1) has in this case a solution in V for a given f in L^2 provided

$$e(w)\equiv\sum_{|\alpha|<m}\left[\int\limits_{Q_+(D^\alpha w)} h_\alpha^+(x)\,D^\alpha w(x)\,dx+\int\limits_{Q_-(D^\alpha w)} h_\alpha^-(x)\,D^\alpha w(x)\,dx\right]>(f,w)$$

holds for all $w\in B_1^L$. This result extends [6, Theorems 3.2 and 3.4] (when $p=m-1$) to the case of Equation (3.1) when $p=m$ (see also [7, Remark 3]).

Remark 3.6. Finally let us observe that if $p<m$ in (3.1), $N: V\to V$ determined by the nonlinear part in (3.1) is compact and so $L+N$ is obviously A-proper. Consequently, in this case Theorem 3.1 is valid if (H1), (H2) and (H4) hold.

We conclude this section with the observation that (H2), (H4) and (H6) appear in [23].

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RESULTS AND PROBLEMS IN THE METRICAL FIXED POINT THEORY *

BY

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§1. **Statement of the problem.** In the last ten years, many papers have appeared which establish various fixed point theorems for self-mappings of a metric space. For example, only in the paper of Rhoades [19] are given 125 such theorems. A great part of these results are generalizations of the well known fixed point theorem of Banach (1922):

Let (X, d) be a complete metric space, and let $f : X \to X$ be a mapping with the property that there is $\alpha \in]0,1[$ such that

$$d(f(x), f(y)) \leqslant \alpha d(x, y), \quad \forall x, y \in X. \tag{1}$$

Then

(i) $$F_f = \{x^*\}$$

(ii) $$x^* = \lim_{n \to \infty} f^n(x_0), \quad \forall x_0 \in X$$

The generalizations mentioned above refer either to the substitution of the contraction condition (1) by some other conditions, or considering various generalized metric spaces instead of (X, d).

Further, we present some of the conditions that appear instead of condition (1):

(i) (*Edelstein* (1962)). For each $x, y, \in X$, $x \neq y$

$$d(f(x), f(y)) < d(x, y).$$

(ii) (*Guseman* (1970)). There exists a number $\alpha \in]0,1[$, such that for each $x \in X$, there exists an integer $p(x)$, such that for each $y \in X$

$$d((f(x)^{p(x)}, (f(y)^{p(x)}) \leqslant \alpha d(x, y).$$

(iii) (*Yen*(1972)). There exist positive integers p, q and a number $\alpha \in]0,1[$ such that

$$d(f^p(x), f^q(y)) \leqslant \alpha d(x, y), \quad \forall x, y \in X.$$

(iv) (*Rakotch* (1962); *Boyd and Wong* (1969); *Browder* (1968). There exists $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\varphi(t) < t$ for $t > 0$, φ is increasing, continuous and

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