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RESULTS AND PROBLEMS IN THE METRICAL FIXED POINT THEORY*

BY

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§1. **Statement of the problem.** In the last ten years, many papers have appeared which establish various fixed point theorems for self-mappings of a metric space. For example, only in the paper of Rhoades [19] are given 125 such theorems. A great part of these results are generalizations of the well known fixed point theorem of Banach (1922):

Let (X, d) be a complete metric space, and let $f: X \to X$ be a mapping with the property that there is $\alpha \in]0,1[$ such that

$$d(f(x), f(y)) \leqslant \alpha d(x, y), \quad \forall x, y \in X. \tag{1}$$

Then

(i) $$F_f = \{x^*\}$$

(ii) $$x^* = \lim_{n \to \infty} f^n(x_0), \quad \forall x_0 \in X$$

The generalizations mentioned above refer either to the substitution of the contraction condition (1) by some other conditions, or considering various generalized metric spaces instead of (X, d).

Further, we present some of the conditions that appear instead of condition (1):

(i) (*Edelstein* (1962)). For each $x, y, \in X$, $x \neq y$

$$d(f(x), f(y)) < d(x, y),$$

(ii) (*Guseman* (1970)). There exists a number $\alpha \in]0,1[$, such that for each $x \in X$, there exists an integer $p(x)$, such that for each $y \in X$

$$d((f(x)^{p(x)}, (f(y)^{p(x)}) \leqslant \alpha d(x, y).$$

(iii) (*Yen*(1972)). There exist positive integers p, q and a number $\alpha \in]0,1[$ such that

$$d(f^p(x), f^q(y)) \leqslant \alpha d(x, y), \quad \forall x, y \in X.$$

(iv) (*Rakotch* (1962); *Boyd and Wong* (1969); *Browder* (1968). There exists $\varphi: \mathbf{R}_+ \to \mathbf{R}_+$ such that $\varphi(t) < t$ for $t > 0$, φ is increasing, continuous and

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$$d(f(x),\ f(y)) \leqslant \varphi(d(x,\ y)),\quad \forall x,\ y,\ \in X.$$

(v) (*Kannan* (1968)). There exists a number $\alpha \in]0,\ 1/2[$ such that

$$d(f(x),f(y)) \leqslant \alpha[d(x,f(x)),f(x)) + d(y,f(y))],\quad \forall x,\ y \in X,$$

(vi) (*Reich* (1971); *Rus* (1971)). There exist $\alpha,\ \beta \in R_+,\ \alpha+2\beta<1$, such that

$$d(f(x),f(y)) \leqslant \alpha d(x,\ y) + \beta[d(x,f(x)) + d(y,f(y))],\ \forall x,\ y \in X,$$

(vii) (*Ćirić* (1971)). There exist $\alpha,\ \beta,\ \gamma \in R_+,\ \alpha+2\beta+2\gamma<1$, such that

$$d(f(x),f(y)) \leqslant \alpha d(x,\ y) + \beta[d(x,f(x)) + d(y,f(y))] + \gamma[d(x,f(y)) + d(y,f(x))],\quad \forall x,\ y \in X,$$

(viii) (*Bianchini* (1972)). There exists $\alpha \in]0,1[$, such that

$$d(f(x),f(y)) \leqslant \alpha \max\ \{d(x,f(x)),d(y,f(y))\},\quad \forall x,\ y \in X,$$

(ix) (*Ćirić* (1974)). There exists $\alpha \in]0,1[$ such that

$$d(f(x),f(y)) \leqslant \alpha\ \max\ \{d(x,\ y),d(x,f(x)),d(y,f(y)),\ d(x,f(y)),d(y,f(x))\},\quad \forall x,\ y \in X,$$

For a better understanding of this accumulation of results, we proposed ourselves to analyse the position of the conditions from the existence and unicity metrical fixed point theorems, in relation with the following problems.

Problem 1. (see [10]) Let $f:(X,d) \to (X,d)$ be a mapping. A point $x \in X$ is a periodic point of f iff there is an integer $n \in N$ such that $f^n(x)=x$. What are the conditions on f such that every periodic point of f is a fixed point?

Problem 2. (see [21]) Let $f:(X,d) \to (X,d)$ be a mapping that satisfies the conditions of some existence and unicity fixed point theorem, and that the fixed point can be obtained by the succesive approximations method ($F_f=\{x^*\}$, $x^*=\lim\limits_{n\to\infty} f^n(x_0)$, $x_0 \in X$). Let $g:X \to X$ be a mapping which approximates the mapping f. More precisely, we assume that

$$d(f(x),g(x)) \leqslant \eta, \text{ with given } \eta \in R,\ \forall x \in X$$

Let $y_n=g^n(x_0)$. The problem is to give an estimate of $d(x^*,\ y_n)$.

Problem 3. (see [6], [7], [9], [14], [15]). Let $f,\ f_n:(X,d) \to (X,d)$ be such that $F_{f_n} \neq \varnothing$ and $F_f=\{x^*\}$. If $f_n \rightrightarrows f$, and $x_n \in F_{f_n}$, which are the conditions to be satisfied by f so that $x_n \to x^*$?

Problem 4. Let $(X,\|\cdot\|)$ be a Banach space and $f:X \to X$ a mapping. What are the conditions on f such that l_x-f is surjective?

Problem 5. ([14], [20], [23]). Let (X,d) be a metric space, Y a topological space and $f:X \times Y \to Y$. Let us assume that the mapping $f(\cdot,\ y)$ has a unique fixed point, x_y^*, for every $y \in Y$. We define the mapping

$$P: Y \to X,\ y \to x_y^*.$$

What are the conditions for the continuity of the mapping P?

For example, analysing in this perspective Banach's theorem we obtain the following.

Theorem 1. *In the conditions of Banach's fixed point theorem we have*

(i) $$F_f=F_{f^n}=\{x^*\},\quad \forall n \in N,$$

(ii) $$x^*=\lim_{n\to\infty} f^n(x_0),\ \forall x_0 \in X,\ \text{and}$$

$$d(x^*,\ x_n) \leqslant \frac{\alpha^n}{1-\alpha}\ d(x_0,\ x_1).$$

(iii) *Let $g:X \to X$ be a mapping which approximates the mapping f. If $d(f(x),\ g(x)) \leqslant \eta,\ \forall x \in X$ and $y_n=g^n(x_0)$, then*

$$d(y_n,\ x^*) \leqslant \frac{\eta}{1-\alpha} + \frac{\alpha^n}{1-\alpha}\ d(x_0,\ x_1),$$

(iv) *If $f_n \rightrightarrows f,\ F_f \neq \varnothing,\ x_n \in F_{f_n}$ then $x_n \to x^*$.*

(v) *If $(X,\|\cdot\|)$ is a Banach space and $d(x,\ y)=\|x-y\|$,* then l_X-f *is surjective.*

Theorem 2. *Let $(X,\ d)$ be a complete metric space, $(Y,\ \tau)$ a topological space and let $f:X \times Y \to X$ be a continuous mapping. If there exists $\alpha \in]0,1[$, such that*

$$d(f(x_1,\ y),\ f(x_2,\ y)) \leqslant \alpha d(x_1,\ x_2),\quad \forall x_1,\ x_2 \in X,\quad y \in Y,$$

then the mapping P is continuous.

Therefore, in the conditions of Banach's theorem we have a positive answer to all the five problems formulated above. The problem which follows to be analysed is

Problem 6. Which of the existence and unicity metrical theorems have the qualities of Banach's theorem?

§2. φ-contractions. Let $(X,\ d)$ be a metric space and $\varphi: R_+ \to R_+$. In this § we will analyse fixed point theorems, where the contraction condition is replaced by the following one

$$d(f(x),f(y)) \leqslant \varphi(d(x,\ y)),\quad \forall x,\ y \in X \tag{2}$$

A mapping that satisfies the condition (2) will be named a φ-contraction. From the fixed point theorems for φ-contraction we mention

Theorem 3. (C.S. Wong [25]). *Let (X,d) be a complete metric space and $f:X \to X$ a mapping. Then the following conditions are equivalent:*

(i) *There exists $\varphi:\mathbf{R}_+ \to \mathbf{R}_+$ such that $\varphi(t)<t$ for $t>0$, φ is increasing, continuous and f is φ-contractive.*

(ii) *There exists $\varphi:\mathbf{R}_+ \to \mathbf{R}_+$ such that $\varphi(t)<t$ for $t>0$, φ is upper semi-continuous on $]0,\ +\infty[$ and f is φ-contractive.*

(iii) *There exists a mapping W: $\mathbf{R}_+ \to \mathbf{R}_+$ such that $W(s)>s$ for $s>0$, W is increasing, continuous and*

$$W(d(f(x),f(y)) \leqslant d(x,\ y),\ \forall x,\ y \in X$$

(iv) *There exists a mapping* $W : \mathbf{R}_+ \to \mathbf{R}_+$ *such that* $W(s) > s$ *for* $s > 0$, W *is lower semicontinuous and*

$$w(d(f(x), f(y)) \leqslant d(x, y), \quad \forall x, y \in X.$$

If any one of (i), (ii), (iii) or (iv) is satisfied, then

(a) $$F_f = \{x^*\},$$

(b) $(f^n(x))_{n \in N}$, $x \in M$, converges uniformly to x^* for any baunded subset $M \subset X$.

In the perspective of Problem 6 we have

Theorem 4. *Let* (X, d) *be a complet metric space,* $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ *and* $f : X \to X$ *such that*

(a) φ *is increasing,*

(b) φ *is continuous,*

(c) $r > \varphi(r)$, $\forall r > 0$,

(d) $r - \varphi(r) \to \infty$ *for* $r \to \infty$,

(e) f *is* φ*-contractive.*

Then

(i) $$F_f = F_{f^n} = \{x^*\}, \; n \in N,$$

(ii) $$f^n(x_0) \to x^*, \quad \forall x_0 \in X \text{ and},$$

$$d(x_n, x^*) \leqslant \varphi^n(r_{x_0}).$$

where

$$r_{x_0} = \sup \{r \mid r - \varphi(r) \leqslant d(x_0, f(x_0))\}$$

(iii) *If* $\varphi(r_1 + r_2) \leqslant \varphi(r_1) + \varphi(r_2)$ *and*

$$d(g(x), f(x)) \leqslant \eta$$

then

$$d(y_n, x^*) \leqslant \eta + \varphi(\eta) + \cdots + \varphi^{n-1}(\eta) + \varphi^n(r_{x_0}).$$

(iv) *If* $f_n \rightrightarrows f$, $F_{f^n} \neq 0$, $x_n \in F_{f^n}$, *then* $x_n \to x^*$.

(v) *If* $(X, \|\cdot\|)$ *is a Banach space and* $d(x, y) = \|x - y\|$ *then* $1_X - f$ *is surjective.*

Proof. (i) *and* (ii). See Browder [3]

(iii). We have

$$d(y_n, x^*) \leqslant d(y_n, x_n) + d(x_n, x^*).$$

On the other hand

$$d(y_n, x_n) \leqslant d(g(y_n - 1), f(y_{n-1})) + d(f(y_{n-1}), f(x_{n-1})) \leqslant \eta + \varphi(d(y_{n-1}, x_{n-1}) \leqslant$$

$$\leqslant \eta + \varphi(\eta + \varphi(d(y_{n-2}, x_{n-2}))) \leqslant \eta + \varphi(\eta) + \cdots + \varphi^{n-1}(\eta).$$

(iv). We have

$$d(x_n, x^*) = d(f_n(x_n), f(x^*)) \leqslant d(f_n(x_n), f(x_n)) + d(f(x_n), f(x^*) \leqslant$$

$$\leqslant d(f_n(x_n), f(x_n)) + \varphi(d(x_n, x^*)).$$

Hence $\lim_{n \to \infty} [d(x_n, x^*) - \varphi(d(x_n, x^*))] = 0$

and by (a)–(d) $\lim_{n \to \infty} d(x_n, x^*) = 0$.

(v) Let y be an element of X. For $h : X \to X$, $x \to h(x) = f(x) + y$ we have

$$d(h(x), h(s)) \leqslant \varphi(d(x, s)), \; \forall x, s \in X.$$

Theorem 5. *Let* (X, d) *be a complete metric space,* (Y, τ) *a topological space and let* $f : X \times Y \to X$ *be a continuous mapping such that*

$$d(f(x_1, y), f(x_2, y)) \leqslant \varphi(d(x_1, x_2)), \quad \forall x_1, x_2 \in X, \; y \in Y$$

where φ *is as in theorem 4. Then the mapping* P *is continuous.*

Proof. We have

$$d(x_{y_1}, x_{y_2}) = d(f(x^*_{y_1}, y_1), \;\; f(x_{y_2}, y_2)) \leqslant d(f(x^*_{y_1}, y_1), f(x^*_{y_2}, y_1)) + d(f(x_{y_2}, y_1),$$

$$f(x^*_{y_2}, y_2)) \leqslant \varphi(d(x^*_{y_1}, x^*_{y_2})) + d(f(x^*_{y_2}, y_1), f(x^*_{y_2}, y_2)).$$

Hence

$$d(x^*_{y_1}, x^*_{y_2}) - \varphi(d(x^*_{y_1}, x^*_{y_2})) \to 0 \quad \text{when} \quad y_1 \to y_2$$

and

$$d(x^*_{y_1}, x^*_{y_2}) \to 0 \quad \text{when} \quad y_1 \to y_2.$$

§3. **Another type of contraction.** In this § we will analyse the following.

Theorem 6. *Let* (X, d) *be a complet metric space, and* $f : X \to X$ *a mapping for which there exist* $\alpha, \beta \in \mathbf{R}_+$, $\alpha + 2\beta < 1$, *such that*

$$d(f(x), \;\; f(y)) \leqslant \alpha d(x, y) + \beta[d(x, f(x)) + d(y, f(y))], \quad \forall x, y \in X$$

Then f *has a unique fixed point. Moreover, for each* $x_0 \in X$, $(f^n(x_0))_{n \in N}$ *converges to the fixed point of* f.

For this theorem we have

Theorem 7. *In the conditions of the theorem 6, we have*

(i) $$F_f = F_{f^n} = \{x^*\}, \quad \forall n \in \mathbf{N},$$

(ii) $$f^n(x_0) \to x^* \quad \forall x_0 \in X \;\; \text{and}$$

$$d(x_n, x^*) \leqslant \frac{K^n}{1-K} d(x_0, x_1),$$

$$\textit{where} \;\; K = \frac{\alpha + \beta}{1 - \beta}.$$

(iii) *Let* $g : X \to X$ *be a mapping which approximates the mapping* f. *If* $d(f(x), g(x)) \leqslant \eta$, $\forall x \in X$ *and* $y_n = g^n(x_0)$, *then*

$$d(y_n, x^*) \leqslant \alpha \frac{n-1}{\eta} + \alpha^{n-2} F(2) + \cdots + F(n) + \frac{K^n}{1-K} d(x_0, x_1).$$

(iv) *If* $f_n \rightrightarrows f$, $F_f \neq \varnothing$, $x_n \in F_{f_n}$, *then* $x_n \to x^*$.

(v) *In general* $1_X - f$ *is not surjective.*

Proof. (i)+(ii). See Reich [17]

(iii). See Rus [21] (iv) See Rus [20], (v) We take $X = R$, $d(x, y) = |x - y|$

and

$$f:\mathbf{R}\to\mathbf{R},\ x\to\begin{cases} 0 & \text{if} \quad x\leqslant 2 \\ -\dfrac{1}{3} & \text{if} \quad x>2 \end{cases}$$

Theorem 8. (See [14]). *Let (X, d) be a complet metric space, (Y, τ) a topological space and let $f: X\times Y\to X$ be a continuous mapping. If there exist $\alpha,\ \beta\in\mathbf{R}_+$, $\alpha+2\beta<1$ auch that*

$$d(f(x_1, y), f(x_2, y))\leqslant \alpha d(x_1, x_2)+\beta[d(x_1, f(x)) + d(x_2, f(x_2))],$$

$$\forall x_1, x_2\in X,\ y\in Y,$$

then the mapping P is continuous.

§4. **The sets with two metrics.** Let X be a nonempty set, d and δ two metrics on X and $f: X\to X$ a mapping. For such mappings, M. G. Maia [13] has given the following.

Theorem 9. *If*

(a) $d(x, y)\leqslant \delta(x, y)$, $\forall x, y\in X$,

(b) *(X, d) is a complete metric space,*

(c) *$f:(X, d)\to(X, d)$ is continuous,*

(d) *there exists $\alpha\in\,]0,1[$ such that*

$$\delta(f(x), f(y))\leqslant\alpha\delta(x, y),\quad \forall x, y\in X,$$

Then f has a unique fixed point.

In the perspective of Problem 6 we have the following

Theorem 10. *If*

(a) *there exists $c>0$, such that*

$$d(f(x), f(y))\leqslant c\delta(x, y),\quad \forall x, y\in X,$$

(b) *(X, d) is a complete metric space,*

(c) *$f:(X, d)\to(X, d)$ is continuous,*

(d) *there exists $\alpha\in\,]0,1[$ such that*

$$\delta(f(x), f(y))\leqslant\alpha\delta(x, y),\quad \forall x, y\in X,$$

then

(i) $$F_f=F_{f^n}=\{x^*\},\quad \forall n\in N,$$

(ii) $f^n(x_0)\to x^*$, $\forall x_0\in X$ *and* $d(x_n, x^*)\leqslant \dfrac{c\alpha^{n-1}}{1-\alpha}\,\delta(x_0, x_1)$.

(iii) *If there exists $c_1>0$, $c\,c_1<1$ such that $\delta(x, y)\leqslant c_1 d(x, y)$, $\forall x, y\in X$* and *$g: X\to X$ approximates the mapping f, then*

$$d(y_n, x^*)\leqslant\eta(1-cc_1)^{-1}+\frac{c\alpha^{n-1}}{1-\alpha}\,\delta(x_0, x_1).$$

(iv) *Let $f_n\overset{X,d}{\rightrightarrows}f$, $F_f\neq\varnothing$, $x_n\in F_{f^n}$. If there exists $c_1>0$, $cc_1<1$ such that $\delta(x, y)\leqslant c_1 d(x, y)$, $\forall x, y\in X$ then $x_n\to x^*$*

(v) *If $(X, \|\cdot\|_1, \|\cdot\|_2)$ is a linear space with two norms and $d(x, y)=\|x-y\|_1$, $\|\delta(x, y)=\|x-y\|_2$, then 1_X-f is surjective.*

Theorem 11. *Let X be a nonempty set, d and δ two metrics on X, and le Y be a topological space and $f: X\times Y\to X$ such that*

(a) *the mapping $f(\cdot, y)$ satisfies the hypotheses of theorem 10, where α and c are not depending of y*

(b) *the mapping $f(x,.)$ is continuous in (X, d)*

(c) *there exists $c_1>0$, $c_1 c\alpha<1$, such that $\delta(x, y)\leqslant c_1 d(x, y)$, $\forall x, y\in X$*

Then the mapping $P: Y\to(X, d)$ is continuous. For the proof of the theorem 10 and theorem 11 see Rus [22].

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STABILITY PROBLEMS FOR SOME VOLTERRA EQUATIONS*)

BY

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We consider some problems that arise in the study of stability properties of the integral equations

$$u(t) = -\int_0^t g(u(t-s))\,K(s)\,ds + f(t), \quad (0 \leq t < \infty), \tag{1}$$

and

$$u'(t) = -\int_0^t g(u(t-s))\,K(s)\,ds + f(t) \quad (u(0) = u_0, \quad 0 \leq t < \infty). \tag{2}$$

We assume the given functions g, K, f are real valued, though some of the methods to be discussed are well known to carry over to certain more general systems (compare [15], [26]). Throughout, we require

$$\begin{aligned} &f(t) \in L^1(0, T) \quad \text{for all} \quad T (0 < T < \infty), \\ &K(t)\,e^{-xt} \in L^1(0, \infty) \quad \text{for all} \quad x > 0. \end{aligned} \tag{3}$$

Define

$$\hat{K}(z) = \int_0^\infty e^{-zt} K(t)\,dt, \quad (\operatorname{Re} z > 0),$$

$$v * K(t) = \int_0^t v(t-s) K(s)\,ds.$$

1. **The linear case** $g(u) = u$. When (1) and (2) are linear, these equations can be inverted and are respectively equivalent to

$$u(t) = f(t) - f * r_1(t), \tag{1'}$$

$$u(t) = u_0 r_2(t) + f * r_2(t), \tag{2'}$$

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