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STABILITY PROBLEMS FOR SOME VOLTERRA EQUATIONS*)

BY

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We consider some problems that arise in the study of stability properties of the integral equations

$$u(t) = -\int_0^t g(u(t-s))\, K(s)\, ds + f(t), \quad (0 \leqslant t < \infty), \tag{1}$$

and

$$u'(t) = -\int_0^t g(u(t-s))\, K(s)\, ds + f(t) \quad (u(0) = u_0, \quad 0 \leqslant t < \infty). \tag{2}$$

We assume the given functions g, K, f are real valued, though some of the methods to be discussed are well known to carry over to certain more general systems (compare [15], [26]). Throughout, we require

$$\begin{gathered} f(t) \in L^1(0, T) \quad \textit{for all} \quad T (0 < T < \infty), \\ K(t)\, e^{-xt} \in L^1(0,\infty) \quad \textit{for all} \quad x > 0. \end{gathered} \tag{3}$$

Define

$$\hat{K}(z) = \int_0^\infty e^{-zt} K(t)\, dt, \quad (\operatorname{Re} z > 0),$$

$$v * K(t) = \int_0^t v(t-s) K(s)\, ds.$$

1. **The linear case** $g(u) = u$. When (1) and (2) are linear, these equations can be inverted and are respectively equivalent to

$$u(t) = f(t) - f * r_1(t), \tag{1'}$$

$$u(t) = u_0 r_2(t) + f * r_2(t), \tag{2'}$$

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where the resolvent kernels r_1, r_2 are defined by

$$\hat{r}_1(z)=\frac{\hat{K}(z)}{1+\hat{K}(z)},\quad \hat{r}_2(z)=\frac{1}{z+\hat{K}(z)}\quad (\operatorname{Re} z>0).$$

We seek conditions on K that imply $r_j\in L^1(0,\infty)$, $j=1$ or 2.

If $K\in L^1(0,\infty)$, this is a Banach algebra problem and the necessary and sufficient condition for $r_1\in L^1$ follows from the classical Wiener-Lévy theorem:

Let $\varphi(w)$ be analytic on $\{\hat{K}(z):\ Re\ z\geq 0\}\cup\{0\}$ with $\varphi(0)=0$. Then there exists $r\in L^1(0,\infty)$ such that

$$r(z)=\varphi(\hat{K}(z))\quad (\operatorname{Re} z\geq 0).$$

Thus, the choice $\varphi(w)=w/(1+w)$ yields

$$r_1\in L^1(0,\infty) \text{ iff } \hat{K}(z)\neq -1\quad (\operatorname{Re} z\geq 0).$$

The Wiener-Lévy theorem does not apply directly to r_2, but the expected result is true:

$$r_2\in L^1(0,\infty) \text{ iff } \hat{K}(z)\neq -z\quad (\operatorname{Re} z\geq 0);$$

for this see Grossman and Miller [5], [22, Theorem 2], or Theorem 1 below.

In some applications using (1) or (2), it is not realistic to assume $K\in L^1(0,\infty)$. For example, equation (2) is used in [13] with $g(u)=u$ and

$$K(t)=\int_0^\infty e^{-x^2 t}h(x)\,dx,\quad (h\in L^1(0,\infty),\ h\geq 0),$$

so that if $h(0+)>0$, $K(t)\sim Ct^{-1/2}$ $(t\to+\infty)$. Nevertheless, using the fact that these kernels are completely monotonic: $(-1)^n(K^n)(t)\geq 0$ for all $n\geq 0$, Levin and Nohel proved that $r_2(t)=O(t^{-3/2})$ at $+\infty$, thus again $r_2\in L^1(0,\infty)$.

A generalization of the Wiener-Lévy theorem that yields $r_j\in L^1$ for certain $K\notin L^1$ is given by

Theorem 1. *Let $K(t)=C(t)+k(t)$ where $k\in L^1(0,\infty)$ and C is nonnegative, nonincreasing and convex. Define*

$$\hat{K}(iy)=\lim_{x\to 0+}\hat{K}(x+iy),$$

$$S=\{(\hat{K}(z),z):\operatorname{Re} z\geq 0\}$$

and assume $\varphi(w,z)$ is analytic on $S\cup\{(0,\infty)\}$, and that $\varphi(0,\infty)=0$. Then there exists $r\in L^1(0,\infty)$ such that

$$\varphi(\hat{K}(z),z)=\hat{r}(z)\quad (\operatorname{Re} z\geq 0). \tag{4}$$

For proof and further discussion, see [22] where however an additional technical condition:

$$tk(t)\in L^1(0,\infty),\text{ or else } C(t)\equiv\text{constant} \tag{5}$$

is imposed on K; (5) is eliminated by Jordan and Wheeler in [9].

We apply Theorem 1 to the linear case of equation (1) or (2) by choosing $\varphi=w/(1+w)$ or $\varphi=1/(z+w)$.

The convexity hypothesis in Theorem 1 cannot be dropped; this follows from examples of Ibragimov [8] and Gripenberg [4]. For further discussion on this point, see Section III below.

Recently Hannsgen has used a variant of the method of [22] to obtain results for certain linear Hilbert space valued equations, by studying the linear case of (2) with $K=K(t,\lambda)=\lambda C(t)$, C as in Theorem 1 and λ a positive parameter; see e.g. [7], where he finds necessary and sufficient conditions for the associated resolvent $r_2=r_2(t,\lambda)$ to be uniformly integrable; the key condition here involves

$$\operatorname{Re}\hat{C}(iy)\geq\frac{\varepsilon}{1+y^2}\ (-\infty<y<\infty) \tag{6}$$

for some $\varepsilon>0$. Condition (6) was introduced by Halanay [6] as a sufficient condition for asymptotic stability of equation (2).

II. Nonlinear $g(u)$. Equations (1) and (2) can be studied in the „near-linear" case $g(u)=u+o(u)$, $(u\to 0)$, see [17] [18] [16] [12]; here the perturbation theory rests on knowing $r_j(t)\in L^1(0,\infty)$ for $j=1$ or 2, and for this Theorem 1 can be applied.

For global problems involving strictly nonlinear g, (1) and (2) have been treated using energy or frequency-domain methods; further references to the large literature here can be found in [10] [19] [25] [14]. We wish to stress some recent work which depends on the frequency-domain methods introduced by Popov, Corduneanu and Halanay. For (2), the key hypothesis on K is that it be „dissipative", in the sense

$$\operatorname{Re}\hat{K}(z)\geq 0\quad (\operatorname{Re} z>0). \tag{7}$$

This condition, used for (2) in Halanay's fundamental paper [6], yields global existence and boundedness of solutions on $0\leq t<\infty$. The usefulness of (7) stems from the fact that it is equivalent to

$$\int_0^T v(t)\,K*v(t)\,dt\geq 0,\quad (0<T<\infty,\ v\in C[0,\infty)), \tag{7'}$$

a condition exploited by Halanay in [6].

Neither (7) or (7′) is easy to check, in general. A simpler condition can be stated in terms of the boundary values of $\hat{K}$: Let

$$u^*(iy)=\limsup_{x\to 0+}\operatorname{Re}\hat{K}(x+iy),\quad u_*(iy)=\liminf_{x\to 0+}\operatorname{Re}\hat{K}(x+iy),$$

$$m(x)=\inf_{-\infty<y<\infty}\operatorname{Re}\hat{K}(x+iy),\quad \lambda=\liminf_{x\to 0+}x^2 m(x).$$

Then (7), (7′) are equivalent to

$$(7'') \qquad u^*(iy) \geqslant 0 \quad \text{a.e.}, \ u_*(iy) > -\infty \ (-\infty < y < \infty), \ \lambda \geqslant 0.$$

That (7″) implies (7) follows from recent work of D a h l b e r g [3]; compare also [19, p. 280] [25, p. 230].

Simple examples show that [7] is not sufficient for asymptotic stability, what is needed is a sharper frequency-domain condition that forbids $\operatorname{Re} \hat{K}(iy) = 0$. A simple but elegant refinement by S t a f f a n s [23] of H a l a n a y 's method in [6] yields a precise condition for asymptotic stability: K satisfies (7″) and

$$(8) \qquad u_*(iy) > 0 \quad (-\infty < y < \infty).$$

(Here we have weakened [25, p. 231] by using Theorem 2 in [3].) If we combine the results of [6] [25] [19], we obtain a simple, fairly sharp, stability result for [2]:

Theorem 2. *Assume* (3), (7″). $f \in L^1(0, \infty)$ *and* $g \in C(-\infty, \infty)$.

(i) *If*

$$(9) \qquad \lim_{u \to \pm\infty} \int_0^u g = +\infty, \qquad g(u) = O\left(\int_0^u g\right) \ (u \to \pm\infty),$$

then (2) *has at least one solution* $u(t)$ *on* $0 \leqslant t < \infty$, *and all solutions of* (2) *are bounded.*

(ii) *Assume* K *also satisfies* (8) *and* $K \in BV\,[1, \infty)$, *and let* $u(t)$ *be a bounded solution of* (2). *Then*

$$\lim_{t \to \infty} g(u(t)) = 0,$$

$$\lim_{t \to \infty} [u'(t) - f(t)] = 0.$$

For some interesting variants, see L e v i n [10], L o n d e n [14], S t a f f a n s [25].

An analysis for (2) with $f(t)$ replaced by $f(t, u(t))$ is given in [20]; the proof involves a study of the energy functional

$$E(u(t)) = \int_0^{u(t)} g(x)\,dx + \frac{1}{2}\int_0^t \int_0^t g(u(s))g\,(u(\xi))K(|s-\xi|)\,d\xi\,ds.$$

When $g(u) = 0$ iff $u = 0$ and $g'(0) > 0$, we can combine Theorems 1 and 2 to obtain estimates on the rate of decay of $u(t)$ in terms of that of $K(t)$ and $f(t)$, thus extending a result of M a c C a m y and W o n g [16].

Equation (1) has also been studied extensively by frequency-domain methods, see for example P o p o v [21], C o r d u n e a n u [2], L o n d e n [14], S t a f f a n s [24], and [19].

III. Kernels $K(t)$ for which Theorem 1 fails. Theorem 1 would have some interesting new consequences, to renewal theory for example, if the convexity hypothesis on $C(t)$ could be dropped. However the analysis in

[22] depends crucially on this convexity, and indeed the theorem fails for certain positive decreasing $K(t) = C(t)$. Examples constructed by I b r a g i m o v [8] for a similar problem can be readily adapted to show this, and we sketch the argument below. Ibragimov constructs a nonnegative sequence $\{g_j\}$ such that $\Sigma g_j < \infty$,

$$(10) \qquad K_1(t) = \sum_{j \geqslant t} g_j \notin L^1(0, \infty), \ \ \hat{K}_1(z) \neq -1 \quad (\operatorname{Re} z \geqslant 0),$$

$$(11) \qquad \hat{K}_1(iy_p) \to 0 \qquad \text{for some } \ y_p \equiv \downarrow 0.$$

Thus $\hat{r}_1(z) = \hat{K}_1(z)/(1 + \hat{K}_1(z))$ satisfies

$$\hat{r}_1(0) = \lim_{x \to 0+} \hat{K}_1(x)/(1 + \hat{K}_1(x)) = 1, \ \lim_{p \to \infty} \hat{r}_1(iy_p) = 0,$$

so that $r_1 \notin L^1(0, \infty)$.

Further, $\hat{r}_2(z) = (z + \hat{K}_1(z))^{-1}$ satisfies $\hat{r}_2(0) = 0$ and $\hat{r}_2(iy)_p) \to \infty\,(p \to \infty)$, so that also $r_2 \notin L^1$.

This example can be modified to yield a kernel K_2 satisfying (10) and

$$(12) \qquad \hat{K}_2(iy_p) \to -1 \quad (y_p \downarrow 0),$$

thus again $\hat{K}_2/(1 + \hat{K}_2) \notin \hat{L}^1$

Independently, G r i p e n b e r g [4] has found an example satisfying (10) and (12)..

A positive result, involving improper Riemann integrability of r_1, for positive decreasing $K \notin L^1$ is given by L e v i n in [11].

One can ask for conditions implying (4) for interesting classes of K other than one given in Theorem 1, and the examples K_1, K_2 above suggest a study of kernels satisfying (8). However, the method of [22, p. 338] shows that $K_0(t) = \cos\,(t^\beta)/(1+t)$ for $\beta \in (0, 1)$ satisfies $\hat{K}_0/(1 + \hat{K}_0) \notin L^1$, although $\operatorname{Re} \hat{K}_0(z) > 0$ for $\operatorname{Re} z \geqslant 0$.

We sketch the construction of kernels satisfying (10)—(12). Following [8], let

$$N_1 = 2^{10}, \ \log N_p = 8\left(\nu + \sum_{j=1}^{p-1} N_j\right)^3 \ (p \geqslant 2\,;\ \nu = 1 \text{ or } 2)\,;$$

here and below logarithms are taken with base 2. For each $p \geqslant 1$, chose T_p to be the number nearest to $(\log N_p)^{2/3}/\log e$ such that N_p/T_p is an integer. Then consider the disjoint sets of integers

$$\Gamma_p = \{j = 4kN_p - 2N_p/T_p : 1 \leqslant k \leqslant N_p\} \ (p \geqslant 1),$$

and define $g_1 = 1$; $g_j = 1/j$ if $j \in \Gamma_p$ for some $p \geqslant 1$, $g_j = 0$ otherwise. Then for $\nu = 1$ or 2,

$$K_\nu(t) = \sum_{j \geqslant t} g\,, \ \ y_p = \pi/N_p$$

satisfies (10)—(12).

To see this, observe that

$$\hat{K}_\nu(iy)=\frac{1}{y}\sum_{j=1}^{\infty} g_j \sin jy-\frac{i}{y}\sum_{j=1}^{\infty} g_j(1-\cos jy) \quad (y\neq 0).$$

If we write

$$\sum_{j=1}^{\infty} g_j \sin jy_p=\Sigma_1+\Sigma_2+\Sigma_3,$$

with

$$\Sigma_1=g_1 \sin y_p+\sum_{m=1}^{p-1}\left(\sum_{j\in\Gamma_m} g_j \sin jy_p\right),$$

$$\Sigma_2=\sum_{j\in\Gamma_p} g_j \sin jy_p, \quad \Sigma_3=\sum_{m=p+1}^{\infty}\left(\sum_{j\in\Gamma_m}\right).$$

then the estimates of [8, p. 175] yield

$$\Sigma_1=y_p\left(1+\sum_{m=1}^{p-1} N_m\right)+o(y_p), \quad \Sigma_3=o(y_p),$$

$$\Sigma_2=-y_p\left(\frac{\log N_p}{2(\log e)T_p}\right)+o(y_p)=-y_p)-y_p\left(\nu+\sum_{m=1}^{p-1} N_m\right)+o(y_p).$$

Thus

$$\Sigma_1+\Sigma_2+\Sigma_3=-y_p(\nu-1)+o(y_p) \quad (p\to\infty\ ;\ \nu=1 \text{ or } 2),$$

and similar, simpler, estimates yield

$$\sum_{j=1}^{\infty} g_j(1-\cos jy_p)=\Sigma_1+\Sigma_2+\Sigma_3=o(y_p)$$

so that

$$\hat{K}_\nu(iy_p)=-(\nu-1)+o(1) \quad (p\to\infty\ ;\ \nu=1 \text{ or } 2).$$

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