Analele științifice ale Universității „Al. I. Cuza" Iași
Supliment la tomul XXV, s. I a 1979
Cu ocazia centenarului nașterii Acad. Prof. Al. Myller

APPROXIMATION BY COMPACT OPERATORS AND THE SPACE $H^\infty + C$ (*)

BY

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I wish to describe a theorem obtained jointly with S. Axler, I. D. Berg, and N. Jewell. It concerns approximation to bounded operators by means of compact operators and has an application to function theory.

If X is a (complex) Banach space, then $L(x)$ will denote the space of all (bounded) linear operators on X, and $K(X)$ will denote the subspace of compact operators. We require the following result.

Basic Lemma. *Let X be any of the following Banach spaces: c_0, $l^p(1 \leq \leq p < \infty)$. Let $T \in L(X)$ and let $\{Q_n\} \subset L(X)$ with both $Q_n \to 0$ and $Q_n^* \to 0$ in the strong operator topology. Let $\varepsilon > 0$ be given. Then there exists an index N such that*

$$\| T + Q_N \| < \varepsilon + \max \left\{ \begin{matrix} \|T\| \\ \|T\|_e \end{matrix} \right. + \| Q_N \| .$$

Here $\|T\|_e$ denotes the essential norm of T:

$$\| T\|_e = \inf\{\|T - K\| : K \in K(X)\}.$$

Presumably the lemma is true for other Banach spaces besides those listed in the theorem. A Schauder basis $\{e_n\}$ in a Banach space X is called a *shrinking basis* if the associated coordinate functionals form a basis for the dual space X^*.

Theorem 1. *Let X be a Banach space for which the basic lemma is true, and assume that X has a shrinking basis. Then to each $T \in L(X)$ there corresponds a $K_0 \in K(X)$ for which $\|T - K_0\| = \|T\|_e$.*

This result was already known for the spaces listed in the basic lemma. However our proof has an application to function theory.

Let L^∞ denote the space of essentially bounded measurable function on the unit circle, and let H^∞ denote the subspace with $\hat{f}(n) = 0 (n < 0)$, where $\hat{f}$ denotes the Fourier coefficients. Also, C will denote the space of continuous

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* Lecture presented at the Romanian-American Seminar on Operator Theory and Applications, 20–24 March 1978.

functions (on the circle), which we regard as a subspace of L^∞. Sarason showed that $H^\infty + C$ is a closed subalgera of L^∞. It is contained in every closed subalgebra of L^∞ that contains H^∞ as a proper subset. We obtain the following result, answering a question posed by Sarason.

Theorem 2. *To each $f \in L^\infty$ there corresponds a $g_0 \in H^\infty + C$ such that*

$$\|f - g_0\| = \inf \{\|f - g\| : g \in H^\infty + C\}.$$

The connection between this result and the previous one is made by means of the theory of Hankel matrices. If $f \in L^\infty$ let

$$H_f = \begin{vmatrix} \hat{f}(-1) & \hat{f}(-2) & \hat{f}(-3) \cdots \\ \hat{f}(-2) & \hat{f}(-3) & \hat{f}(-4) \cdots \\ \hat{f}(-3) & \hat{f}(-4) & \hat{f}(-5) \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}$$

(viewed as an operator on l^2).

Hartman showed that $\|H_f\| = \operatorname{dist}(f, H^\infty)$. Nehari showed that H_f is compact if and only if $f \in H^\infty + C$, and Adamian, Arov, and Krein showed that $\|H_f\|_e = \operatorname{dist}(f, H^\infty + C)$. It can be shown that Theorem 2 is equivalent to the assertion that each Hankel operator has a closest compact operator which is itself Hankel.

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Supliment la tomul XXV, s. I a 1979
Cu ocazia **centenarului naşterii Acad. Prof. Al. Myller**

NON-BOUNDED INTERTWINING OPERATORS OF BUNDLE SHIFTS *)

BY

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1. **Bundle shifts.** In what follows Ω will be a bounded connected open set in the complex plane whose boundary $\partial\Omega$ consists of finite number of analytic Jordan curves. Denote by D the unit disk and by μ the normalized Lebesque measure on ∂D.

From the construction of the universal covering space for Ω we obtain the following:

— A group G of linear fractional transformations A that map D onto D.

— An open G-invariant subset ω of ∂D such that $\mu(\partial D \setminus \omega) = 0$.

— A simply connected G-invariant subset D' containing $D \setminus \omega$.

— An open set Ω' containing the closure of Ω.

— A holomorphic covering map Π from D' onto Ω' with the group of deck transformations G and such that $\Pi(D) = \Omega$ and $\Pi(\omega) = \partial\Omega$.

Let m be the harmonic measure, supported in $\partial\Omega$, of the point $\Pi(0) = z_0$ in Ω. The measure μ lifts to the universal cover the measure m, in the sense that, for any $f \in L^1(m)$, we have

$$\int_{\partial\Omega} f(z)\, dm(z) = \int_{\partial D} f(\Pi(\lambda))\, d\mu(\lambda). \tag{1.1}$$

Let $\mathcal{E}$ be a separable complex Hilbert space. Denote by $H^2_{\mathcal{E}}(\Omega)$ the space of all $\mathcal{E}$-valued analytic functions on Ω which are dominated in norm by positive harmonic functions. Then, putting

$$\|h\| = \inf \{[u(z_0)]^{1/2},\ u \text{ positive harmonic in } \Omega,\ u(z) \geq \|h(z)\|,\ z \in \Omega\} \tag{1.2}$$

we obtain a norm on $H^2_{\mathcal{E}}(\Omega)$ with respect to which, $H^2_{\mathcal{E}}(\Omega)$ becomes a Hilbert space. Any function h in $H^2_{\mathcal{E}}(\Omega)$ has m a.e. nontangentially boundary limit $\tilde{h}$ and the map $h \to \tilde{h}$ is an isometric isomorphism of $H^2_{\mathcal{E}}(\Omega)$ with a subspace $H^2_{\mathcal{E}}(\partial\Omega)$ of $L^2_{\mathcal{E}}(m)$. Hence

*) Lecture presented at the Romanian-American Seminar on Operator Theory and Applications, 20–24 March 1978.