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SPECIAL PERTURBATIONS OF THE $\bar{\partial}$-OPERATOR*)

BY

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1. Introduction. In the present work we shall describe the construction of a certain operator-valued integral kernel which occurs naturally in the study of the analytic functional calculus for commuting systems of operators in Hilbert spaces. This construction depends strongly on the structure of the $\bar{\partial}$-operator, regarded as a closed operator on certain Hilbert spaces of vector-valued differential forms. We do not intend to present the above mentioned construction in detail; a complete version of this work will be published elsewhere.

Let X be a complex Hilbert space and $\mathcal{C}(X)$ ($\mathcal{L}(X)$) the set of all densely defined, closed (bounded) operators acting in X. For any $T \in \mathcal{C}(X)$ we denote by $\mathcal{D}(T)$, $\mathcal{R}(T)$, $\mathcal{K}(T)$ the domain of definition, the range and the kernel of T, respectively. In this paper we shall deal mainly with operators $T \in \mathcal{C}(X)$ having the property $\mathcal{R}(T) \subset \mathcal{K}(T)$; such an operator will be called *exact* when one actually has $\mathcal{R}(T) = \mathcal{K}(T)$.

Let us consider a system of indeterminates $s = (s_1, \ldots, s_n)$. The exterior algebra over the complex field $\mathbf{C}$ generated by $s_1, \ldots, s_n$ will be denoted by $\Lambda[s]$. For any integer p, $0 \leq p \leq n$, we denote by $\Lambda^p[s]$ the space of all homogeneous exterior forms of degree p in $s_1, \ldots, s_n$. The space $\Lambda[s]$ has a natural structure of Hilbert space in which the elements

$$s_{j_1} \wedge \cdots \wedge s_{j_p} \quad (1 \leq j_1 < \cdots < j_p \leq n;\quad p = 0, 1, \ldots, n) \tag{1.1}$$

form an orthonormal basis. We consider here that $1 \in \mathbf{C} = \Lambda^0[s]$ is in the family (1.1). Let us define the operators

$$S_j \xi = s_j \wedge \xi \quad (\xi \in \Lambda[s];\ j = 1, \ldots, n). \tag{1.2}$$

Then the adjoints of the operators (1.2) are given by the formula

$$S_j^*(\xi_j' + s_j \wedge \xi_j'') = \xi_j'' \quad (j = 1, \ldots, n), \tag{1.3}$$

where $\xi_j' + s_j \wedge \xi_j''$ is the canonical decomposition of an arbitrary element $\xi \in \Lambda[s]$. Note the anticommutation relations

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(1.4) $$S_jS_k+S_kS_j=0 \quad S_jS_k^*+S_k^*S_j=\varepsilon_{jk} \quad (j,k=1,\ldots,n),$$

where ε_{jk} is the Kronecker symbol.

For an arbitrary complex linear space L, we denote by $\Lambda[s,L]$ the tensor product $L\otimes\Lambda[s]$. If α is any endomorphism of L then the action of α is extended on $\Lambda[s,L]$ by the endomorphism $\alpha\otimes 1$. We identify the endomorphisms α and $\alpha\otimes 1$ and keep the notation α for both of them. Analogously, if θ is any endomorphism of $\Lambda[s]$ then the endomorphism $1\otimes\theta$, acting on $\Lambda[s,L]$ will be denoted also by θ.

In the case of a Hilbert space X, the space $\Lambda[s,X]$ is also a Hilbert space. The action of any $T\in\mathcal{C}(X)$ will be extended in $\Lambda[s,X]$ by $T\otimes 1$, denoted simply by T, defined on $\mathcal{D}(T)\otimes\Lambda[s]=\Lambda[s,\mathcal{D}(T)]$.

Let Ω be an open set in $\mathbf{C}^m$ and $C^\infty(\Omega,X)$ $(A(\Omega,X))$ the set of all X-valued, indefinitely differentiable (analytic) functions on Ω. Consider a commuting system $\alpha=(\alpha_1,\ldots,\alpha_n)$ in $A(\Omega,\mathcal{L}(X))$, i.e. a system of operator-valued analytic functions such that $\alpha_j(z)\alpha_k(w)=\alpha_k(w)\alpha_j(z)$ for any $j,k=1,\ldots,n$ and z,w in Ω. The system α can be associated with the endomorphism

(1.5) $$\delta_\alpha\xi(z)=(\alpha_1(z)S_1+\ldots+\alpha_n(z)S_n)\xi(z),$$

where $\xi\in\Lambda[s,C^\infty(\Omega,X)]$. From the relations (1.4) it follows easily that $(\delta_\alpha)^2=0$. We may consider also the usual $\bar\partial$-operator

(1.6) $$\frac{\partial}{\partial\bar z_1}d\bar z_1+\ldots+\frac{\partial}{\partial\bar z_m}d\bar z_m,$$

acting in the space $\Lambda[d\bar z,C^\infty(\Omega,X)]$, where $z=(z_1,\ldots,z_m)$ are the complex coordinates and $d\bar z=(d\bar z_1,\ldots,d\bar z_m)$ the corresponding system of differentials. Then the endomorphism $\delta_\alpha+\bar\partial$ acts in the space $\Lambda[(s,d\bar z),C^\infty(\Omega,X)]$, with $(s,d\bar z)=(s_1,\ldots,s_n,d\bar z_1,\ldots,d\bar z_m)$ and has the property $(\delta_\alpha+\bar\partial)^2=0$.

The aim of this work is to study the exactness of the operators of the type $\delta+\bar\partial$ in certain Hilbert spaces of square integrable exterior forms. We shall emphasize the role played by the operator $T+T^*$, where $T\in\mathcal{C}(X)$ is an operator with the property $\mathcal{R}(T)\subset\mathcal{K}(T)$. It is such an operator which leads us to a natural kernel yielding integral representation formulas in Hilbert spaces (see also [5]).

2. **The $\bar\partial$-operator in Hilbert Spaces.** From now on, X will be a fixed complex Hilbert space. Let Ω be an open, relatively compact subset of $\mathbf{C}^m$ and $L^2(\Omega)$ the usual space of all (classes of) complex-valued square integrable functions on Ω, with respect to the Lebesgue measure. Let us denote by X_Ω the completion $L^2(\Omega)\bar\otimes X$ of the tensor product $L^2(\Omega)\otimes X$ with respect to the canonical hilbertian norm. In other words, X_Ω is the space of all (classes of) X-valued functions, strongly measurable on Ω and whose norm is square integrable [4].

Let us define now the operator $\bar\partial$ in $\Lambda[\bar\zeta,X_\Omega]$, where $\bar\zeta=(\bar\zeta_1,\ldots,\bar\zeta_m)$ is a system of indeterminates. We denote by $\mathcal{D}(\bar\partial)\subset\Lambda[\bar\zeta,X_\Omega]$ the set of those $\xi\in\Lambda[\bar\zeta,X_\Omega]$ such that there exists an $\eta\in\Lambda[\bar\zeta,X_\Omega]$ satisfying

(2.1) $$\int\varphi(z)\eta(z)d\lambda(z)=-\int\left(\frac{\partial\varphi}{\partial\bar z_1}(z)\bar\zeta_1+\cdots\frac{\partial\varphi}{\partial\bar z_m}(z)\bar\zeta_m\right)\wedge\xi(z)d\lambda(z),$$

for any $\varphi\in C_0^\infty(\Omega)$, where $d\lambda$ is the Lebesgue measure and $C_0^\infty(\Omega)$ is the subspace of $C^\infty(\Omega)$ $(=C^\infty(\Lambda\Omega,\mathbf{C}))$ of compactly supported functions. We define $\bar\partial\xi=\eta$ and the formula (2.1) shows that the operator $\bar\partial$ is a weak extension of the operator (1.6) (we prefer to use system $\bar\zeta=(\bar\zeta_1,\ldots,\bar\zeta_m)$ instead of $d\bar z=(d\bar z_1,\cdots,d\bar z_m)$ in order to stress the independence of this system on the points in Ω and its orthonormality). As in the scalar case, the operator $\bar\partial$ is closed and densely defined. From the formula (2.1) it follows that $\mathcal{R}(\bar\partial)\subset\subset\mathcal{K}(\bar\partial)$. Let us denote by $\bar\partial_s$ the „scalar" operator $\bar\partial$, i.e., the operator $\bar\partial$ obtained for $X=\mathbf{C}$.

2.1. **Theorem.** *The operator $\bar\partial$ is the closure of the operator $\bar\partial_s\oplus 1$. Analogously, the operator $\bar\partial^*$ is the closure of the operator $\bar\partial_s^*\otimes 1$.*

Outline of the proof. It can be shown that any $\zeta\in\mathcal{D}(\bar\partial)$ there is a sequence $\zeta_j\in\mathcal{D}(\bar\partial_s)\otimes X$ such that $\zeta_j\to\zeta$ and $(\bar\partial_s\otimes 1)\zeta_j\to\bar\partial\zeta$ as $j\to\infty$, in $\Lambda[\bar\zeta,X_\Omega]$. Similarly, it $\xi\in\mathcal{D}(\bar\partial^*)$ then there is a sequence $\zeta_j\in\mathcal{D}(\bar\partial_s^*)\oplus X$ such that $\zeta_j\to\zeta$ and $(\bar\partial_s^*\otimes 1)\zeta_j\to\bar\partial^*\zeta$ as $j\to\infty$, in $\Lambda[\zeta,X_\Omega]$.

Theorem 2.1. suggests that many significant properties of the operator $\bar\partial_s$ can be reformulated and proved for the operator $\bar\partial$ too. In particular, from the deep theory concerning the Neumann problem developed by J. J. Kohn [1] we may obtain that if Ω is a strongly pseudoconvex then the range of the $\bar\partial$-operator is closed.

3. **Special Perturbations of $\bar\partial$.** Let U be an arbitrary open set in $\mathbf{C}^m$ and $\alpha=(\alpha_1,\ldots,\alpha_n)\subset(U,\mathcal{L}(X))$ a commuting system. We denote by $\sigma_U(\alpha,X)$ the set of all points $z\in U$ such that the system $\alpha(z)=(\alpha_1(z),\ldots,\alpha_n(z))$ is *singular* as a commuting system of linear operators [3], [5]. The set $\sigma(\alpha,X)$ is closed in U (it may be either empty or equal to U), therefore $U/\sigma_U(\alpha,X)$ is open [3], [5]. We associate the system $\alpha=(\alpha_1\ldots,\alpha_n)$ with the system of indeterminates $s=(s_1,\ldots,s_n)$ and the system of areolar derivatives $(\partial/\partial\bar z_1,\cdots,\partial/\partial\bar z_m)$ in U with the system $\bar\zeta=(\bar\zeta_1,\ldots,\bar\zeta_m)$. If δ_α is given by (1.5) and $\Omega\subset U/\sigma_U(\alpha,X)$ is an arbitrary open set then $(\delta_{\alpha(z)}+\delta_{\alpha(z)}^*)^{-1}$ is a function in $C^\infty(\Omega,\mathcal{L}(\Lambda[s,X]))$ [5]. When $\bar\Omega$ is compact and contained in, $U/\sigma_U(\alpha,X)$ we may consider the operator $\delta_\alpha+\bar\partial$, acting in the space $\Lambda[(s,\bar\zeta)X_\Omega]$.

Let us start with the unbounded variant of a result in [5].

3.1. **Lemma.** *Assume that $T\in\mathcal{C}(X)$ has the property $\mathcal{R}(T)\subset\mathcal{K}(T)$. Then T is exact if and only if $T+T^*$ has a bounded inverse on X.*

This is a simple but useful characterization, which leads us to the main result of this section (Theorem 3.4).

3.2. **Corollary.** *If T is exact, then we have*

$$(T+T^*)^{-1}Tx=T^*(T+T^*)^{-1}x \quad (x\in\mathcal{D}(T))$$

and

$$(T+T^*)^{-1}T^*y=T(T+T^*)^{-1}y \quad (y\in\mathcal{D}(T^*)).$$

3.3. **Lemma.** *Consider an open set $U\subset\mathbf{C}^m$ and a commuting system $\alpha=(\alpha_1,\ldots\alpha_n)\subset A(U,\mathcal{L}(X))$. If $\bar\Omega$ is an open, relatively compact subset in $U/\sigma_U(\alpha,X)$, then the operator $D_\alpha=\delta_\alpha+\bar\partial$ is exact in $\Lambda[(s,\bar\zeta),X_\Omega]$.*

Outline of the proof. With some moditications due to the unboundeness, of $\bar\partial$, the proof is similar to one in [5].

Consider a form $\eta \in \mathcal{D}(D_\alpha)$ such that $D_\alpha \eta = 0$. With no loss of generality one supposes that η is homogeneous of degree $p \leq n+m$ in $s_1, \dots, s_n, \bar\zeta_1, \dots, \bar\zeta_m$. Then $\eta = \eta_0 + \dots + \eta_p$, where η_j is of degree j in $\bar\zeta_1, \dots, \bar\zeta_m$ and of degree $p-j$ in $s_1, \dots, s_n$. One looks for a solution ξ of the equation $D_\alpha \xi = \eta$, where $\xi = \xi_0 + \dots + \xi_{p-1}$, ξ_j being of degree j in $\bar\zeta_1, \dots, \bar\zeta_m$ and of degree $p-j-1$ in $s_1, \dots, s_n$. By identifying the forms of the same type, one obtains a system of equations with supplementary conditions, which can be solved step by step, by using Corollary 3.2 and the special structure of the operator δ_α.

We can state now the main result of this work. By a *smooth form we mean any element of* $\Lambda[(s, \bar\zeta), X_\Omega]$ whose coefficients are in $C^\infty(\Omega, X)$.

3.4. Theorem. *With the conditions of Lemma 3.3, the operator $D_\alpha + D_\alpha^*$ has a bounded inverse R_α in the space $\Lambda[(s, \bar\zeta), X_\Omega]$. Moreover, if ζ is a smooth form then $R_\alpha \zeta$ is also smooth.*

Outline of the proof. The invertibility of $D_\alpha + D_\alpha^*$ follows from Lemmas 3.1 and 3.3. The second assertion is related to the problem of regularity of the solutions of the elliptic differential operators. In order to apply well-known results of this type, we have to reduce the problem to the scalar case.

Assume first that $\zeta \in \Lambda[(s, \bar\zeta), X_\Omega]$ is a smooth form such that $(\delta_\alpha + \bar\partial)\zeta = 0$ in Ω. By Corollary 3.2 we have then $(\delta_\alpha + \bar\partial) R_\alpha \zeta = \zeta$ and $(\delta_\alpha^* + \bar\partial^*) R_\alpha \zeta = 0$. It will be enough to prove that for any $\zeta \in C_0^\infty(\Omega)$ the form $\eta = \psi R_\alpha \zeta$ is smooth. Since $R_\alpha \zeta \in \mathcal{D}(\bar\partial^*)$ we have also $\eta \in \mathcal{D}(\bar\partial^*)$ on account of the formula

$$\bar\partial^* \eta = -\left(\frac{\partial\psi}{\partial z_1} \bar Z_1^* + \dots + \frac{\partial\psi}{\partial z_m} \bar Z_m^*\right) R_\alpha \zeta + \psi \bar\partial^* R_\alpha \zeta,$$

where $\bar Z_j$ is the operator associated with $\bar\zeta_j$ by (1.2).

Next we show that if $\gamma = \bar\partial^* \eta$ then we have the equality

$$\int \varphi(z)\gamma(z)\, d\lambda(z) = \int \left(\sum_{j=1}^{m} \frac{\partial\varphi}{\partial z_j} \bar Z_j^*\right) \eta(z)\, d\lambda(z).$$

An induction argument is then used in order to get that the functions $\langle \eta_I(x), x \rangle$ belong to any Sobolev space, where η_I is a coefficient of η and $x \in X$ is arbitrary; this procedure is inspired by some results from [2]. Then the coefficients η_I are themselves in $C^\infty(\Omega, \Lambda[s, X])$.

When $\zeta \in \Lambda[(s, \bar\zeta), X_\Omega]$ is an arbitrary smooth form then, by the previous case, $\zeta_1 = (\delta^* + \bar\partial_\alpha^*) R_\alpha \zeta = R_\alpha(\bar\partial + \bar\partial)\zeta$ is smooth, therefore $\zeta_2 = (\delta_\alpha + \bar\partial) R_\alpha \zeta = \zeta - \zeta_1$ is also smooth, and a similar procedure can be applied in order to obtain the result in its full generality.

If $\zeta \in \Lambda[(s, \bar\zeta), X_\Omega]$ is arbitrary then we define its integral on a Borel set $M \subset \Omega$ in the following way:

Denote by ζ_m the part of ζ of degree m in $\bar\zeta_1, \dots, \bar\zeta_m$ and by $\tilde\zeta_m$ the form obtained from ζ_m by substituting $\bar\zeta_1, \dots, \bar\zeta_m$ with $d\bar z_1, \dots, d\bar z_m$. Then we put, by definition

(3.1)
$$\int_M \zeta(z) \wedge dz_1 \wedge \dots \wedge dz_m = \int_M \tilde\zeta_m(z) \wedge dz_1 \wedge \dots \wedge dz_m.$$

The right side of the formula (3.1) makes sense and it is an element of $\Lambda[s, X]$.

The integral given by (3.1) is not enough for our purpose. We need a more complicated concept, valid for smooth surfaces of real dimension $2m-1$. Assume that $K \subset U$ is a compact subset, where $U \subset \mathbf{C}^m$ is open. Take then three open sets V_1, V and V_2 such that $K \subset V_1 \subset \bar V_1 \subset V \subset \bar V \subset V_2 \subset \bar V_2 \subset U$ with $\bar V_2$ compact in U, such that the boundary Σ of V is a smooth surface. We say that Σ is an *admissible surface surrounding K in U*(5).

Define $\Omega = V_2 \setminus \bar V_1$ and consider $\zeta \in \mathcal{D}(\bar\partial)$ in $\Lambda[(s, \bar\zeta), X]$. If $\psi \in C_0^\infty(U)$ is such that supp $\psi \subset \Omega$, $\psi = 1$ in a neighbourhood of Σ then $\psi\zeta \in \mathcal{D}(\bar\partial)$ and $\psi\xi$ has an extension in U as a compactly supported form. Then we define

(3.2)
$$\int_\Sigma \zeta(z) \wedge dz_1 \wedge \dots \wedge dz_m = \int_V \bar\partial\psi\zeta(z) \wedge dz_1 \wedge \dots \wedge dz_m,$$

where the right side is given by (3.1). Plainly, the formula (3.2) is suggested by the Stokes formula. If we approximate the form ζ with smooth forms then one can easily see that (3.2) does not depend on the particular choice of the funtion ψ.

Let us denote by X_V^{loc} the space of all X-valued locally square integrable functions in $U \subset \mathbf{C}^m$. Denote also by $Z_0[(s, \bar\zeta), X_U^{loc}]$ the subspace of $\Lambda[(s, \bar\zeta), X_U^{loc}]$ of those forms η of degree $\leq m (= n)$ such that $(\delta_\alpha + \bar\partial)\eta = 0$, which can be approximated with exterior forms $\Sigma_j \eta_j x_j$, where the coefficients η_j are smooth functions with values in the commutant of the set $\{\alpha_1(z), \dots, \alpha_n(z);\ z \in U\}$ in $\mathcal{L}(X)$, $x_j \in X$ and $(\delta_\alpha + \bar\partial)\eta_j = 0$ for all j.

3.5. Theorem. *Assume that $a = (a_1, \dots, a_n)$ is a commuting system of operators in $\mathcal{L}(X)$ and define $\alpha_j(z) = z_j - a_j$, $j = 1, \dots, n$, $z = (z_1, \dots, z_n) \in \mathbf{C}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n)$. If $U \subset \mathbf{C}^n$ is any open set containing $\sigma_{\mathbf{C}^n}(\alpha, X)$ then there exists a continuous linear map μ_α from $Z_0(s, \bar\zeta), X_U^{loc}]$ into X such that for any $f \in A(U)$ and $\eta \in Z_0[(s, \bar\zeta), X_U^{loc}]$ we have $\mu_\alpha(f\eta) = \mu_\alpha(Sf)\ \mu_\alpha(\eta)$, where $S = S_1 \dots S_n$.*

Outline of the proof. For any $\eta \in Z_0[(s, \bar\zeta), X_U^{loc}]$ we define

(3.3)
$$\mu_\alpha(\eta) = \int_\Sigma R_\alpha \eta \wedge dz_1 \wedge \dots \wedge dz_m,$$

where Σ is an admissible surface surrounding $\sigma_{\mathbf{C}^n}(\alpha, X)$ in U. The integral (3.3) does not depend on the particular choice of the surface Σ. The stated property is a „Fubini type" result which follows by adapting some techniques from [3] and [5].

4. Some final Comments. The most important consequence of Theorems 3.4 and 3.5 is a new canonical formula for the analytic functional calculus for commuting systems of linear operators in Hilbert spaces, which can be derived from (3.3) (see [5] for another canonical formula). However, Theorem 3.5 is an extension of what is known as the „multiplicativity property" of the analytic functional calculus. In fact, for commuting systems of linear operators, the map (3.3) is an $A(U)$-module homomorphism. As one can see, actually for $n=1$, Theorem 3.5 gives more than the usual analytic functional calculus.

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