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ON THE CONNECTION OBJECTS OF SECOND ORDER

BY

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In the mathematical literature there are numerous works dealing with the notion of connection of first and higher orders. Among these we mention those of Ehresmann, Kobayashi, Oproiu, Pohl and others, in which there are presented also the prolongations of the first order connections.

It arises the question to examine the general geometric objects to which lead these connections of higher order.

In this note we deal with the definition and study research of several properties of the connection of the second order, without imposing restrictions on the symmetry of first order linear connection which enters in its structure and without requiring that these connections come out from a (first order) connection.

On this occasion, we try to make evident the subordinate geometric objects of the second order connection to characterize the considered objects and to define the notion of generalized parallelism.

1. Consider M an n-dimensional, differentiable manifold and in every point of it a collection of fields of pairs $(\Gamma^i_{jk}, \Gamma^i_{jkh})$, defined on the domains of local charts which with respect to a coordinate transformatoin $x \to \overline{x}$ satisfy the relations :

$$\overline{\Gamma}^i_{jk} = \Gamma^p_{qr} \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \overline{x}^j} \frac{\partial x^r}{\partial \overline{x}^k} + \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial^2 x^p}{\partial \overline{x}^j \partial \overline{x}^k},$$

$$(1.1) \qquad \overline{\Gamma}^i_{jkh} = \Gamma^p_{qrs} \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \overline{x}^j} \frac{\partial x^r}{\partial \overline{x}^k} \frac{\partial x^s}{\partial \overline{x}^h} + \Gamma^p_{qr} \frac{\partial \overline{x}^i}{\partial x^p} \left(\frac{\partial^2 x^q}{\partial \overline{x}^j \partial \overline{x}^k} \cdot \frac{\partial x^r}{\partial \overline{x}^h} + \frac{\partial^2 x^q}{\partial \overline{x}^j \partial \overline{x}^h} \cdot \frac{\partial x^r}{\partial \overline{x}^k} + \frac{\partial^2 x^r}{\partial \overline{x}^k \partial \overline{x}^h} \cdot \frac{\partial x^q}{\partial \overline{x}^j} \right) + \frac{\partial \overline{x}^i}{\partial x^p} \cdot \frac{\partial^3 x^p}{\partial \overline{x}^j \partial \overline{x}^k \partial \overline{x}^h}.$$

Proposition 1.1. *The collection of pairs $(\Gamma^i_{jk}, \Gamma^i_{jkh})$ with the transformation rule (1.1) is a linear unhomogeneous geometric object, of minimal class 3.*

Proof. Let $\overline{x}^i = \overline{x}^i(x^1, \dots, x^n)$, $\overline{\overline{x}}^i = \overline{\overline{x}}^i(\overline{x}^1, \dots, \overline{x}^n)$ be two succesive coordinate transformations.

According to (1.1) :

$$\bar{\Gamma}^i_{jk} = \Gamma^p_{qr} \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial x^r}{\partial \bar{x}^k} + \frac{\partial \bar{x}^i}{\partial x^p} \cdot \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^k},$$

$$\begin{aligned} (1.2) \qquad \bar{\Gamma}^i_{jkh} = \Gamma^p_{qrs} \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^s}{\partial \bar{x}^h} + \Gamma^p_{qr} \frac{\partial \bar{x}^i}{\partial x^p} \Bigg(\frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k} \cdot \frac{\partial x^r}{\partial \bar{x}^h} + \\ + \frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^h} \cdot \frac{\partial x^r}{\partial \bar{x}^k} + \frac{\partial^2 x^r}{\partial \bar{x}^k \partial \bar{x}^h} \cdot \frac{\partial x^q}{\partial \bar{x}^j} \Bigg) + \frac{\partial \bar{x}^i}{\partial x^p} \cdot \frac{\partial^3 x^p}{\partial \bar{x}^j \partial \bar{x}^k \partial \bar{x}^h}. \end{aligned}$$

Using again (1.1) and the chain rule of differentiation one has:

$$\bar{\Gamma}^i_{jk} = \Gamma^p_{qr} \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial x^r}{\partial \bar{x}^k} + \frac{\partial \bar{x}^i}{\partial x^p} \cdot \frac{\partial^2 x^p}{\partial \bar{x}^j \partial \bar{x}^k}.$$

$$\begin{aligned} \bar{\Gamma}^i_{jkh} = \Gamma^p_{qrs} \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^s}{\partial \bar{x}^h} + \Gamma^p_{qr} \cdot \frac{\partial \bar{x}^i}{\partial x^p} \Bigg(\frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k} \cdot \frac{\partial x^r}{\partial \bar{x}^h} + \\ + \frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^h} \cdot \frac{\partial x^r}{\partial \bar{x}^k} + \frac{\partial^2 x^r}{\partial \bar{x}^h \partial \bar{x}^k} \cdot \frac{\partial x^q}{\partial \bar{x}^j} \Bigg) + \frac{\partial \bar{x}^i}{\partial x^p} \cdot \frac{\partial^3 x^p}{\partial \bar{x}^j \partial \bar{x}^k \partial \bar{x}^h}. \end{aligned}$$

Definition 1.1. *The geometric object $(\Gamma^i_{jk}, \Gamma^i_{jkh})$ given by the previous proposition is called the object of the generalized connection of second order.*

To give an example, we shall use the objects of connection of second order obtained by O p r o i u in [4], but without condition of symmetry of linear connexion.

Proposition 1.2. *If Γ^i_{jk} is a linear connexion on M and*

$$(1.3) \qquad \underset{01}{\Gamma}{}^i_{jkh} = \frac{\partial \Gamma^i_{jk}}{\partial x^h} + \Gamma^m_{jk} \Gamma^i_{mh},$$

is Oproiu's geometric object on M, then the pair $(\Gamma^i_{jk}, \underset{01}{\Gamma}{}^i_{jkh})$ is a geometric object with the transformation rule (1.1).

Proof. On can see without difficulty that under a change of coordinates $\bar{x}^i = \bar{x}^i(x^1, x^2, \ldots, x^n)$, the object $\underset{01}{\Gamma}{}^i_{jkh}$ has the transformation law (1.1.).

Another example is $(\Gamma^i_{jk}, \underset{02}{\Gamma}{}^i_{jkh})$, [7] where

$$(\underset{02}{\Gamma}{}^i_{jkh} = \frac{1}{2} (\underset{01}{\Gamma}{}^i_{jkh} + \underset{01}{\Gamma}{}^i_{jhk}).$$

Finally, another example is the object of connection of the second order given by [7], namely, $(\Gamma^i_{jk}, \underset{03}{\Gamma}{}^i_{jkh})$, where:

$$\begin{aligned} (1.4) \qquad \underset{03}{\Gamma}{}^i_{jkh} = \frac{1}{3} \left(\frac{\partial \Gamma^i_{jk}}{\partial x^h} + \frac{\partial \Gamma^i_{jh}}{\partial x^k} + \frac{\partial \Gamma^i_{kh}}{\partial x^j} + \Gamma^m_{jk} \Gamma^i_{mh} + \Gamma^m_{jh} \Gamma^i_{mk} + \Gamma^m_{kh} \Gamma^i_{mj} \right) = \\ = \frac{1}{3} (\underset{01}{\Gamma}{}^i_{jkh} + \underset{01}{\Gamma}{}^i_{jhk} + \underset{01}{\Gamma}{}^i_{khj}). \end{aligned}$$

Theorem 1.3. *The pair $(\Gamma^i_{jk}, \Gamma^i_{jkh})$ defines an object of generalized connection of second order if and only if for every vector field $\{v^i\}$, the pair $(Dv^i/dt, \tilde{D}^2 v^i/dt^2)$, given by*

$$\begin{aligned} \frac{Dv^i}{dt} &= \frac{dv^i}{dt} + \Gamma^i_{jk} v^j \frac{dx^k}{dt}, \\ (1.5) \qquad \frac{\tilde{D}^2 v^i}{dt^2} &= \frac{d^2 v^i}{dt^2} + \Gamma^i_{jkh} v^j \frac{dx^k}{dt} \frac{dx^h}{dt} + 2\Gamma^i_{jk} \frac{dv^j}{dt} \cdot \frac{dx^k}{dt} + \Gamma^i_{jk} v^j \frac{d^2 x^k}{dt^2} \end{aligned}$$

is a pair of contravariant vectors, where, $x^i = x^i(t)$, $t \in (a, b)$ is an arbitrary curve in M.

Proof. Suppose that $(\Gamma^i_{jk}, \Gamma^i_{jkh})$ is a geometric object with the transformation law (1.1). Then as it is known, we have:

$$(1.6) \qquad \frac{D\bar{v}^i}{dt} = \frac{\partial \bar{x}^i}{\partial x^j} \cdot \frac{Dv^j}{dt}.$$

We have, also:

$$\begin{aligned} \frac{\tilde{D}^2 \bar{v}^i}{dt^2} &= \frac{d^2 \bar{v}^i}{dt^2} + \bar{\Gamma}^i_{jkh} \bar{v}^j \cdot \frac{d\bar{x}^k}{dt} \cdot \frac{d\bar{x}^h}{dt} + 2\bar{\Gamma}^i_{jk} \frac{d\bar{v}^j}{dt} \cdot \frac{d\bar{x}^k}{dt} + \bar{\Gamma}^i_{jk} \bar{v}^j \frac{d^2 \bar{x}^k}{dt^2}, \\ (1.7) \qquad \bar{v}^i &= \frac{\partial \bar{x}^i}{\partial x^a} \cdot v^a, \quad \frac{d\bar{v}^i}{dt} = \frac{\partial^2 \bar{x}^i}{\partial x^a \partial x^b} \cdot v^a \cdot \frac{dx^b}{dt} + \frac{\partial \bar{x}^i}{\partial x^a} \cdot \frac{dv^a}{dt}; \end{aligned}$$

$$\begin{aligned} &\cdot v^a \cdot \frac{dx^b}{dt} \cdot \frac{dx^c}{dt} + 2 \frac{\partial^2 \bar{x}^i}{\partial x^a \partial x^b} \cdot \frac{dv^a}{dt} \cdot \frac{dx^b}{dt} + \\ (1.8) \qquad &+ \frac{\partial^2 \bar{x}^i}{\partial x^a \partial x^b} \cdot v^a \cdot \frac{d^2 x^b}{dt^2} + \frac{\partial \bar{x}^i}{\partial x^a} \cdot \frac{d^2 v^a}{dt^2}. \end{aligned}$$

$$\frac{d\bar{x}^i}{dt} = \frac{\partial \bar{x}^i}{\partial x^a} \cdot \frac{dx^a}{dt}; \quad \frac{d^2 \bar{x}^i}{dt^2} = \frac{\partial^2 \bar{x}^i}{\partial x^a \partial x^b} \cdot \frac{dx^a}{dt} \cdot \frac{dx^b}{dt} + \frac{\partial \bar{x}^i}{\partial x^a} \cdot \frac{d^2 x^a}{dt^2}.$$

Replacing in (1.7) and taking into account (1.1), it follows:

$$(1.9) \qquad \frac{\tilde{D}^2 \bar{v}^i}{dt^2} = \frac{\partial \bar{x}^i}{\partial x^a} \cdot \frac{\tilde{D}^2 v^a}{dt^2}.$$

Conversely, supposing that $(Dv^i/dt, \tilde{D}^2 v^i/dt^2)$ given by (1.5) define contravariant vectors, then from (1.6) it follows at once the first relation (1.1), and from (1.9) we find:

$$\begin{aligned} \frac{D^2 \bar{v}^i}{dt^2} &= \frac{d^2 \bar{v}^i}{dt^2} + \bar{\Gamma}^i_{jkh} \cdot \bar{v}^j \cdot \frac{d\bar{x}^k}{dt} \frac{d\bar{x}^h}{dt} + 2\Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} + \Gamma^i_{jk} v^j \frac{d^2 x^k}{dt^2} = \\ (1.10) \qquad &= \frac{\partial x^i}{\partial x^a} \left(\frac{d^2 v^a}{dt^2} + \Gamma^a_{bcd} v^b \frac{dx^c}{dt} \frac{dx^d}{dt} + 2\Gamma^a_{bc} \frac{dv^b}{dt} \cdot \frac{dx^c}{dt} + \Gamma^a_{bc} v^b \frac{d^2 x^c}{dt} \right). \end{aligned}$$

Taking into account that :

$$v^a = \frac{\partial x^a}{\partial \bar{x}^i} \cdot \bar{v}^i ; \quad \frac{dv^a}{dt} = \frac{\partial^2 x^a}{\partial \bar{x}^i \partial \bar{x}^j} \cdot \bar{v}^i \cdot \frac{dx^j}{dt} + \frac{\partial x^a}{\partial \bar{x}^i} \cdot \frac{d\bar{v}^i}{dt} ,$$

$$\frac{d^2 v^a}{dt^2} = \frac{\partial x^a}{\partial \bar{x}^i} \cdot \frac{d^2 v^i}{dt^2} + 2 \frac{\partial^2 x^a}{\partial \bar{x}^i \partial \bar{x}^j} \cdot \frac{dv^i}{dt} \cdot \frac{dx^j}{dt} + \frac{\partial^2 x^a}{\partial \bar{x}^i \partial \bar{x}^j} \cdot \bar{v}^i \cdot \frac{d^2 \bar{x}^j}{dt^2} +$$

$$+ \frac{\partial^3 x^a}{\partial \bar{x}^i \partial \bar{x}^j \partial \bar{x}^k} \cdot \bar{v}^i \cdot \frac{d\bar{x}^j}{dt} \cdot \frac{d\bar{x}^k}{dt} \cdot \frac{dx^a}{dt} = \frac{\partial x^a}{\partial \bar{x}^i} \cdot \frac{d\bar{x}^i}{dt} ; \quad \frac{d^2 x^a}{dt^2} =$$

$$= \frac{\partial^2 x^a}{\partial \bar{x}^i \partial \bar{x}^j} \cdot \frac{d\bar{x}^i}{dt} \cdot \frac{d\bar{x}^j}{dt} + \frac{\partial x^a}{\partial \bar{x}^i} \cdot \frac{d^2 \bar{x}^i}{dt^2} ,$$

substituing in (1.10) and identifing the coefficients of $v^j d\bar{x}^k/dt$. $d\bar{x}^h/dt$ the second relation (1.1) follows.

Definition 1.2. *A pair* (τ^i_k, τ^i_{jkh}) *with the transformation law :*

$$\bar{\tau}^i_{jk} = \tau^p_{qr} \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial x^q}{\partial x^j} \frac{\partial x^r}{\partial x^k} , \quad \bar{\tau}^i_{jkh} = \tau^p_{qrs} \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^s}{\partial \bar{x}^h} +$$

$$+ \tau^p_{qr} \frac{\partial \bar{x}^i}{\partial x^p} \left(\frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k} \cdot \frac{\partial x^r}{\partial \bar{x}^h} + \frac{\partial^2 x^r}{\partial \bar{x}^k \partial \bar{x}^h} \cdot \frac{\partial x^q}{\partial \bar{x}^j} + \frac{\partial^2 x^q}{\partial \bar{x}^h \partial \bar{x}^j} \cdot \frac{\partial x^r}{\partial \bar{x}^k} \right) .$$

is called a differentiable, linear, homogeneous geometric object of second order with the transformation law subordinated to (1.1) of the object $(\Gamma^i_{jk}, \Gamma^i_{jkh})$.

Proposition 1.4. If $(\underset{a}{\Gamma}{}^i_{jk}, \underset{a}{\Gamma}{}^i_{jkh})$, $(a = 1, 2)$ *define two generalized connections of second order, then the pair*

$$(\tau^i_{jk}, \tau^i_{jkh}) = (\underset{1}{\Gamma}{}^i_{jk} - \underset{2}{\Gamma}{}^i_{jk}, \underset{1}{\Gamma}{}^i_{jkh} - \underset{2}{\Gamma}{}^i_{jkh})$$

is a differentiable, linear, homogeneous geometric object with the transformation law subordinate to the rule (1.1).

The proof is elementary ; it follows by writing the transformation law of $\underset{a}{\Gamma}{}^i_{jk}$, $\underset{a}{\Gamma}{}^i_{jkh}$ and getting their difference.

Remark. If $\underset{1}{\Gamma}{}^i_{jk} = \underset{2}{\Gamma}{}^i_{jk}$, then $(0, \underset{1}{\Gamma}{}^i_{jkh} - \underset{2}{\Gamma}{}^i_{jkh})$ defines a tensor field of type (1.3).

For instance, let us consider Oproiu's connections of second order $(\Gamma^i_{jk}, \underset{01}{\Gamma}{}^i_{jkh})$, $(\Gamma^i_{kj}; \underset{01}{\Gamma}{}^i_{kjh})$. ; then their difference is $(T^i_{jk}, \underset{01}{T}{}^i_{jkh})$, where :

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}, \quad T^i_{jkh} = \frac{\partial T^i_{jk}}{\partial x^h} + T^m_{jk} \Gamma^i_{mh} . \tag{1.13}$$

Proposition 1.5. *The pair* $(T^i_{jk}, \underset{01}{T}{}^i_{jkh})$ *defines a linear, homogeneous, geometric object of second order with the transformation law subordinated to* (1.1)

Remark. The geometric objects $(\Gamma^i_{jk}, \underset{01}{\Gamma}{}^i_{jkh})$, $(\Gamma^i_{jk}, \underset{01}{\Gamma}{}^i_{jkh})$ give the tensor field $(0), R^i_{jkh})$, where R^i_{jkh} is the curvature tensor of the connexion Γ^i_{jk}.

Theorem 1.6. *Given the components of an object of generalized second order connection* Γ^i_{jkh}, *there exists a linear connection* Γ^i_{jk} *so that the pair* $(\Gamma^i_{jk}, \Gamma^i_{jkh})$ *is the Oproiu's linear connection of the second order, if and only if, in the system :*

$$\begin{cases} A^i_{jkhl_1} - A^i_{jkl_1h} = 0 \\ A^i_{jkhl_1l_2} - A^i_{jkl_1hl_2} = 0 \end{cases} \tag{1.14}$$

Where

$$A^i_{jkhl_1} = \frac{\partial \Gamma^i_{jkh}}{\partial x^{l_1}} + \Gamma^m_{jkh} \Gamma^i_{ml_1} - \Gamma^m_{jk} \Gamma^i_{mhl_1} ,$$

$$A^i_{jkhl_1l_2} = \frac{\partial^2 \Gamma^i_{jkh}}{\partial x^{l_1} \partial x^{l_2}} + \frac{\partial \Gamma^i_{jkh}}{\partial x^{l_2}} \cdot \Gamma^i_{ml_1} + \Gamma^m_{jkh} \Gamma^i_{ml_1l_2} - \Gamma^m_{jkh} \Gamma^s_{ml_1} \Gamma^i_{sl_2} -$$

$$- \Gamma^i_{mhl_1} \Gamma^m_{jkl_2} + \Gamma^s_{jk} \Gamma^m_{sl_2} \Gamma^i_{mhl_1} - \Gamma^m_{jk} \frac{\partial \Gamma^i_{mhl_1}}{\partial x^{l_2}}$$

there exists a rank N, *so that the system of the first* N *equations from* (1.14) *is compatible with respect to* Γ^i_{jk} *and the solution of this system verifies the equation of the order* $(N + 1)$ *of the system* (1.14).

Proof. Consider the system of equations with total differentials :

$$\frac{\partial \Gamma^i_{jk}}{\partial x^h} \Gamma^i_{jkh} - \Gamma^m_{jk} \Gamma^i_{mh} .$$

Applying a well-known theorem of E i s e n h a r t, the conditions of compatibility of this system there are, as it is easily seen, the conditions of compatibility of the system (1.14).

2. We shall deal further with several questions suggested by theorem 1.2.

Proposition 2.1. *For the pair* $(\Gamma^i_{jk}, \frac{1}{2} (\underset{01}{\Gamma}{}^i_{jkh} + \underset{01}{\Gamma}{}^i_{jkh}))$ *and every vector along a curve C, the vectors* $(Dv^i/dt, \bar{D}^2 v^i/dt^2)$ *coincide with* $(Dv^i/dt, \bar{D}^2 v^i/dt^2 = D/dt \, (Dv^i/dt))$.

The proof is a question of calculus.

From the theorem 1.3, it follows the possibility to consider several types of parallelism of vectors with respect to generalized connection of the second order.

Definition 2.1. *A vector field* v^i *is a parallel vector field of second order along the curve C in the manifold M with respect to the generalized connection of second order* $(\Gamma^i_{jk}, \Gamma^i_{jkh})$, *if :*

$$\frac{\bar{D}^2 v^i}{dt^2} = 0 . \tag{2.1}$$

Recall that the vector field v^i is parallel of first order if:

$$\frac{Dv^i}{dt^2} = 0. \tag{2.2}$$

Remark. The case of fields $v^i(x^j)$ for which $Dv^i/dt = 0$, $\widetilde{D}^2v^i/dt^2 = 0$ may be either trivial or leads to an impossibility, because these fields are characterized by (2.2) and by the linear, homogeneous system of n equation with n unknows:

$$\frac{1}{2}\{(\Gamma^i_{jkh} + \Gamma^i_{jhk}) - (\underset{01}{\Gamma^i_{jkh}} + \underset{01}{\Gamma^i_{jhk}})\} v^j \frac{dx^k}{dt} \cdot \frac{dx^h}{dt} = 0.$$

It is obvious that this system of equations has geometric character, because

$$\frac{1}{2}\{(\Gamma^i_{jkh} + \Gamma^i_{jhk}) - (\underset{01}{\Gamma^i_{jkh}} + \underset{01}{\Gamma^i_{jhk}})\}$$

is an tensor of type (1.3).

The existence and unicity theorem for parallel vector fields of first order is known. For the parallelism of second order we have:

Theorem 2.2. *Given a curve (C), $x^i = x^i(t)$, $t \in (a, b)$ and in the point $x^i(t_0)$, $t_0 \in (a, b)$, the pair of vectors $(v^i_0, (Dv^i/dt)_0)$ there is an interval $(a_1, b_1) \subset (a, b)$ and a unique vector field defined along the curve $x^i = x^i(t)$, $t \in (a_1, b_1)$, parallel of second order along C and so that $v^i(t_0) = v^i_0$, $(\widetilde{D}v^i/dt)(t_0) = (\widetilde{D}v^i/dt)_0$.*

The proof is usual from the theory of ordinary differential equations.

Similarly, one can define the geodesics of the first and second order, taking into the precedent considerations, $v^i = dx^i/dt$.

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A CONTROL PROBLEM RELATED TO A VOLTERRA PREY-PREDATOR SYSTEM

BY

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Introduction. As it is well known, the evolution of a Volterra prey-predator system is described ([6], chap. I) by the differential equations:

$$\begin{aligned} \frac{dN_1}{dt} &= N_1(a_1 - b_1 N_2), \\ \frac{dN_2}{dt} &= N_2(-a_2 + b_2 N_1), \end{aligned} \tag{1}$$

where N_1, N_2 are the numbers of individuals of the prey - respectively the predator-populations and a_1, a_2, b_1, b_2 are positive constants; the solution of this system is periodical; in the phase-space (N_1, N_2), each curve $N_1 = N_1(t)$, $N_2 = N_2(t)$ determined uniquely by the initial values N_1^0, N_2^0, is closed; its equation is

$$N_1^{-a_2} e^{b_2 N_1} = C N_2^{a_1} e^{-b_1 N_2}; \tag{2}$$

the constant C is determined by the initial values. By increasing t the point $(N_1(t), N_2(t))$ moves in the direct sense on the above closed curve; all these curves include in their interior the point $E_0(a_2/b_2, a_1/b_1)$. The time necessary to cross each curve is finite, but depends on the curve.

We shall denote by (Γ) the family of curves described by the solutions of (1), i.e. (2).

Suppose now that in the ecosystem (N_1, N_2) is introduced a hunter population and that the number of hunted individuals from each population is proportional to the number of all individuals existing in the respective population. Denoting by u the proportionality factor and supposing

$$0 \leq u \leq 1 < a_1, \tag{3}$$

the differential system describing the new ecosystem is:

$$\begin{aligned} \frac{dN_1}{dt} &= N_1(a_1 - u - b_1 N_2), \\ \frac{dN_2}{dt} &= N_2(-a_2 - u + b_2 N_1). \end{aligned} \tag{4}$$