Recall that the vector field v^i is parallel of first order if :

$$\frac{Dv^i}{dt^2} = 0, \tag{2.2}$$

Remark. The case of fields $v^i(x^j)$ for which $Dv^i/dt = 0$, $\tilde{D}^2v^i/dt^2 = 0$ may be either trivial or leads to an impossibility, because these fields are characterized by (2.2) and by the linear, homogeneous system of n equation with n unknows :

$$\frac{1}{2}\{(\Gamma^i_{jkh} + \Gamma^i_{jhk}) - (\underset{01}{\Gamma^i_{jkh}} + \underset{01}{\Gamma^i_{jhk}})\} v^j \frac{dx^k}{dt} \cdot \frac{dx^h}{dt} = 0.$$

It is obvious that this system of equations has geometric character, because

$$\frac{1}{2}\{(\Gamma^i_{jkh} + \Gamma^i_{jhk}) - (\underset{01}{\Gamma^i_{jkh}} + \underset{01}{\Gamma^i_{jhk}})\}$$

is an tensor of type (1.3).

The existence and unicity theorem for parallel vector fields of first order is known. For the parallelism of second order we have :

Theorem 2.2. *Given a curve (C), $x^i = x^i(t)$, $t \in (a, b)$ and in the point $x^i(t_0)$, $t_0 \in (a, b)$, the pair of vectors $(v^i_0, (Dv^i/dt)_0)$ there is an interval $(a_1, b_1) \subset (a, b)$ and a unique vector field defined along the curve $x^i = x^i(t)$, $t \in (a_1, b_1)$, parallel of second order along C and so that $v^i(t_0) = v^i_0$, $(\tilde{D}v^i/dt)(t_0) = (\tilde{D}v^i/dt)_0$.*

The proof is usual from the theory of ordinary differential equations.

Similarly, one can define the geodesics of the first and second order, taking into the precedent considerations, $v^i = dx^i/dt$.

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A CONTROL PROBLEM RELATED TO A VOLTERRA PREY-PREDATOR SYSTEM

BY

ADOLF HAIMOVICI

Introduction. As it is well known, the evolution of a Volterra prey-predator system is described ([6], chap. I) by the differential equations :

$$\begin{aligned} \frac{dN_1}{dt} &= N_1(a_1 - b_1 N_2), \\ \frac{dN_2}{dt} &= N_2(-a_2 + b_2 N_1), \end{aligned} \tag{1}$$

where N_1, N_2 are the numbers of individuals of the prey - respectively the predator-populations and a_1, a_2, b_1, b_2 are positive constants ; the solution of this system is periodical ; in the phase-space (N_1, N_2), each curve $N_1 = N_1(t)$, $N_2 = N_2(t)$ determined uniquely by the initial values N_1^0, N_2^0, is closed ; its equation is

$$N_1^{-a_2} e^{b_2 N_1} = C N_2^{a_1} e^{-b_1 N_2} ; \tag{2}$$

the constant C is determined by the initial values. By increasing t the point $(N_1(t), N_2(t))$ moves in the direct sense on the above closed curve ; all these curves include in their interior the point $E_0(a_2/b_2, a_1/b_1)$. The time necessary to cross each curve is finite, but depends on the curve.

We shall denote by (Γ) the family of curves described by the solutions of (1), i.e. (2).

Suppose now that in the ecosystem (N_1, N_2) is introduced a hunter population and that the number of hunted individuals from each population is proportional to the number of all individuals existing in the respective population. Denoting by u the proportionality factor and supposing

$$0 \leqslant u \leqslant 1 < a_1, \tag{3}$$

the differential system describing the new ecosystem is :

$$\begin{aligned} \frac{dN_1}{dt} &= N_1(a_1 - u - b_1 N_2), \\ \frac{dN_2}{dt} &= N_2(-a_2 - u + b_2 N_1). \end{aligned} \tag{4}$$

To the same system (4) leads the problem of fishing in a region where two species of fishes exist, namely a herbivorous one and a carnivosous one.

Our aim is to *determine the control function* $u: R_+ \to [0, 1]$, *in order that if the initial values of* N_1, N_2 *are respectively* N_1^0, N_2^0, *after a time interval of length* T

a) *the values of* N_1, N_2 *be* N_1^T, N_2^T,

b) *the length of the time interval* T *be minimal.*

Analogous problems, concerning harvesting problems can be found in [1, 2, 3, 5].

2. Preliminaries. To solve this problem (see for instance [4], chap. I) we must consider:

a) the adjoint system:

$$(5)\qquad \begin{aligned} \frac{d\psi_1}{dt} &= -(a_1 - u - b_1 N_2)\,\psi_1 - b_2 N_2 \psi_2, \\ \frac{d\psi_2}{dt} &= b_1 N_1 \psi_1 - (-a_2 - u + b_2 N_1)\,\psi_2. \end{aligned}$$

b) the hamiltonian function:

$$H(N, \psi, u) = N_1 \psi_1 (a_1 - b_1 N_2) + N_2 \psi_2 (-a_2 + b_2 N_1) - u(N_1 \psi_1 + N_2 \psi_2).$$

Denoting

$$(6)\qquad S(N, \psi) = N_1 \psi_1 + N_2 \psi_2,$$

it is obvious that the maximum of H, by given N_1, N_2, ψ_1, ψ_2 and $0 \leqslant u \leqslant 1$, is realized by

$$(7)\qquad \begin{aligned} u &= 0 \quad \text{if} \quad \operatorname{sgn} S = +1, \\ u &= 1 \quad \text{if} \quad \operatorname{sgn} S = -1, \end{aligned}$$

from which it follows that u is a piecewise constant function. Moreover, since ψ_1, ψ_2 is the solution of a linear differential system, with coefficients continuous in each interval where u is constant, it follows that no component of a non-trivial solution can vanish in a convergent set of points $\{t_i\}$, and that the distance between two consecutive values t_i where ψ_i vanishes is strictly positive.

3. Denoting

$$(8)\qquad y_1 = N_1 \psi_1, \qquad y_2 = N_2 \psi_2,$$

the system (5) becomes:

$$(9)\qquad \begin{aligned} \frac{dy_1}{dt} &= -b_2 N_1 y_2, \\ \frac{dy_2}{dt} &= b_1 N_2 y_1, \end{aligned}$$

a particular solution of which is

$$(10)\qquad z_1(u, t) = b_2 N_1 - a_2 - u, \quad z_2(u, t) = b_1 N_2 - a_1 + u.$$

After knowing this solution, the general integral of (9) can be found by quadratures; it is:

$$(11)\qquad \begin{aligned} y_1(u, t) &= (C_1 \varphi_1(0, u, t) + C_2)\, z_1(u, t), \\ y_2(u, t) &= (C_1 \varphi_1(0, u, t) + C_2)\, z_2(u, t) - \frac{C_1}{z_1(u, t)}, \end{aligned}$$

where

$$(12)\qquad \varphi_1(t', u, t) = \int_{t'}^{t} \frac{b_2 N_1(\tau)}{[z_1(u, \tau)]^2}\, d\tau.$$

We will suppose the initial point (N_1^0, N_2^0) chosen so that $z_1(u, 0)$, $z_2(u, 0) < 0$.

Obviously, the above solution is defined only in the interval $[0, t_1)$ where t_1 is the first value of t, for which $z_1(u, t) = 0$.

To prolonge the solution (11), we remark first that, denoting

$$(13)\qquad \varphi_2(t', u, t) = \int_{t'}^{t} \frac{b_1 N_2(\tau)}{[z_2(u, \tau)]^2}\, d\tau,$$

we have

$$(14)\qquad \varphi_1(t', u, t) - \varphi_2(t', u, t) = \frac{1}{z_1(u, t)\, z_2(u, t)} - \frac{1}{z_1(u, t')\, z_2(u, t')},$$

and replacing in (11) φ_1 by φ_2 given in (14), the solution is defined in $[0, t_2)$, t_2 being the first value of t where $z_2(u, t) = 0$.

To prolonge the solution for $t \geqslant t_2$, we will take a point $t' \in (t_1, t_2)$, and then, considering

$$(15)\qquad \varphi_2(0, u, t) = \varphi_2(0, u, t') + \varphi_2(t', u, t).$$

we will replace in (11) $\varphi_2(t', u, t)$ by

$$(16)\qquad \varphi_1(t', u, t) - \frac{1}{z_1(u, t)\, z_2(u, t)} + \frac{1}{z_1(u, t')\, z_2(u, t')},$$

with which (11) makes sense. By the same method, we can prolonge the solution (11) for each value of t.

4. Let now

$$(17)\qquad \psi_i(0) = \psi_i^0, \qquad N_i(0) = N_i^0, \qquad (i = 1, 2),$$

and suppose, chosen N_i^0, as we have already remarked, so that

$$(18)\qquad z_i(u, 0) < 0.$$

Moreover, suppose chosen ψ_1^0 so that

$$N_1^0\psi_1^0+N_2^0\psi_2^0<0. \tag{19}$$

It follows from (7), $u(0)=1$.

Denote by (Δ) the family of solutions of (4), corresponding to $u=1$. It results that the trajectories of this system are like (2):

$$N_1^{-a_2-1}\, e^{b_2 N_1}=C N_2^{a_1-1}\, e^{b_1 N_2}\,; \tag{Δ}$$

let Δ_0 be the trajectory passing through $O(N_1^0, N_2^0)$. From (11), taking into account the initial values, we obtain:

$$y_1(1,t)=\left\{\{(\psi_1^0 N_1^0 z_2(1,0)-\psi_2^0 N_2^0 z_1(1,0))\,\varphi_1(0,1,t)+\frac{\psi_1^0 N_1^0}{z_1(1,0)}\right\} z_1(1,t),$$

$$y_2(1,t)=\left\{(\psi_1^0 N_1^0 z_2(1,0)-\psi_2^0 N_2^0 z_1(1,0))\,\varphi_1(0,1,t)+\frac{\psi_1^0 N_1^0}{z_1(1,0}\right\} z_2(1,t)-\frac{\psi_1^0 N_1^0 z_2(1,0)-\psi_2^0 N_2^0 z_1(1,0)}{z_1(1,t),}, \tag{20}$$

the prolongation of this solution being performed as in the previous paragraph.

With these functions, one can construct

$$\Sigma(1,t,\alpha,\beta)=N_1(1,t)\,\psi_1(1,t)+N_2(1,t)\,\psi_2(1,t)=$$

$$=\alpha\left\{(z_2(1,0)-z_1(1,0))\,\Phi(1,t)+\frac{z_1(1,t)+z_2(1,t)}{z_1(1,0)}\right\}+$$

$$+\beta\left\{(z_1(1,0)+z_2(1,0))\,\Phi(1,t)+\frac{z_1(1,t)+z_2(1,t)}{z_1(1,0)}\right\},$$

where

$$2\alpha=\psi_1^0 N_1^0+\psi_2^0 N_2^0<0$$

$$2\beta=\psi_1^0 N_1^0-\psi_2^0 N_2^0.$$

$$\Phi(1,t=(z_1(1,t)+z_2(1,t))\,\varphi_1(0,1,t)-\frac{1}{z_1(1,t)}=$$

$$=(z_1(1,t)+z_2(1,t))\,\varphi_2(0,1,t)+\frac{1}{z_2(1,t)}-\frac{z_1(1,t)+z_2(1,t)}{z_1(1,0)\,z_2(1,0)}.$$

Obviously, $\alpha<0$; we will take $\alpha=-1$, without restricting the generality and denote $\Sigma(1,t,-1,\beta)$ by $\Sigma(1,t,\beta)$.

5. Our aim is now to find the point $t=\theta(\beta)$ where $\Sigma(1,t,\beta)=0$. We will prove the

Theorem. *If $z_1(1,0)$, $z_2(1,0)<0$, it exists a value t^* of t, $t^*>t_2$, such that, for each β, it exists a unique value $0<\theta(\beta)<t^*$ where $\Sigma(1,\theta(\beta),\beta)=0$ and namely if $\beta>(z_2(1,0)-z_1(1,0))/(z_1(1,0)+z_2(1,0)$, $\theta(\beta)<\bar{t}$, $\bar{t}$ being the value for which $z_1(\bar{t})+z_2(\bar{t})=0$, $z_2(\bar{t})<0$ — and for $\beta<(z_2(1,0)-z_1(1,0))/(z_1(1,0)+z_2(1,0))$, $\theta(\beta)>\bar{t}$.*

Proof. We write $\Sigma(1,t,\beta)$ under the form:

$$\Sigma(1,t,\beta)=A(z_1(1,t)+z_2(1,t))\left\{\varphi_1(0,1,t)-\frac{1}{z_1(1,t)\,[z_1(1,t)+z_2(1,t)]}+\frac{-1+\beta}{z_1(1,0)\,A}\right\}, \tag{21}$$

for $z_1(1,t)+z_2(1,t)\neq 0$.

where

$$A=-(z_2(1,0)-z_1(1,0))+\beta(z_1(1,0)+z_2(1,0)). \tag{22}$$

The function

$$F(t)=\varphi_1(0,1,t)-\frac{1}{z_1(1,t)\,[z_1(1,t)+z_2(1,t)]}+\frac{-1+\beta}{z_1(1,0)\,A}=$$

$$=\varphi_2(0,1,t)+\frac{1}{z_2(1,t)\,[z_1(1,t)+z_2(1,t)]}-\frac{1}{z_1(1,0)\,z_2(1,0)}+\frac{-1+\beta}{z_1(1,0)\,A}$$

is increasing for each t (this can be proved by considering the derivative of F) and

$$F(0)=\frac{-2}{A(z_1(1,0)+z_2(1,0))},$$

$$F(t-0)=+\infty,\quad F(t+0)=-\infty.$$

Suppose now $A<0$, i.e.

$$\beta>\frac{z_2(1,0)-z_1(1,0)}{z_2(1,0)+z_2(1,0)}. \tag{23}$$

In this case, it exists a value $\theta(\beta)<\bar{t}$, such that $F(\theta(\beta))=0$.

If $A>0$, i.e.

$$\beta<\frac{z_2(1,0)-z_1(1,0)}{z_2(1,0)+z_2(1,0)}, \tag{24}$$

taking the prolongation of φ_2, one sees that $F(\bar{\bar{t}}-0)=+\infty$, $\bar{\bar{t}}$ being the first point where $z_1(1,\bar{\bar{t}})+z_2(1,\bar{\bar{t}})=0$ and $z_2(1,\bar{\bar{t}})>0$; and so it is easily seen that $\bar{t}\leqslant\theta(\beta)\leqslant\bar{\bar{t}}$.

6. Denote by t_0 a value of t where

$$\Sigma(1,t,\beta)=0.$$

Then, if $t_0<t_1$,

$$\varphi_1(0,1,t_0)-\frac{1}{z_1(1,t_0)\,[z_1(1,t_0)+z_2(1,t_0)]}+\frac{-1+\beta}{z_1(1,0)\,A}=0, \tag{25}$$

and if $t_1<t_0<t_2$.

$$\varphi_2(0,1\,t_0) + \frac{1}{z_2(1,t_0)\,[z_1(1,t_0)+z_2(1,t_0)]} - \frac{1+\beta}{z_2(1,0)\,A} = 0.$$

Since $\varphi_2(0,1,t) + \dfrac{1}{z_2(1,t)\,[z_1(1,t)+z_2(1,t)]}$ is increasing with respect to t, it follows that, if we mentain β fixed, and take $t_1 > t_0$ we have

$$(25') \qquad \varphi_2(0,1,t_1) + \frac{1}{z_2(1,t_1)\,[z_1(1,t_1)+z_2(1,t_1)]} - \frac{1+\beta}{z_1(1,0)\,A} > 0.$$

Let $P_0 = (N_1(t_0), N_2(t_0))$ be the point of the trajectory $\Delta_0 \subset (\Delta)$ passing through (N_1^0, N_2^0), obtained for $t = t_0$. As in this point $\Sigma(1, t, \beta)$ changes its sign, it follows that for $t > t_0$, u will take the value 0, and the optimal trajectory will be an arc of the curve $\Gamma_0 \subset (\Gamma)$ passing through $(N_1(t_0), N_2(t_0))$; $y_1(0,t)$, $y_2(0,t)$ will be a continuous prolongation of $y_1(1,t)$, $y_2(1,t)$ beginning with the initial values

$$y_1(0,t_0) = y_1(1,t_0),\quad y_2(0,t_0) = y_1(1,t_0).$$

(This will imply the continuity of the functions ψ_1, ψ_2).

These functions are, for $t > t_0$:

$$y_1(0,t) = \Big\{[y_1(1,t_0)\,z_2(1,t_0) - y_2(1,t_0)\,z_1(0,t_0)]\,\varphi_1(t_0,0,t) + $$
$$+ y_1(1,t_0)\,\frac{1}{z_1(0,t_0)}\Big\}\,z_1(0,t)\,.$$

$$y_2(0,t) = \Big\{[y_1(1,t_0)\,z_2(0,t_0) - y_2(1,t_0)\,z_1(0,t_0)]\,\varphi_1(t_0,0,t) + $$
$$+ y_1(1,t_0)\,\frac{1}{z_1(0,t_0)}\Big\}\,z_2(0,t) - \frac{y_1(1,t_0)\,z_2(0,t_0) - y_2(1,t_0)\,z_1(0,t_0)}{z_1(0,t)}\,,$$

and, as a consequence

$$\Sigma(0,t,\beta) = \Big\{[(y_1(1,t_0)\,z_2(0,t_0) - y_2(1,t_0)\,z_1(0,t_0)]\,\varphi_1(t_0,0,t) - $$
$$+ y_1(1,t_0)\,\frac{1}{z_1(0,t_0)}\Big\}\,(z_1(0,t) + z_2(0,t)) + \frac{y_1(1,t_0)\,z_2(0,t_0) - y_2(1,t_0)\,z_1(0,t_0)}{z_1(0,t)}\,.$$

Taking into account (20), (22) and (10), it follows:

$$y_1(1,t_0)\,z_2(0,t_0) - y_2(1,t_0)\,z_1(0,t_0) =$$
$$= -A\,\Big\{\varphi_1(0,1,t_0)\,(z_1(1,t_0) + z_2(1,t_0)) - \frac{z_1(0,t_0)}{z_1(1,t_0)} + $$
$$+ \frac{1-\beta}{z_1(1,0)}\,(z_1(1,t_0) + z_2(1,t_0))$$

and

$$y_1(1,t_0) = A\left(\varphi_1(0,1,t_0) + \frac{-1+\beta}{z_1(1,0)}\right) z_1(1,t_0),$$

and then considering (25), it results

$$y_1(1,t_0)\,z_2(0,t_0) - y_2(1,t_0)\,z_1(0,t_0) = A\,(z_1(1,t_0) + z_2(1,t_0))$$

and

$$y_1(1,t_0) = A\,\frac{1}{z_1(1,t_0) + z_2(1,t_0)}\,,$$

so that (27) becomes.

$$(27) \qquad \Sigma(0,t,\beta) = A\,(z_1(0,t) + z_2(0,t))\,\Big\{\varphi_1(t_0,0,t) - \frac{1}{z_1(0,t)\,[z_1(0,t)+z_2(0,t)]} + \frac{1}{z_1(0,t_0)\,[z_1(0,t_0)+z_2(0,t_0)]}\Big\}\,.$$

If we chose $t_0 = \overline{t}$, it follows

$$\Sigma(1,t,\beta) = -A\,\frac{1}{z_1(1,t)} = 0,$$

and so $A = 0$. It results in this case:

$$\Sigma(1,t,\beta) = \frac{-1+\beta}{z_1(1,0)}\,(z_1(1,t) + z_2(1,t)),$$
$$-(z_2^0 - z_1^0) + \beta(z_2^0 + z_1^0) = 0.$$

This time, Σ changes its sign in $\overline{\overline{t}}$, and his prolongation is

$$\Sigma(0,t,\beta) = \frac{y_1(1,t_0)}{z_1(0,t_0)}\,(z_1(0,t) + z_2(0,t))\,;\ t_0 = \overline{\overline{t}}$$

this prolongation can also be prolonged.

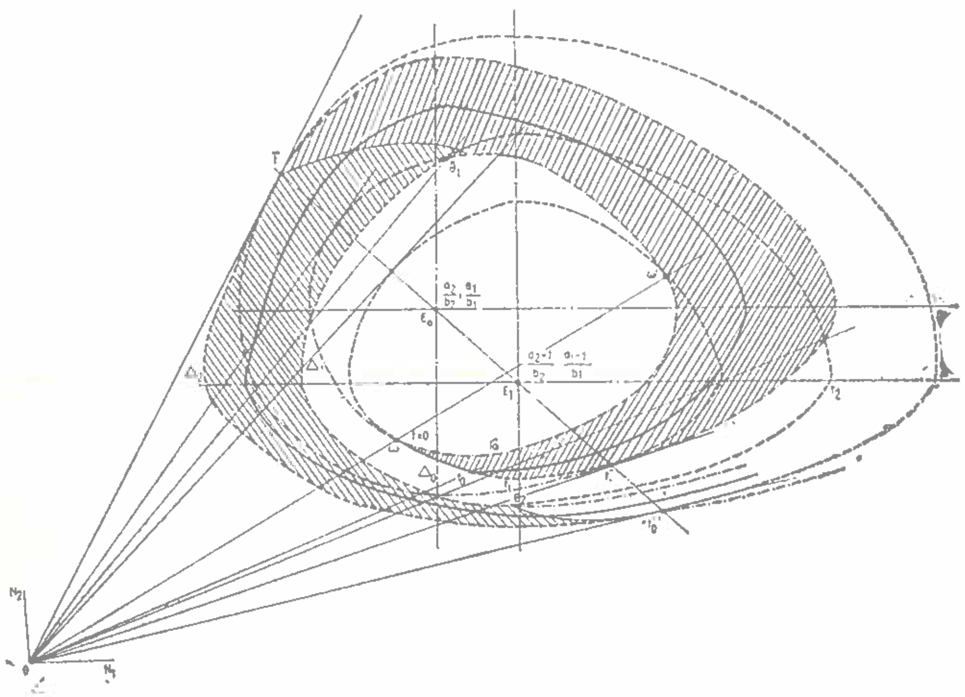
7. Taking into account the tangents at the two arcs of curves Γ_0 and Δ_0 in (N_1^0, N_2^0) one sees that the arc of Γ_0 passes in the interior of the domain bounded by Δ_0. Moreover, it exists a second point of intersection, ω', between Δ_0 and Γ_0, and this point is on the line passing through the origin of the coordinates and the first point of intersection.

Consider now another solution ψ_1, ψ_2 of the adjoint system (5), such that

$$\psi_1^0 N_2^0 + \psi_2^0 N_2^0 > 0.$$

In this case $u = 0$, and the trajectory Γ_0 passing through (N_1^0, N_2^0) will belong to the family (Γ). With this solution we will construct, as in the previous case, the functions $y_1(0, t)$, $y_2(0, t)$ solution of (9), and then

$$\Sigma(0, t, \beta) = y_1(0, t) + y_2(0, t) = [\alpha(z_2(0, 0) - z_1(0, 0)) + \beta(z_2(0, 0) + z_1(0, 0))]$$
$$\left[\varphi_1(0, 0, t)(z_1(0, t) + z_2(0, t)) - \frac{1}{z_1(0, t)}\right] + \frac{\alpha+\beta}{z_1(0, 0)}(z_1(0, t) + z_2(0, t)).$$



To chose between the two trajectories Γ_0 and Δ, we will apply the condition for the optimal trajectory to satisfy the condition

$$M(N, \psi) = \max_u H(N, \psi, u) \geqslant 0,$$

where N and ψ are solutions of (1) and (5) respectively.

The result is stated in the

Theorem. *The optimal trajectory of the stated problem is obtained for* $A \leqslant 0$.

Proof. We have obviously:

$$H(N, \psi, u) = \psi_1 N_1 (a_1 - u - b_1 N_2) + \psi_2 N_2 (-a_2 - u + b_2 N_1) - y_1 z_1 (u, t) +$$
$$+ y_2 z_2 (u, t) = y_2 (u, 0) z_1 (u, 0) - y_1 (u, 0) z_2 (u, 0) =$$
$$= [\alpha (z_2 (u, 0) - z_1 (u, 0))) + \beta (z_1 (u, 0) + z_2 (u, 0))] = -A(u) \geqslant 0.$$

For $u = 1$, $\alpha = (\psi_1^0 N_1^0 + \psi_2^0 N_2^0)/2 < 0$, and $A_{u=1} < 0$, we have

$$\Sigma(1, t_0, \beta) = 0 \text{ for } 0 < t_0 < \bar{t}$$

and

$$\Sigma(1, t, \beta) < 0 \text{ for } 0 < t < t_0,$$

and for $u = 0$, $\alpha = (\psi_1^0 N_1^0 + \psi_2^0 N_2^0)/2 > 0$ and $A_{u=0} < 0$, we have

$$\Sigma(0, t_0, \beta) = 0 \text{ for } \bar{t} \leqslant t_0 \leqslant \bar{t},$$

and

$$\Sigma(0, t, \beta) > 0 \text{ for } 0 \leqslant t < t_0$$

We denote by $\sigma_1, \sigma_2, \ldots$ the curves — sets of points where the functions $\Sigma(0, t, \beta)$, $\Sigma(1, t, \beta)$ change the sign.

8. If we denote by Γ_1 the curve belonging to (Γ) and tangent in $\bar{t}$ to Δ_0, then througheach point of the domain M_1 bounded by the arc of Δ_0 between 0 and t_0, by Γ_0 and Γ_1 and the arc $\sigma_1 = \bar{t}0_1$, it passes an arc of optimal trajectory belonging to the family (Γ), through each point of the domain M_2 bounded by this last arc, by the curves Δ_1 and Δ_2 of (Δ) passing through t and θ_1, respectively and the arc σ_2 it passes an arc of optimal trajectory belonging to Δ, and so on.

It follows that the final point $(N_1^T N_2^T)$ reachable from (N_1^0, N_2^0) belongs to the set $\bigcup M_i$, and the optimal trajectory is formed by arcs of optimal trajectories of the above type.

The same method, can be used to find the optimal trajectories, when the initial values of N_1 and N_2 are in other positions then these considered in the previous paragraph.

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