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AN ASYMPTOTIC DOSING PROBLEM FOR A SYSTEM OF FUNCTIONAL DIFFERENTIAL EQUATIONS

BY

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1. Throughout this note, we shall use as general notations $N=\{0, 1,\dots\}$, $N'=\{1, 2,\dots\}$, $R_+=[0, +\infty[$, $R_+^0=]0, +\infty[$, $\overline{R}_+=[0, +\infty]$, $F(X, Y)=\{f; f: X\to Y\}$, T^n (resp. $T^{(n)}$) $=$ the n-th iterate of $T\in F(X, X)$ (X, Y arbitrary sets, $n\in N$), $C(X, Y)=\{f\in F(X, Y)$; f is continuous$\}$ (X, Y arbitrary topological spaces). A function $f: R_+\to R_+$ is said to have property (P) iff $f(0)=0$ and $f^n(t)\to 0$ as $n\to\infty$, for all $t\in R_+$. The set of all functions $f\in F(R_+, R_+)$ having property (P) will be denoted by P.

Let (X, d) be a generalized metric space [2], and let $T: X\to X$ be a given mapping. By a *(simple) fixed point* of T we mean a point $z\in X$ satisfying $z=Tz$. An useful result concerning (simple) fixed points may be stated as follows.

Theorem 1.1. [5]. *Let (X, d) be a complete g.m.s., and let $T: X\to X$ and $f\in P$ be such that*

$$x, y\in X,\ \tau>0 \text{ and } d(x, y)\leqslant\tau \text{ imply } d(Tx, Ty)\leqslant f(\tau) \tag{1.1}$$

$$X(T)=\{x\in X\ ;\ d(x, Tx)<+\infty\}\neq\varnothing \tag{1.2}$$

Then, there exist $S: X(T)\to X$ and $\rho: X(T)\to R_+$ such that (for every $x\in X(T)$) (a) $X(x)=\{y\in X\ ;\ d(x, y)<+\infty\}$ is T-invariant and Sx is the unique fixed point of T in $X(x)$, (b) the iterative process $(T^m x\ ;\ m\in N)$, $X_{(x)}$ converges to Sx in the sense that $d(T^m x, Sx)\leqslant f^{(m)}(\tau)$, for all $\tau\geqslant\rho(x)$ and all $m\in N$.

2. Again let (X, d) be a g.m.s., and let $T: X^2\to X$ be a given mapping. By a *double fixed point* of T we mean a point $z\in X$ satisfying $z=T(z, z)$. Let $x\in X$. By a *simple fixed point iterative process* generated by x and T we mean a sequence $(x_m\ ;\ m\in N)\subset X$ defined by the recursive procedure

$$x_0=x,\ x_1=T(x_0, x_1),\ x_2=T(x_1, x_2),\dots \tag{2.1}$$

(that is, x_1 is a simple fixed point of $T(x_0,.)$, x_2 is a simple fixed point of $T(x_1,.)$, and so on).

The main aim of this note is to give some sufficient metric conditions implying convergence of a simple fixed point iterative process to a double fixed point on one hand and an evaluation of this convergence, on the other hand. Under these requirements, the main result of this note may be stated as follows.

Theorem 2.1. *Let (X, d) be a complete g.m.s., and let $T: X^2 \to X$ and $f \in P$ be such that*

(2.2) $\quad (x_1, y_1), (x_2, y_2) \in X^2, \ \tau > 0$ *and* $d(x_1, x_2) \leq \tau, \ d(y_1, y_2) \leq \tau$

imply $d(T(x_1, y_1), T(x_2, y_2)) \leq f(\tau)$

(2.3) *for every* $x \in X$ *there is a* $y \in X(x)$ *with* $d(y, T(x, y)) < +\infty$.

Then there exist $S: X \to X$ *and* $\rho: X \to R_+$ *such that (for every* $x \in X$*)* (a) Sx *is the unique double fixed point of* T *in* $X(x)$, (b) *the (unique) simple fixed point iterative process* $(x_m; m \in N) \subset X(x)$ *generated by* x *and* T *converges to* Sx *and the error of approximation is given by* $d(x_m, Sx) \leq f^{(m)}(\tau)$, *for all* $\tau \geq \rho(x)$, $m \in N$.

Proof. Define a mapping $U: X \to P(X) = \{Y \subset X;\ Y \neq \varnothing\}$ by

(2.4) $$Ux = \{y \in X(x);\ y = T(x, y)\}, \quad x \in X.$$

It is a simple matter to verify that (taking into account (2.3) and theorem 1.1 with T replaced by $T(x,.)$) Ux consists of a single element, for every $x \in X$. So, U is an ordinary mapping from X into itself, and, moreover,

(2.5) $$d(x, Ux) < +\infty, \text{ for all } x \in X \text{ (i.e., } X(U) = X).$$

(2.6) $$Ux = T(x, Ux), \text{ for all } x \in X.$$

Now, let $x, y \in X$, $\tau > 0$ be such that $d(x, y) \leq \tau$. From (2.5), $d(Ux, Uy) < +\infty$ and theorefore, taking into account (2.2) + (2.6), $d(Ux, Uy) \leq f$ (max $(\tau, d(Ux, Uy))$), which implies $d(Ux, Uy) \leq f(\tau)$. Hence,

$$x, y \in X, \ \tau > 0, \ d(x, y) \leq \tau \text{ imply } d(Ux, Uy) \leq f(\tau)$$

and therefore, theorem 1.1 is applicable (with T replaced by U), completing the proof. Q.E.D.

Remark. The above result easily extends to simple fixed point iterative processes generated by elements of X^n and mappings from X^n into X and to their corresponding n-fixed points ($n \in N'$ being an arbitrary fixed natural number). On the other hand, a close analysis of the technical aspects of our theorem leads us to the conclusion that it may be compared with some reslts of S. B. P r e s i ć [3] and M. R. T a s k o v i ć [4].

3. In this paragraph, an important application of the main result to an asymptotic dosing problem for a system of functional differential equations will be presented. Let $n \in N'$ be a given natural number and let $(R^n, \|\cdot\|)$ be the euclidean n-dimensional space endowed with a given norm $\|\cdot\|$. We denote by $\leq$ (resp. $<$) the relation in R^n defined by: $x \leq y$ iff $y - x \in R^n_+$ (resp. $x < y$ iff $y - x \in (R^0_+)^n$). Also, for $a, b \in R^n, a < b$, the n-dimensional intervals $[a, b],]a, b], [a, b[,]a, b[$ are defined usually. Furthermore, for a given $\omega > 0$, denote $X = C([0, \omega], R^n)$, $Y = C([0, \omega]^2, R^n)$, $X_0 = C([0, \omega], R_+)$, $Y_0 = C([0, \omega]^2, R_+)$, $X'_0 = \{g \in X_0;\ g^{-1}(R^0_+)$ is dense in $[0, \omega]\}$. For every $x \in X$ define $\|x\| \in X_0$ by

$$\|x\|(t) = \|x(t)\|, \text{ for all } t \in [0, \omega]$$

and, if $g \in X'_0$, define $\|\cdot\|_g: X \to \overline{R}_+$ by

$$\|x\|_g = \inf\{\lambda \in R_+;\ \|x\| \leq \lambda g\}, \text{ if } \{\lambda \in R_+;\ \|x\| \leq \lambda g\} \neq \varnothing$$
$$= +\infty \qquad\qquad\qquad , \text{ if } \{\lambda \in R_+;\ \|x\| \leq \lambda g\} = \varnothing$$

for every $x \in X$. It is simply to verify that $(X, \|\cdot\|_g)$ is a *generalized Banach space* (respectively, a complete g.m.s., by the standard construction of its metric). For every $g \in X'_0$ denote also $X_g = \{x \in X;\ \|x\|_g < +\infty\}$.

Now, let $x \mapsto k(x)$ be a mapping of X onto itself, let $x^0 \in R^n$, $d \in R^n$ and consider the sequence of Cauchy problems

(P.1) $\quad x_1'(t) = k(x_1)(t), \quad t \in [0, \omega]; \quad x_1(0) = x^0 + d$

(P.2) $\quad x_2'(t) = k(x_2)(t), \quad t \in [0, \omega]; \quad x_2(0) = x_1(\omega) + d$

(P.3) $\quad x_3'(t) = k(x_3)(t), \quad t \in [0, \omega]; \quad x_3(0) = x_2(\omega) + d$

. . . .

and, in the same time, the „*bilocal*" Cauchy problem

(BCP) $$x'(t) = k(x)(t),\ t \in [0, \omega];\ x(0) = x(\omega) + d.$$

By an *asymptotic dosing problem* we mean a problem of finding sufficient conditions concerning ω, k, x^0, d, in order that the sequence of solutions of $((P.m);\ m \in N')$ be convergent to a solution of (BCP). The fundamental assumptions we need in our considerations are expressed below.

i) A first assumption we make upon our data is that there exist mappings $x \mapsto A(x)$ of R^n into X and $x \mapsto B(x)$ of X into Y such that, every solution of the Cauchy problem

(CP) $$x'(t) = k(x)(t),\ t \in [0, \omega],\ x(0) = x^0$$

is, necessarily, a solution of the integral equation

(IE) $$x(t) = A(x^0)(t) + \int_0^t B(x)(t, s)\, ds,\ t \in [0, \omega]$$

and viceversa (the existence of such an equivalence relation in a number of particular cases will be discussed in a forthcoming paper). If we admit such an assumption, it is clear that the sequence $(x_m;\ m \in N')$ of solutions of $((P.m);\ m \in N')$ satisfies the recurrence relation

(3.1) $$x_{m+1}(t) = A(x_m(\omega) + d)(t) + \int_0^t B(x_{m+1})(t, s)\, ds,\ t \in [0, \omega]$$

for every $m \in N'$ and therefore, if we define a mapping $T: X^2 \to X$ by

(3.2) $$T(x, y)(t) = A(x(\omega) + d)(t) + \int_0^t B(y)(t, s)\, ds,\ t \in [0, \omega],\ x, y \in X$$

the sequence of functions $(x_m;\ m \in N)$ appears as a simple fixed point iterative process generated by $x_0 \in X$, $x_0(t) = x^0$ $(t \in [0, \omega])$ and T.

ii) A second assumption we make upon our data is that there exists a couple of mappings $x \mapsto a(x)$ of R_+ into X_0 and $x \mapsto b(x)$ of X_0 into Y_0, a $g \in X'_0$ and a $f \in P$, such that

(3.3) $\quad x, y \in R^n, \ \tau > 0, \ \|x - y\| \leq \tau$ implies $\|A(x) - A(y)\| \leq a(\tau)$

(3.4) $x, y \in X^2, h \in X_0, \| x-y \| \leqslant h$ implies $\| B(x)-B(y) \| \leqslant b(h)$

(3.5) $a(\tau g(\omega))(t)+\int_0^t b(\tau g)(t, s)\, ds \leqslant f(\tau)\, g(t),\ t \in [0, \omega].$

In this case, the mapping T defined above is, necessarily, f-contractive in the sense of §2. Indeed, let $(x_1, y_1), (x_2, y_2) \in X^2$ be such that $\| x_1-x_2 \|_g \leqslant \leqslant \tau$ and $\| y_1-y_2 \|_g \leqslant \tau$, $\tau>0$ being a positive number. From the definition of $\|\cdot\|_g$, $\| x_1-x_2 \| \leqslant \tau g$, $\| y_1-y_2 \| \leqslant \tau g$ and then, from (3.3)—(3.5),

$$\| T(x_1, y_1)(t)-T(x_2, y_2)(t) \| \leqslant \| A(x_1(\omega)+d)(t)-A(x_2(\omega)+$$

$$+d)(t) \|+\int_0^t \| B(y_1)(t, s)-B(y_2)(t, s) \| ds \leqslant a(\tau g(\omega))(t)+\int_0^t b(\tau g)(t, s)\, ds \leqslant$$

$$\leqslant f(\tau)\, g(t), \text{ for all } t \in [0, \omega]$$

and so, (again by the definition of $\|\cdot\|_g$)

$$\| T(x_1, y_1)-T(x_2, y_2) \|_g \leqslant f(\tau)$$

which was to be proved.

iii) The last assumption used here is that, for every $x \in X$ there is a $y \in X$ such that

(3.6) $\| x(t)-y(t) \| \leqslant \mu g(t),\ t \in [0, \omega]$ and

(3.7) $\| y(t)-A(x(\omega)+d)(t)-\int_0^t B(y)(t, s)\, ds \| \leqslant \mu g(t),\ t \in [0, \omega]$

for some $\mu>0$. This is equivalent to saying that, for every $x \in X$ there is a $y \in x+X_g$ such that $\| y-T(x, y) \|_g<+\infty$.

Suppose now that assumptions (i)—(iii) are admitted. Then, clearly all hypotheses of the main result are valid and hence, we proved.

Theorem 4.1. *Under the assumptions (i)—(iii), the asymptotic dosing problem formulated above has a positive answer, that is, the sequence $(x_m ; m \in N')$ of solutions of $((P.m) ; m \in N')$ converges in the sense of $\|\cdot\|_g$ to a solution x of the bilocal Cauchy problem (BCP).*

Remark. The asymptotic dosing problem formulated here may be regarded not only as a *convergence process* but also as a *localization* of the limit x of $(x_m ; m \in N')$ with respect to a given n-dimensional interval $]a, b[$ (where $a, b \in R^n$, $a<b$ are given) i.e., the question is to find sufficient conditions implying

(3.8) $a<x(t)<b,\ t \in [0, \omega].$

Concerning this aspect, it should be noted that, from a localization of $x(0)$ we may derive a localization of the whole x by a *flow-invariance technique* [6] for this class of functional differential equations. We also note that, in the *ordinary case, i.e.*, a case defined by

(3.9) $k(x)(t)=f(t, x(t)),\ t \in [0, \omega],\ x \in X$

($f: R_+ \times R^n \to R^m$ being a continuous function) our asymptotic dosing problem may be compared with some results of V. Capra [1], and, in the *linear autonomous ordinary case*, defined by

(3.10) $f(t, x)=Ax,\ x \in R^n$

($A=\| a_{ij} \|_{n \times n}$ being a $n \times n$ real matrix) a number of important aspects of our problem were treated in [7] together with some interesting applications to general pharmacology.

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