Analele ştiinţifice ale Universităţii „Al. I. Cuza" Iaşi
Supliment la tomul XXV, s. I a 1979
Cu ocazia centenarului naşterii Acad. Prof. Al. Myller

THE CAUCHY PROBLEM IN RELATIVISTIC THEORY OF RADIATION FIELD

BY

GH. PROCOPIUC

1. Introduction. In this paper, we first deduce the relativistic equations of a radiation field. Then, we establish an existence and uniqueness theorem for the Cauchy problem for the matter-radiation field schema, as in [1], [2]. To this effect we use a result of L e r a y relating to the hyperbolic systems [3]. We prove that from the system defined by Einsten's, equations, the radiation field equations, together with the conservation conditions, we can get a hyperbolic Leray system, for which the assumptions of the Leray's theorem of existence and uniqueness of solution are satisfied.

In what follows we use the harmonic coordinates introduced by D a r m o i s.

2. The Momentum-Energy of a Radiation Field. Let V_4 be the space-time of general relativity. As we established in the special theory of relativity, in [4], to a radiation field we can associate an energy-momentum tensor :

$$(2.1)\qquad \tau^{(r)}_{\alpha\beta} = \frac{1}{c}\int_0^\infty d\nu \int_{4\pi} I\,(\nu,\Omega)\,\Omega_\alpha \Omega_\beta\, d\Omega, \qquad \alpha, \beta = 0, 1, 2, 3 ;$$

where $I(\nu, \Omega)$ is the specific intensity of radiation field, defined as in [5, p.2], Ω_α a null direction vector,

$$(2.2)\qquad g^{\alpha\beta}\,\Omega_\alpha \Omega_\beta = 0.$$

From (2.2), it follows that :

$$(2.3)\qquad g^{\alpha\beta}\,\tau^{(r)}_{\alpha\beta} = 0.$$

Let $\mathbf{u}$ be an unitary vector at $\mathbf{x} \in V_4$, defining a time-like direction in this point,

$$(2.4)\qquad g^{\alpha\beta}\, u_\alpha u_\beta = 1,$$

and let us consider at $x \in V_4$ an orthonormalized frame $\{\mathbf{e}_{(\rho)}\}$ such that $\mathbf{e}_{(0)} = \mathbf{u}$; $\{\mathbf{e}_{(\rho)}\}$ is a rest frame with respect to $\mathbf{u}$. In this, from (2.1), we have :

$$(2.5)\qquad \tau^{(r)}_{00} = E^{(r)}, \qquad \tau^{(r)}_{0j} = S_j, \qquad \tau^{(r)}_{jk} = \delta_{jk}\, p^{(r)}, \qquad p^{(r)} = \frac{1}{3} E^{(r)}, \; j, k = 1, 2, 3 ;$$

where $E^{(r)}$ is the energy density of radiation field, **S** — the radiative flux. Let us find the condition in order that the energy-momentum tensor (2.5) be a normal tensor [6, p. 15].

An eigenvector $\mathbf{v}^{\beta}$ for $\tau^{(r)}_{\alpha\beta}$ satisfies to the relation: $(\tau^{(r)}_{\alpha\beta} - \lambda g_{\alpha\beta})\, v^{\beta} = 0$, where λ is an eigenvalue of tensor $\tau^{(r)}_{\alpha\beta}$ relative to $g_{\alpha\beta}$, i.e. a root of the equation:

$$(\lambda + p^{(r)})\,[\lambda^2 - (E^{(r)} - p^{(r)})\,\lambda - E^{(r)} p^{(r)} + |\mathbf{S}|^2] = 0, \tag{2.6}$$

where $|\mathbf{S}|^2 = -S^{\alpha} S_{\alpha} \geqslant 0$. If

$$|\mathbf{S}| < 2p^{(r)}, \tag{2.7}$$

all the roots of equation (2.6) are real, and defining ξ and η by:

$$\xi + \eta = 4p^{(r)}, \qquad \xi\eta = |\mathbf{S}|^2, \qquad \xi > \eta$$

we can write:

$$\lambda_1 = \xi - p^{(r)}, \qquad \lambda_1 = \eta - p^{(r)}, \qquad \lambda_3 = \lambda_4 = -p^{(r)}. \tag{2.8}$$

The eigenvectors corresponding to (2.8), up to an arbitrary factor, are:

$$\mathbf{v}_{(0)} = (\xi, S^j), \qquad \mathbf{v}_{(1)} = (\eta, S^j),$$

$$\mathbf{v}_{(2)} = (0, S^2, -S^3, 0), \qquad \mathbf{v}_{(3)} = (0, S^3, 0, -S^1).$$

We obtain:

$$\mathbf{v}^2_{(0)} = \xi^2 - |\mathbf{S}|^2 = \xi(\xi - \eta) > 0, \qquad \mathbf{v}^2_{(1)} = \eta^2 - |\mathbf{S}|^2 = \eta(\eta - \xi) < 0,$$

$$\mathbf{v}^2_{(2)} = -[(S^1)^2 + (S^2)^2] < 0, \qquad \mathbf{v}^2_{(3)} = -[(S^1)^2 + (S^3)^2] < 0.$$

We have from here:

Theorem 2.1. *The energy-momentum tensor of a radiation field that satisfies the condition (2.7) is a normal tensor.*

The spatial flux vector **S** is invariant to transformations of frame $\{\mathbf{e}_{(\rho)}\}$, which leaves $\mathbf{e}_{(0)}$ unchanged. Consequently, we can choose **e** such that $S^2 = S^3 = 0$. Then,

$$\mathbf{e}_{(0')} = \frac{1}{\sqrt{\xi^2 - |\mathbf{s}|^2}}(\xi \mathbf{e}_{(0)} + S^1 \mathbf{e}_{(1)}), \qquad \mathbf{e}_{(1')} = \frac{1}{\sqrt{|\mathbf{s}|^2 - \eta^2}}(\eta \mathbf{e}_{(0)} + S^1 \mathbf{e}_{(1)}),$$

$$\mathbf{e}_{(2')} = \mathbf{e}_{(2)}, \qquad \mathbf{e}_{(3')} = \mathbf{e}_{(3)}$$

it is an orthonormalized frame in **x**, consisting of eigenvectors, that is a principal frame for the tensor $\tau^{(r)}_{\alpha\beta}$. So that we have:

Theorem 2.2. *For any energy-momentum tensor of a radiation field that satisfies the condition (2.7), there is an orthonormalized frame in which $\tau^{(r)}_{\alpha\beta}$ is reduced to the diagonal form,*

$$\| \tau^{(r)}_{\alpha\beta} \| = \begin{Vmatrix} \xi - p^{(r)} & 0 & 0 & 0 \\ 0 & p^{(r)} - \eta & 0 & 0 \\ 0 & 0 & p^{(r)} & 0 \\ 0 & 0 & 0 & p^{(r)} \end{Vmatrix}.$$

Finally, let us observe that in the case $|\mathbf{S}| = 2p^{(r)}$, the characteristic equation (2.6) admits four real roots, $\lambda_1 = \lambda_2 = p^{(r)}$, $\lambda_3 = \lambda_4 = -p^{(r)}$, too. The eigensubspace, corresponding to the double eigenvalue $p^{(r)}$, has the dimension one, the eigenvector generating this subspace being a null vector. Because the eigenvectors corresponding to the double eigenvalue $-p^{(r)}$ are time-like, it results that in this case $\tau^{(r)}_{\alpha\beta}$ is not a normal tensor.

3. Relativistic Radiation Field. Constitutive Equations. Inside matter, we define a radiation field by a symmetric tensor field $\tau^{(r)}$ of order two and a vector field **F**. The tensor $\tau^{(r)}$ is called the energy-momentum tensor of radiation field and the vector **F**—the power-force density of radiation field.

If **u** is an unitary vector defining a time-like direction, we put:

$$E^{(r)} = u^{\alpha} u^{\beta} \tau^{(r)}_{\alpha\beta}, \qquad S^{\alpha} = (g^{\alpha\beta} - u^{\alpha} u^{\beta})\, u^{\gamma} \tau^{(r)}_{\beta\gamma}; \tag{3.1}$$

$E^{(r)}$ is called the energy density of radiation field and **S**—the radiative flux corresponding to **u**. We have:

$$g_{\alpha\beta} u^{\alpha} S^{\beta} = 0. \tag{3.2}$$

Let us consider again at $\mathbf{x} \in V_4$, the rest frame $\{\mathbf{e}_{(\rho)}\}$ with respect to **u**. In this frame, we have $u^0 = 1$, $u^i = 0$ and according to (3.1):

$$E^{(r)} = \tau^{(r)}_{00}, \qquad S_0 = 0, \qquad S_i = \tau^{(r)}_{0i}.$$

Thus, we can rewrite the components of the energy-momentum tensor, given by (2.5), as follows:

$$\tau^{(r)}_{00} = E^{(r)} u_0 u_0 - (g_{00} - u_0 u_0)\, p^{(r)} + u_0 S_0 + u_0 S_0,$$

$$\tau^{(r)}_{0i} = E^{(r)} u_0 u_i - (g_{0i} - u_0 u_i)\, p^{(r)} + u_0 S_i + u_i S_0,$$

$$\tau^{(r)}_{jk} = E^{(r)} u_j u_k - (g_{jk} - u_j u_k)\, p^{(r)} + u_j S_k + u_k S_j,$$

which can be written in an unique tensorial formula,

$$\tau^{(r)}_{\alpha\beta} = E^{(r)} u_{\alpha} u_{\beta} - (g_{\alpha\beta} - u_{\alpha} u_{\beta})\, p^{(r)} + u_{\alpha} S_{\beta} + u_{\beta} S_{\alpha}, \tag{3.3}$$

which is true in an arbitrary frame.

Similarly, in the rest frame $\{\mathbf{e}_{(\rho)}\}$ the vector **F** is given by the constitutive equations [4]:

$$F_0 = -\sigma'_a (E^{(r)} - a T^4), \qquad F_i = -\sigma'_a S_i, \tag{3.4}$$

where T is the proper temperature, a is the radiation constant and σ'_a is the absorbtion coefficient.

We can rewrite the equations (3.4) as

$$F_0 = -\sigma'_a [S_0 + (E^{(r)} - a T^4)\, u_0], \qquad F_i = -\sigma'_a [S_i + (E^{(r)} - a T^4)\, u_i]$$

and thus we obtain,

$$F_{\alpha} = -\sigma'_a [S_{\alpha} + (E^{(r)} - a T^4)\, u_{\alpha}], \tag{3.5}$$

which represents the constitutive equations in an arbitrary frame.

We also observe that,

$$u^\beta F_\beta = -\sigma'_a (E^{(r)} - a T^4), \qquad (g^{\alpha\beta} - u^\alpha u^\beta) F_\beta = -\sigma'_a S^\alpha. \tag{3.6}$$

4. Relativistic Equations of Radiation Field and its Characteristics. We assume that, in a domain of V_4, the radiation field satisfies the equations:

$$\delta \tau^{(r)} = -\mathbf{F}, \tag{4.1}$$

where δ is the operator of codifferentiation for the tensors, that is the contracted covariant derivative, except the sign:

$$\nabla_\alpha \tau^{(r)\alpha\beta} = F^\beta. \tag{4.2}$$

We have, with (3.3):

$$u_\beta \nabla_\alpha \tau^{(r)\alpha\beta} \equiv \nabla_\alpha [(E^{(r)} + p^{(r)}) u^\alpha + S^\alpha] + u^\alpha u^\beta \nabla_\alpha S_\beta - u^\alpha \partial_\alpha p^{(r)} = u^\beta F_\beta. \tag{4.3}$$

$$\begin{aligned}(g^{\alpha\beta} - u^\alpha u^\beta) \nabla_\gamma \tau^{(r)\gamma}_\alpha \equiv (g^{\alpha\beta} - u^\alpha u^\beta) u^\gamma \nabla_\gamma S_\alpha - (g^{\alpha\beta} - u^\alpha u^\beta) \partial_\alpha p^{(r)} + \\ + [(E^{(r)} + p^{(r)}) u^\alpha + S^\alpha] \nabla_\alpha u^\beta + S^\beta \nabla_\alpha u^\alpha = (g^{\alpha\beta} - u^\alpha u^\beta) F_\alpha.\end{aligned} \tag{4.4}$$

Thus, according to (3.6), we obtain for the relativistic equations of radiation field.

$$u^\alpha \partial_\alpha E^{(r)} + \nabla_\alpha S^\alpha + (E^{(r)} + p^{(r)}) \nabla_\alpha u^\alpha - u^\alpha u^\beta \nabla_\alpha u_\beta = -\sigma'_a (E^{(r)} - a T^4), \tag{4.5}$$

$$\begin{aligned}u^\alpha \nabla_\alpha S^\beta - (g^{\alpha\beta} - u^\alpha u^\beta) \partial_\alpha p^{(r)} + (E^{(r)} + p^{(r)}) u^\alpha \nabla_\alpha u^\beta + S^\alpha \nabla_\alpha u^\beta + \\ + S^\beta \nabla_\alpha u^\alpha + u^\alpha u^\beta S^\gamma \nabla_\alpha u_\gamma = -\sigma'_a S^\beta.\end{aligned} \tag{4.6}$$

Now we shall study the Cauchy problem for the gravitation and radiation fields, in the case of a matter-radiation field schema. Giving the values of gravitation and radiation fields on a hypersurface Σ, we intend to determine these fields outside Σ, knowing that the gravitational potentials $g_{\alpha\beta}$ satisfy *the Einstein equations*:

$$S_{\alpha\beta} = \varkappa T_{\alpha\beta}, \qquad \varkappa = const, \tag{4.7}$$

where

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + \tau^{(r)}_{\alpha\beta}, \tag{4.8}$$

ρ being the proper energy density, and the tensor

$$S_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R,$$

according to Bianchi's identities, satisfies:

$$\nabla_\alpha S^\alpha_\beta = 0. \tag{4.9}$$

Because of the unitary character of $\mathbf{u}$, the conservation equations (4.9) are equivalent to the system defined by *the equation of continuity*:

$$\nabla_\alpha (\rho u^\alpha) = \sigma'_a (E^{(r)} - a T^4), \tag{4.10}$$

and *the equations of motion*:

$$\rho u^\alpha \nabla_\alpha u^\beta = \sigma'_a S^\beta. \tag{4.11}$$

On the hypersurface Σ, with the local equation $x^0 = 0$, we give the values of the potentials $g_{\alpha\beta}$ and of their first derivatives, the values of u_α which define an unitary vector and the value of ρ. We assume that these data are such that $g_{\alpha\beta} v^\alpha v^\beta$ is normal hyperbolic, that Σ is not tangent to the elementary cones ($g^{00} \neq 0$) and it is not tangent to $\mathbf{u}$ ($u^0 \neq 0$). We also suppose that the system of local coordinates $\{x^\rho\}$ is harmonic on Σ, that is, the x^ρ are local solutions of the Laplace equation for the Riemannian manifold:

$$H^\rho = \Delta x^\rho = g^{\alpha\beta} \Gamma^\rho_{\alpha\beta} = 0, \qquad \text{for } x^0 = 0. \tag{4.12}$$

We know that we can deduce from the data the values on Σ of the S^0_α. From (4.8) and (3.3), for $\beta = 0$, we have:

$$(\rho + E^{(r)} + p^{(r)}) u^0 u^\alpha + u^0 S^\alpha + u^\alpha S^0 = \varkappa^{-1} S^{0\alpha} + p^{(r)} g^{0\alpha}. \tag{4.13}$$

From here, with (2.4), we obtain:

$$[\rho + E^{(r)} + p^{(r)}] u^0 + S^0 = \varkappa^{-1} S^0_\alpha u^\alpha + p^{(r)} u^0. \tag{4.14}$$

It follows

$$S^\alpha = \frac{1}{u^0} (g^{\alpha\beta} - u^\alpha u^\beta) [\varkappa^{-1} S^0_\beta + p^{(r)} g^0_\beta], \tag{4.15}$$

$$\rho + E^{(r)} + p^{(r)} = \frac{1}{(u^0)^2} (2u^0 u^\beta - g^{0\beta}) [\varkappa^{-1} S^0_\beta + p^{(r)} g^0_\beta]. \tag{4.16}$$

The last equation is a condition for the possible values of $E^{(r)}$ on Σ. If

$$(u^0)^2 + \frac{1}{3} (g^{00} - u^0 u^0) \neq 0, \tag{4.17}$$

we obtain the value of $E^{(r)}$ on Σ, and then the relation (4.15) gives the values of S^α on Σ, which define a vector orthogonal to u_α.

In the harmonic coordinates, the equations (4.7) can be written:

$$R^{(h)}_{\alpha\beta} = \varkappa (T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T), \tag{4.18}$$

where,

$$R^{(h)}_{\alpha\beta} = -\frac{1}{2} g^{\lambda\mu} \partial_{\lambda\mu} g_{\alpha\beta} + f_{\alpha\beta} (g_{\lambda\mu}, \partial_\rho g_{\lambda\mu}). \tag{4.19}$$

The $f_{\alpha\beta}$ are given regular functions, and $T = g^{\alpha\beta} T_{\alpha\beta}$.

We note that, according to (4.11) and (3.2), $u^\alpha u^\beta \nabla_\alpha u_\beta = 0$ and therefore that u_β, which is initially unitary, remains unitary.

We assume that the Cauchy data are given in terms of formal series in the local coordinates and we shall look for formal series corresponding to these Cauchy data and solutions of the system (4.19), (4.10), (4.11), (4.5) and (4.6). Since $g^{00} \neq 0$, from (4.19) we deduce the values on Σ of the derivatives $\partial_{00} g_{\alpha\beta}$. The equation (4.11), for $\beta = 0$, gives for $u^0 \neq 0$ the value on Σ of ∂u^0, then the equation (4.10) gives $\partial_0 \rho$.

From (4.5) and (4.6) for $\beta=0$, we have:

$$
(4.20)\qquad \begin{cases} u^0\partial_0 E^{(r)}+\partial_0 S^0=\varphi \text{ (d.C.)} \\ -\dfrac{1}{3}(g^{00}-u^0u^0)\,\partial_0 E^{(r)}+u^0\partial_0 S^0=\Psi^0 \text{ (d.C.)} \end{cases}
$$

where φ and ψ^0 are functions with known values on Σ. If:

$$
(4.21)\qquad (u^0)^2+\frac{1}{3}(g^{00}-u^0u^0)\neq 0,
$$

that is the same condition as (4.17), from (4.20) we obtain the values on Σ of the derivatives $\partial_0 E^{(r)}$ and $\partial_0 S^0$. For $u^0\neq 0$, the equations (4.11) for $\beta=i$ give $\partial_0 u^i$ and (4.6) for $\beta=i$ give $\partial_0 S^i$.

Thus, under assumptions:

$$
(4.22)\qquad g^{00}\neq 0,\qquad u^0\neq 0,\qquad (u^0)^2+\frac{1}{3}(g^{00}-u^0u^0)\neq 0,
$$

we have obtained on Σ the values of $\partial_{00}g_{\alpha\beta}$, $\partial_0\rho$, $\partial_0 u^\alpha$, $\partial_0 E^{(r)}$, $\partial_0 S^\alpha$ and the same conclusions hold for the determination of the consecutive derivatives; being sufficient to derive in x^0 the different considered equations.

We see that, in these assumptions, the formal series that we looked for are determined in a unique way. If the Cauchy data are analytic, it follows that we obtain one and only one analytic solution of the Cauchy problem corresponding to the system (4.19), (4.10), (4.11), (4.5), (4.6).

If $\varphi=0$ is the local equation of a regular hypersurface, we obtain from (4.22) three types of characteristics: *the gravitational waves*, solutions of equation $g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi=0$, the hypersurface generated by *the stream lines*, solutions of $u^\alpha\partial_\alpha\varphi=0$, and *the radiative wave fronts*, solutions of:

$$
\left[u^\alpha u^\beta+\frac{1}{3}(g^{\alpha\beta}-u^\alpha u^\beta)\right]\partial_\alpha\varphi\partial_\beta\varphi=0\,.
$$

Since the wave velocity **v** with respect to the time-direction defined by **u** is given [4, p. 36] by the formula:

$$
\frac{v^2}{c^2}=-\frac{u^\alpha u^\beta\partial_\alpha\varphi\partial_\beta\varphi}{(g^{\alpha\beta}-u^\alpha u^\beta)\partial_\alpha\varphi\partial_\beta\varphi}\,.
$$

for the radiative wave, we have $v^{(r)}=\pm c/\sqrt{3}$.

5. Existence and Uniqueness Theorem. The unknown functions are $g_{\alpha\beta}$, ρ, u^α, $E^{(r)}$, S^α, that is 20 functions that satisfy the partial differential system defined by *the Einstein equations*:

$$
(5.1)\qquad R^{(h)}_{\alpha\beta}=\varkappa\left[\rho\left(u_\alpha u_\beta-\frac{1}{2}g_{\alpha\beta}\right)+(u_\alpha u_\beta-g_{\alpha\beta})\,p^{(r)}+u_\alpha S_\beta+u_\beta S_\alpha\right],
$$

the equation of continuity:

$$
(5.2)\qquad u^\alpha\partial_\alpha\rho+\rho\nabla_\alpha' u^\alpha=\sigma_a'(E^{(r)}-aT^4)\,,
$$

the equation of motion:

$$
(5.3)\qquad \rho u^\alpha\nabla_\alpha u^\beta=\sigma_a' S^\beta,
$$

and *the equations of radiation field*:

$$
(5.4)\qquad u^\alpha\partial_\alpha E^{(r)}+\nabla_\alpha S^\alpha+(E^{(r)}+p^{(r)})\nabla_\alpha u^\alpha-S_\beta u^\alpha\nabla_\alpha u^\beta=-\sigma_a'(E^{(r)}-aT^4)\,,
$$

$$
(5.5)\qquad \begin{aligned} &u^\alpha\nabla_\alpha S^\beta-(g^{\alpha\beta}-u^\alpha u^\beta)\,\partial_\alpha p^{(r)}+(E^{(r)}+p^{(r)})\,u^\alpha\nabla_\alpha u^\beta+S^\alpha\nabla_\alpha u^\beta+\\ &+S^\beta\nabla_\alpha u^\alpha+u^\alpha u^\beta S_\gamma\nabla_\alpha u^\gamma=-\sigma_a' S^\beta. \end{aligned}
$$

We shall prove the existence and uniqueness of the solution of the Cauchy problem for the system (5.1)—(5.5), using a Leray's theorem for strictly hyperbolic systems [7, § 6]. But the system (5.1)—(5.5) is not a Leray system. Let us first deduce from (5.1)—(5.5) a strictly hyperbolic Leray system.

We substitute for (5.1) the derivatives of these equations along the current lines, the trajectories of **u**, taking into account (5.2):

$$
(1)\qquad -\frac{1}{2}u^\gamma g^{\lambda\mu}\partial_{\lambda\mu\gamma}g_{\alpha\beta}=G_{\alpha\beta}\,(2\text{ in }g^{\alpha\beta},\ 1\text{ in }\rho,\ 0\text{ in }u^\alpha,\ 1\text{ in }E^{(r)},\ 1\text{ in }S^\alpha),
$$

where the notation shows the maximum order of the derivatives appearing in the functions $G_{\alpha\beta}$.

Similarly, from (5.2) we obtain:

$$
(2)\qquad u^\alpha u^\beta\partial_{\alpha\beta}\rho=R(2\text{ in }g^{\alpha\beta},\ 1\text{ in }\rho,\ 1\text{ in }u^\alpha,\ 1\text{ in }E^{(r)},\ 1\text{ in }S^\alpha).
$$

And (5.3) can be rewritten:

$$
(3)\qquad u^\alpha\partial_\alpha u^\beta=C^\beta(1\text{ in }g^{\alpha\beta},\ 0\text{ in }\rho,\ 0\text{ in }u^\alpha,\ 0\text{ in }S^\alpha).
$$

Let us then observe that, according to (5.2), the equatuions (5.4) and (5.5) become:

$$
(5.4')\qquad u^\alpha\partial_\alpha E^{(r)}+\nabla_\alpha S^\alpha+(E^{(r)}+p^{(r)})\nabla_\alpha u^\alpha=\frac{\sigma_a'}{\rho}S_\beta S^\beta-\sigma_a'(E^{(r)}-aT),
$$

$$
(5.5')\qquad \begin{aligned} &-\frac{1}{3}(g^{\alpha\beta}-u^\alpha u^\beta)\,\partial_\alpha E^{(r)}+u^\alpha\nabla_\alpha S^\beta+S^\alpha\nabla_\alpha u^\beta+S^\beta\nabla_\alpha u^\alpha=-\frac{\sigma_a'}{\rho}[(E^{(r)}+\\ &+p^{(r)})\,S^\beta+S_\gamma S^\gamma u^\beta]-\sigma_a' S^\beta. \end{aligned}
$$

We take the contracted covariant derivative ∇_β of (5.5'). Thus, we have:

$$
(5.6)\qquad \begin{aligned} &-\frac{1}{3}(g^{\alpha\beta}-u^\alpha u^\beta)\nabla_\alpha\nabla_\beta E^{(r)}+u^\alpha\nabla_\beta\nabla_\alpha S^\beta+S^\alpha\nabla_\beta\nabla_\alpha u^\beta+S^\beta\nabla_\beta\nabla_\alpha u^\alpha=\\ &=A(1\text{ in }g^{\alpha\beta},\ 1\text{ in }\rho,\ 1\text{ in }u^\alpha,\ 1\text{ in }E^{(r)},\ 1\text{ in }S^\alpha). \end{aligned}
$$

But, according to the Ricci identity,

$$
\nabla_\beta\nabla_\alpha S^\beta=\nabla_\alpha\nabla_\beta S^\beta+R_{\alpha\beta}S^\beta,
$$

it follows that:

$$
\begin{aligned} u^\alpha\nabla_\beta\nabla_\alpha S^\beta&=u^\alpha\nabla_\alpha(\nabla_\beta S^\beta)+u^\alpha R_{\alpha\beta}S^\beta=-u^\alpha u^\beta\nabla_\alpha\nabla_\beta E^{(r)}+\\ &+B(2\text{ in }g^{\alpha\beta},\ 1\text{ in }\rho,\ 1\text{ in }u^\alpha,\ 1\text{ in }E^{(r)},\ 1\text{ in }S^\alpha), \end{aligned}
$$

which replaced in (5.6), gives:

$$\left[u^\alpha u^\beta + \frac{1}{3}\left(g^{\alpha\beta} - u^\alpha u^\beta\right)\right] \nabla_\alpha \nabla_\beta E^{(r)} = S^\alpha \nabla_\beta \nabla_\alpha u^\beta + S^\alpha \nabla_\alpha \nabla_\beta u^\beta + \tag{5.7}$$

$$+ C\,(2 \text{ in } g^{\alpha\beta},\ 1 \text{ in } \rho,\ 1 \text{ in } u^\alpha,\ 1 \text{ in } E^{(r)},\ 1 \text{ in } S^\alpha).$$

Now we take the derivative of (5.7) along the current lines, and using (5.2), we find:

$$u^\gamma \left[u^\alpha u^\beta + \frac{1}{3}(g^{\alpha\beta} - u^\alpha u^\beta)\right] \partial_{\alpha\beta\gamma} E^{(r)} = D\,(3 \text{ in } g^{\alpha\beta},\ 2 \text{ in } \rho, \tag{4}$$

$$2 \text{ in } u^\alpha,\ 2 \text{ in } E^{(r)},\ 2 \text{ in } S^\alpha).$$

We apply now to the equation (5.5′) the differential operator

$$\left[u^\alpha u^\beta + \frac{1}{3}(g^{\alpha\beta} - u^\alpha u^\beta)\right] \nabla_\alpha \nabla_\beta.$$

We obtain:

$$u^\alpha \left[u^\lambda u^\mu + \frac{1}{3}(g^{\lambda\mu} - u^\lambda u^\mu)\right] \nabla_\lambda \nabla_\mu \nabla_\alpha S^\beta - \frac{1}{3}(g^{\alpha\beta} - u^\alpha u^\beta)\left[u^\lambda u^\mu + \right.$$

$$\left. + \frac{1}{3}(g^{\lambda\mu} - u^\lambda u^\mu)\right] \nabla_\lambda \nabla_\mu \nabla_\alpha E^{(r)} + S^\alpha \left[u^\lambda u^\mu + \frac{1}{3}(g^{\lambda\mu} - u^\lambda u^\mu)\right] \nabla_\lambda \nabla_\mu \nabla_\alpha u^\beta +$$

$$+ S^\beta \left[(u^\lambda u^\mu + \frac{1}{3}(g^{\lambda\mu} - u^\lambda u^\mu)\right] \nabla_\lambda \nabla_\mu \nabla_\alpha u^\alpha = E^\beta\,(3 \text{ in } g^{\alpha\beta}, 2 \text{ in } \rho,$$

$$2 \text{ in } u^\alpha,\ 2 \text{ in } E^{(r)},\ 2 \text{ in } S^\alpha).$$

But, with (5.2) and (5.7), it follows that:

$$u^\alpha \left[u^\lambda u^\mu + \frac{1}{3}(g^{\lambda\mu} - u^\lambda u^\mu)\right] \nabla_\lambda \nabla_\mu \nabla_\alpha S^\beta + \frac{1}{3} S^\beta g^{\lambda\mu} \nabla_\lambda \nabla_\mu \nabla_\alpha u^\alpha +$$

$$+ \frac{1}{3} S^\alpha g^{\lambda\mu} \nabla_\lambda \nabla_\mu \nabla_\alpha u^\beta - \frac{1}{3} g^{\alpha\beta} S^\eta \nabla_\alpha \nabla_\lambda \nabla_\eta u^\lambda - \frac{1}{3} g^{\alpha\beta} S^\eta \nabla_\alpha \nabla_\eta \nabla_\lambda u^\lambda = \tag{5.8}$$

$$= F^\beta\,(3 \text{ in } g^{\alpha\beta},\ 2 \text{ in } \rho,\ 2 \text{ in } u^\alpha,\ 2 \text{ in } E^{(r)},\ 2 \text{ in } S^\alpha).$$

Taking now the derivative of (5.8) along the current lines, we have:

$$u^\alpha u^\gamma \left[u^\lambda u^\mu + \frac{1}{3}(g^{\lambda\mu} - u^\lambda u^\mu)\right] \partial_{\alpha\gamma\lambda\mu} S^\beta = G^\beta\,(4 \text{ in } g^{\alpha\beta},\ 3 \text{ in } \rho,\ 3 \text{ in } u^\alpha, \tag{5}$$

$$3 \text{ in } E^{(r)},\ 3 \text{ in } S^\alpha).$$

The system (1)—(5) is a Leray system. The matrix of the system is a diagonal matrix defined by:

$$a(1) = -\frac{1}{2} u^\gamma g^{\lambda\mu} \partial_{\lambda\mu\gamma}; \qquad a(2) = u^\alpha u^\beta \partial_{\alpha\beta}; \qquad a(3) = u^\alpha \partial_\alpha;$$

$$a(4) = u^\gamma \left[u^\alpha u^\beta + \frac{1}{3}(g^{\alpha\beta} - u^\alpha u^\beta)\right] \partial_{\alpha\beta\gamma};\ a(5) = u^\alpha u^\gamma \left[u^\lambda u^\mu + \right.$$

$$\left. + \frac{1}{3}(g^{\lambda\mu} - u^\lambda u^\mu)\right] \partial_{\alpha\gamma\lambda\mu}$$

This system satisfies the assumptions of Leray's theorem on the derivatives with the following indices for the unknowns

$$s(g_{\alpha\beta}) = 5, \quad s(\rho) = 4, \quad s(u^\alpha) = 4, \quad s(E^{(r)}) = 4, \quad s(S^\alpha) = 4,$$

and with the following indices for the equations:

$$t(1) = 3, \quad t(2) = 3, \quad t(3) = 4, \quad t(4) = 2, \quad t(5) = 1.$$

Indeed, the orders of the five operators are 3, 2, 1, 3, 4, respectively. The maximum orders of derivation which can appear in the coefficients of the operators and in the right members, according to previous indices, are:

	$g_{\alpha\beta}$	ρ	u^α	$E^{(r)}$	S^α
(1)	2	1	1	1	1
(2)	2	1	1	1	1
(3)	1	0	0	0	0
(4)	3	2	2	2	2
(5)	4	3	3	3	3

These maximum orders are compatible with the system (1)—(5).

The Cauchy data on Σ are, as we know, the values of the $g^{\alpha\beta}$, of their first derivatives, the values of ρ and u^α. We assume that these data are such that: A). the quadratic form $g_{\alpha\beta}\xi^\alpha\xi^\beta$ is normal hyperbolic; the hypersurface Σ is space-like in each point with respect to the corresponding elementary cone; B). we have on Σ, $H^\rho = 0$; C). the relations $S^0_\alpha = \varkappa T^0_\alpha$ and the values of ρ and u^α define S^β and an admissible value for $E^{(r)}$.

The equations (5.1) give the values on Σ of the second derivatives of the $g_{\alpha\beta}$, the equations (3) the values of the first derivatives of the u^α, and then the equations (5.2) the value of first derivative of ρ. The equations (5.4) and (5.5), in the condition (4.17) give the values of first derivatives of $E^{(r)}$ and S^α. The equation (1) gives the values of third derivatives of the $g_{\alpha\beta}$, (3) the second derivative of ρ, and by differentiation, (3) the second derivatives of u^α. The equation (5.7) gives the second derivative of $E^{(r)}$, and by differentiation, (5.5) the second derivatives of S^α. By differentiation, (1) gives the derivatives of $g_{\alpha\beta}$ of order four, (2) the third derivatives of ρ, and by a new differentiation, (3) gives the third derivatives of u^α. Finally, the equations (4) and (5.8) give the third derivatives of $E^{(r)}$ and S^α. We have thus obtained the values on Σ of the derivatives of order $\leqslant s(\sigma) - 1$. Therefore, we obtain for the system (1)—(5), which is a strictly hyperbolic

Leray system, a Cauchy problem. If the initial data are sufficiently regular, the theorem of Leray shows that:

Theorem 5.1. *In the previous assumptions the Cauchy problem for the system* (1)–(5) *has one and only one solution* $(g^{\alpha}_{\beta}, \rho, u^{\alpha}, E^{(r)}, S^{\alpha})$.

To prove that $(g_{\alpha\beta}, \rho, u^{\alpha}, E^{(r)}, S^{\alpha})$ gives a solution of the system (5.1)–(5.5), let us observe that, after a reason given in [7], if the Cauchy data are analytic, the system defined by these equations has, under our assumptions, one and only one solution, which satisfies necessarily the system (1)–(5) and, according to the choice of the adopted Cauchy data, coincides with the solution of this system given by theorem of Leray. By passing to limit it shows that this last solution satisfies the system (5.1)–(5.5), when the data are not assumed to be analytic, but only sufficiently differentiable.

REFERENCES

1. Fourès-Bruhat Y. — *Théorèmes d'Existence en Mécanique des Fluides Relativistes*, Bull. Soc. Math. France, 86, 1958, pp. 155–175.
2. Lichnerowicz A. — *Théorèmes d'existence et d'unicité pour un fluid thermodynamique relativiste*, C. R. Acad. Sc. Paris, t. 260, 1965, pp. 3291–3295.
3. Leray J. — *Hyperbolic differential equations*, Lecture Notes, Institute for Advanced Study. Princeton, 1953.
4. Procopiuc Gh. — *The Equations of Radiation Field. Invariance Groups*, An. şt. Univ. Iaşi, t. XXIV, f. 2, s. I a, 1978, pp. 389–398.
5. Pomraning G. C. — *The Equations of Radiation Hydrodynamics*, Pergamon Press, New York, 1973.
6. Lichnerowich A. — *Théories Relativistes de la Gravitation et de l'Eelctromagnétisme*, Masson, Paris, 1955.
7. Lichnerowich A. — *Relativistic Hydrodynamics and Magnetohydrodynamics*, W. A. Benjamin, New York, 1967.

Received 21. V. 1979

Departament of Mathematics
Polytechnic Institute, Iaşi
R. S. Romania

Analele ştiinţifice ale Universităţii „Al. I. Cuza" Iaşi
Supliment la tomul XXV, s. I a 1979
Cu ocazia centenarului naşterii Acad. Prof. A. Myller

SUR LES FLOTS DYNAMIQUES

BY

ELEONOR CIUREA

1. Formulation du problème. Soit $G=(N, A)$ un graphe connexe dont N est l'ensemble des sommets et A l'ensemble des arcs. A chaque arc $(x, y)\in A$ on attache les nombres entiers positifs $h(x, y)$ et $c(x, y\,;\,t)$ qui représentent le temps de parcours et respectivement la capacité au moment t.

On note par s la source et avec d le puits du réseau G.

Le problème du flot dynamique maximal pour p périodes de temps peut être formulé de la manière suivante: Déterminer les flux $f(x, y\,:\,t)$ qui doivent satisfaire:

$$\sum_{t=0}^{t=p}\sum_{y\in N}(f(s, y\,;\,t)-f(y, s\,;\,t-h(y, s)))=v(p), \tag{1 a}$$

$$\sum_{y\in N}(f(x, y\,;\,t)-f(y, x\,;\,t-h(y, x)))=0, \tag{1b}$$
$$(x\neq s, d\,; \quad t=0, j, \ldots, p),$$

$$\sum_{t=0}^{t=p}\sum_{y\in N}(f(d, y\,;\,t)-f(y, d\,;\,t-h(y, d)))=-v(p) \tag{1 c}$$

$$0\leqslant f(x, y\,;\,t)\leqslant c(x, y\,;\,t)\,;\ t=0, 1,\ldots, p-h\,(x, y), \tag{1 d}$$

$$\max v(p) \tag{2}$$

Si $f(x, y\,;\,t)$ et $v(p)$ satisfont à (1), on dit que $f(t)$ est un flot dynamique de s à d en G pour p périodes avec la valeur $v(p)$. Si, de plus $v(p)$ est maximal, alors $f(t)$ est un flot dynamique maximal en G pour les p périodes.

Lorsque les capacités se conservent dans le temps, autrement dit pour chaque arc $(x, y)\in A$ on a

$$c(x, y\,;\,t)=c(x, y), \ t=0, 1, \ldots, p-h(x, y). \tag{3}$$

Ford et Fulkerson [2], ont généré un flot dynamique maximal en G, en partant d'un flot statique de G. Dans le cas général, les auteurs ont montré qu'un flot dynamique en G est équivalent au flot statique dans un réseau agrandi $G(p)=(N(p), A(p))$.

La construction du réseau $G(p)$ est la suivante: (i) pour chaque sommet $x\in N$ on définit $p+1$ sommets $x(t)\in N(p)$, $t=0, 1, \ldots, p$ et p arcs