$$l' - l = \partial_{01}^{p,q+1} D_{01}^{p,q} \Delta^{p,q} + D_{01}^{p,q-1} \partial_{01}^{p,q} \Delta^{p,q}$$

où $D_{01}^{p,q} = \sum_i \alpha_i D_{01,i}^{p,q}$.

Etant donnée la (p, q)-chaîne $l = \sum_i \alpha_i \sigma_i$, telle que $\alpha_i \neq 0$, posons $| l | = \bigcup_i | \sigma_i |$, où $| \sigma_i | = \sigma_i(\Delta^{p,q})$. Alors nous pouvons énoncer la

Proposition 2.3. $| l_1 + l_2 | \subseteq | l_1 | \bigcup | l_2 |$; $| \partial_{10}^{p,q} l | \subseteq | l |$ $| \partial_{01}^{p,q} l | \subseteq | l |$; $| al | \subseteq | l |$, $a \in R$.

BIBLIOGRAPHIE

1. Godement R. — *Topologie algébrique et théorie des faisceaux*. Hermann, Paris 1973.
2. Pitiş Gh. — *Une théorie de cohomologie sur la catégorie des variétés feuilletées*. Studia Univ. Babeş-Bolyai (à paraître).
3. Vaisman I. — *Variétés riemanniennes feuilletées*. Czechosl. Math. Journal 21 (1971) 46—75.
4. Vaisman, I. — *Cohomology and differential forms*. Marcel Dekker, Inc., New-York, 1973.

Université de Braşov
Faculté de Mathématiques
Str. Karl Marx 50
2200 BRAŞOV

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Supliment la tomul XXVII, s. I a, 1981

FOLIATIONS OF ALMOST AFFINE SPACES

BY

ANDREI RUGINĂ

To the memory of Professor Mendel Haimovici

1. **Introduction**. The notion of almost affine space was introduced by R. Miron in [10], as a result of his investigations concerning the independence [12, 14, 15, 16, 17] of the systems of axioms for the vector spaces and point-wise affine spaces [20, 23].

Let M be a usual affine space over a skew-field K. One drops the axioms $1 * x = x$, $\forall x$, and $x + y = y + x$, $\forall x, y$ for the tangent vector space TM of the K-affine space M, and obtains a new structure on M, called a structure of K-almost affine space ; TM is called the associated K-almost vector space of M. We have described in [21] a geometrical model for this structure, which has some interesting incidence properties. Having as a main purpose to rediscover these properties in any almost affine space, starting from the axioms, we introduce in this paper some geometrical objects, analogous to the usual affine lines, which satisfy a minimal number of incidence properties, similar to the usual ones. We point out the conditions of compatibility between the modular component uTM of TM, and the groupal one, zTM, which ensure some other natural properties.

2. **K-almost affine spaces.** In this section we recall the definition of the notion of K-almost affine space, and its main properties. Let $M = \{A, B, C, \ldots\}$ be a set whose elements are called *points* and let ρ be an equivalence relation on the set $M \times M$. The quotient set $(M \times M)/\rho = TM$ is called the set of *vectors* of M. Let K be a skew-field, and let 1 be its identity.

Definition 1. *The set $M \neq \Phi$ is called a K-almost affine space if* :

(*a*) *An equivalence relation ρ on the set $M \times M$, which defines the quotient set $(M \times M)/\rho = TM$ of the* **vectors** *of M is given* ;

(*b*) *There exists a mapping $*$: $K \times TM \to TM$; $*(\alpha, x) = \alpha * x$, $\forall \alpha \in K$, $\forall x \in TM$* ;

(*c*) *The following two groups of axioms are verified* :

First group of axioms.

I_1. $\forall A \in M$, $\forall x \in TM$, $\exists B \in M : \overrightarrow{AB} = x$.

I_2. $\forall A, B, A' \in M : \overrightarrow{AB} = \overrightarrow{A'B} \Rightarrow A = A'$.

I_3. $\forall A, B, C, A', B', C' \in M$, $\overrightarrow{AB} = \overrightarrow{A'B'}$, $\overrightarrow{BC} = \overrightarrow{B'C'} \Rightarrow \overrightarrow{AC} = \overrightarrow{A'C'}$.

Second group of axioms.

II_1. $\forall\alpha\in K,\ \forall x, y\in TM,\ \alpha\ast(x+y)=\alpha\ast x+\alpha\ast y$.

II_2. $\forall\alpha,\beta\in K,\ \forall x\in TM,\ (\alpha+\beta)\ast x=\alpha\ast x+\beta\ast x$.

II_3. $\forall\alpha,\beta\in K,\ \forall x\in TM,\ \alpha\ast(\beta\ast x)=(\alpha\,.\,\beta)\ast x$.

where the sum of vectors is defined as follows : let $x, y\in TM$ *and* $A\in M$; $\exists B\in M:\overrightarrow{AB}=x$ *and* $\exists C\in M:\overrightarrow{BC}=y$; *then the vector* $\overrightarrow{AC}$ *is called the sum of the vectors* x *and* y, *and we denote this by* $\overrightarrow{AC}=x+y$.

Definition 2. *The set* M *is called a K-Weylean almost affine space if in definition 1 we replace the axiom* I_3 *by (Raschewsky's) axiom :*

I_3' $\quad\forall A, B, A', B'\in M:\overrightarrow{AB}=\overrightarrow{A'B'}\Rightarrow\overrightarrow{AA'}=\overrightarrow{BB'}$.

Definition 3. *The set of vectors* TM *of a K-almost affine space* M *is called its associated (tangent) K-almost linear (vector) space.*

Theorem 1. *Let* M *be a K-almost affine space,* TM *its tangent K-almost linear space, and* $zTM=\{x\in TM:\alpha\ast x=0,\ \forall\alpha\in K\}$, $uTM=\{x\in TM:\exists y\in TM:x=1\ast y\}$. *Then :*

(a) zTM *is an almost normal subspace of* TM, *in the sense that* $(zTM, +)$ *is a normal subgroup of* $(TM, +)$

(b) uTM *is a K-vector space.*

(c) $TM=zTM+uTM$, *where the sum is semidirect.*

(d) $TM/zTM\simeq uTM$.

(e) *The mapping* $P:TM\to uTM$ *defined by* $P(x)=1\ast x$ *is a projection.*

Theorem 2. *The sum* $TM=zTM+uTM$ *is direct iff* uTM *is a normal K-almost subspace of* TM.

One defines in a natural manner the notion of K-almost affine subspace of a given K-almost affine space M.

Definition 4. *A family* $\mathcal{F}=\{M_i\}_{i\in I}$ *of almost affine subspaces of a K-almost affine space* M *is called a foliation of* M, *if :*

(a) $\forall i\in I,\ M_i\neq\varnothing$;

(b) $\forall A\in M,\ \exists i_0\in I:A\in M_{i_0}$;

(c) $\forall i, j\in I:M_i\| M_j$, *where the parallelism* $M_i\| M_j$ *means* $TM_i\subset TM_j$.

An element $M_i\in\mathcal{F}$ *is called a leaf of the foliation* $\mathcal{F}$.

Let $\mathcal{F}_0, \mathcal{F}_1$ be the foliations of the subspaces having as tangent K-almost linear spaces zTM, respectively uTM.

Theorem 3. *The intersection of two leafs* $M_0\in\mathcal{F}_0$ *and* $M_1\in\mathcal{F}_1$ *contains one point and only one.*

Theorem 4. (First fundamental theorem). *The K-almost affine space* M *is generated by the set of points* $M_0\cup M_1$, *where* $M_i\in\mathcal{F}_i$ $(i=0, 1)$.

A foliation $\mathcal{F}$ of M is called *normal* if the K-almost linear subspace associated to any leaf of $\mathcal{F}$ is a normal subspace of TM. If the foliation $\mathcal{F}$ is normal, one defines the quotient K-almost affine space $M/\mathcal{F}$.

Theorem 5. (Second fundamental theorem) *The quotient almost affine subspace* $M/\mathcal{F}_0$ *is isomorphic to a fixed leaf* $M_1\in\mathcal{F}_1$, *and* M_1 *is a K-affine space.*

Theorem 6. (Third fundamental theorem). *The K-almost affine space* M *is the direct product of a fixed leaf* $M_0\in\mathcal{F}_0$ *with a fixed leaf* $M_1\in\mathcal{F}_1$ *iff* $\mathcal{F}_1$ *is a normal foliation.*

3. Almost affine lines. In the remainder of this paper we shall consider, for simplicity, that M is a K-almost affine space with the property that uTM is a one-dimensional K-linear space, that is, $\forall u\neq 0$, $uTM=\{\lambda\ast u:\lambda\in K\}$; hence, the leaves of the foliation $\mathcal{F}_1$ are K-affine lines. Let $0\in M$ be a fixed point.

Theorem 7. $\mathcal{F}_0=\{Z(u)\}_{u\in uTM}=\{\widetilde{Z}(u)\}_{u\in uTM}$, *where* $Z(u)$ *and* $\widetilde{Z}(u)$ *are sets of points defined by :*

$$Z(u)=\{P\in M:\overrightarrow{OP}=z+u,\ z\in zTM\},\ \widetilde{Z}(u)=\{P\in M:\overrightarrow{OP}=u+z,\ z\in zTM\}.$$

Proof. Let A be an arbitrary point of M ; we denote by $M_0(A)$ the (unique) leaf of $\mathcal{F}_0$ through A. The vector $\overrightarrow{OA}=x$ can be decomposed uniquely as follows : $x=z_A+u=u+\tilde{z}_A$, where $u\in uTM$, $z_A, \tilde{z}_A\in zTM$ (generally, $z_A\neq\tilde{z}_A$). Hence $A\in Z(u)$ and $A\in\widetilde{Z}(u)$. $\forall B\in M_0(A)$ we have $\overrightarrow{AB}=z\in zTM$ and $\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}=z_A+u+z=z_A+z'+u\Rightarrow B\in Z(u)$, and $\overrightarrow{OB}=u+\tilde{z}_A+z\Rightarrow B\in\widetilde{Z}(u)$. Hence $M_0(A)\subseteq Z(u)$ and $M_0(A)\subseteq\widetilde{Z}(u)$. Conversely, $\forall B\in Z(u)$ we have $\overrightarrow{OB}=z_B+u=u+\tilde{z}_B\Rightarrow B\in\widetilde{Z}(u)$. Then $\overrightarrow{AB}=-\overrightarrow{OA}+\overrightarrow{OB}=-(u+\tilde{z}_A)+u+\tilde{z}_B=-\tilde{z}_A-u+u+\tilde{z}_B=-\tilde{z}_A+\tilde{z}_B\in\tilde{z}TM\Rightarrow B\in M_0(A)\Rightarrow Z(u)\subseteq M_0(A)$. In the same way we show that $\tilde{Z}(u)\subseteq M_0(A)$. The two pairs of inclusions show us that $M_0(A)=Z(u)=\widetilde{Z}(u)$. Now, when A gets over M, $x=\overrightarrow{OA}$ gets over TM and $u=1\ast x$ gets over uTM, hence $\mathcal{F}_0=\{Z(u)\}_{u\in uTM}=\{\tilde{Z}(u)\}_{u\in uTM}$.

We shall call the sets of points of the form $Z(u)$ or $\widetilde{Z}(u)$, Z-almost affine lines of M, in short, *z-lines*.

Let us consider now the sets of points : $D(z_l, z_r)=\{P\in M:\overrightarrow{OP}=z_l+\lambda\ast u+z_r,\ \lambda\in K\}$, where $z_r, z_r\in zTM$, and u is a fixed vector belonging to $uTM/\{0\}$.

Theorem 8. $\mathcal{F}_1=\{D(z, 0)\}_{z\in zTM}$.

Proof. Let $A\in M$ be an arbitrary point and let $M_1(A)$ be the leaf of $\mathcal{F}_1$ through A. Then $\overrightarrow{OA}=x$ can be decomposed uniquely :

$x=z_A+\lambda_A\ast u=\lambda_A\ast u+\tilde{z}_A\Rightarrow A\in D(z_A, 0)$; $\forall B\in D(z_A, 0)$, $\overrightarrow{OB}=z_A+\lambda_B\ast u\Rightarrow\overrightarrow{AB}=-\overrightarrow{OA}+\overrightarrow{OB}=-\lambda_B\ast u-z_A+z_A+\lambda_A\ast u=(-\lambda_B+\lambda_A)\ast u\in uTM\Rightarrow B\in M_1(A)\Rightarrow D(z_A, 0)\subseteq M_1(A)$. Conversely, $\forall B\in M_1(A)\Rightarrow\overrightarrow{AB}\in uTM\Rightarrow\exists\lambda\in K:\overrightarrow{AB}=\lambda\ast u$; from $\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}$ we have $OB=z_A+\lambda_A\ast u+\lambda\ast u=z_A+(\lambda_A+\lambda)\ast u\Rightarrow B\in D(z_A, 0)\Rightarrow M_1(A)\subseteq D(z_A, 0)$. Hence $M_1(A)=D(z_A, 0)\Rightarrow\mathcal{F}_1=\{D(z, 0)\}_{z\in zTM}$.

Let us consider now the families of sets of points :

$\mathcal{F}(z^*)=\{D(z, z^*)\}_{z\in zTM}$ and $\widetilde{\mathcal{F}}(z^*)=\{D(z^*, z)\}_{z\in zTM}$; by theorem 8 above we have $\mathcal{F}(0)=\mathcal{F}_1$.

Theorem 9. *Any family of the form* $\mathcal{F}(z^*)$ *or* $\tilde{\tilde{\mathcal{F}}}(z^*)$, $z^* \in zTM$ *is a partition of* M.

Proof. Let $A \in M$ be an arbitrary point, and $\overrightarrow{OA} = x$; the vector $y = x - z^*$ has a unique decomposition $y = z_A + \lambda_A \ast u \Rightarrow x = z_A + \lambda_A \ast u + z^* \Rightarrow A \in D(z_A, z^*)$, with $D(z_A, z^*)$ uniquely determined in $\mathcal{F}(z^*)$. Let $A \in D(z_1, z^*) \cap D(z_2, z^*) \Rightarrow \overrightarrow{OA} = z_1 + \lambda_1 \ast u + z^* = z_2 + \lambda_2 \ast u + z^* \Rightarrow z_1 + \lambda_1 \ast u = z_2 + \lambda_2 \ast u \Rightarrow z_1 = z_2$ and $\lambda_1 = \lambda_2$; hence two elements of $\mathcal{F}(z^*)$ are disjoint or coincident. In the same way we see that $\tilde{\mathcal{F}}(z^*)$ is a partition of the space M.

Theorem 10. *Any leaf* $Z(u_0) \in \mathcal{F}_0$ *meets every set* $D(z_l, z_r)$ *in exactly one point.*

Proof. The point $P_0 \in D(z_l, z_r)$ given by $\overrightarrow{OP_0} = z_1 + u_0 + z_r$ belongs to $Z(u_0)$, because $u_0 + z_r = \tilde{z}_r + u_0$, $\tilde{z}_r \in zTM$, $\tilde{z}_r$ uniquely determined $\Rightarrow \overrightarrow{OP_0} = z_l + \tilde{z}_r + u_0 \Rightarrow P_0 \in Z(u_0)$, and P_0 is uniquely defined.

Theorem 11. *The set of the vectors of any* $D(z, z^*) \in \mathcal{F}(z^*)$ *is of the form* $V = \{-z^* + \lambda \ast u + z^*,\ \lambda \in K\} = -z^* + uTM + z^*$, *and* V *is a subgroup of* $(TM, +)$.

Proof. $\forall A, B \in D(z, z^*)$, $\overrightarrow{OA} = z + \lambda_A \ast u + z^*$, $\overrightarrow{OB} = z + \lambda_B \ast u + z^* \Rightarrow \overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB} = -z^* - \lambda_A \ast u - z + z + \lambda_B \ast u + z^* = -z^* + (-\lambda_A + \lambda_B) \ast u + z^* \in V$; conversely, $\forall x \in V$, $x = -z^* + \lambda \ast u + z^*$, we have the points P_0 and P_1 given by $\overrightarrow{OP_0} = z + 0 \ast u + z^*$, $\overrightarrow{OP_1} = z + \lambda \ast u + z^*$, $P_0, P_1 \in D(z, z^*)$, and $\overrightarrow{P_0P_1} = -\overrightarrow{OP_0} + \overrightarrow{OP_1} = -z^* - z + z + \lambda \ast u + z^* = -z^* + \lambda \ast u + z^* = x$; finally,, $\forall x_1, x_2 \in V$, $x_1 = -z^* + \lambda_1 \ast u + z^*$, $x_2 = -z^* + \lambda_2 \ast u + z^*$, we have $-x_1 + x_2 = -z - \lambda_1 \ast u + z^* - z^* + \lambda_2 \ast u + z^* = -z^* + (-\lambda_1 + \lambda_2) \ast u + z^* \in V \Rightarrow V$ is a subgroup of $(TM, +)$. In the same way one proves:

Theorem 12. *The set of the vectors of* $D(z^*, z) \in \tilde{\mathcal{F}}(z^*)$ *is of the form* $V = \{-z + \lambda \ast u + z,\ \lambda \in K\} = -z + uTM + z$, *and* V *is a subgroup of* $(TM, +)$.

Theorem 13. $\forall D(z_l, z_r)$ *there exists an application* $\tilde{\ast} : K \times TM \to TM$ *such that* M *is a* K*-almost affine space with respect to the same equivalence relation* ρ *and to the scalar multiplication* $\tilde{\ast}$, *and* $D(z_l, z_r) \in \tilde{\mathcal{F}}_1$, *where* $\tilde{\mathcal{F}}_1$ *is the maximal unitary foliation of the new structure.*

Proof. Let us take $\alpha \tilde{\ast} x = -z_r + \alpha \ast x + z_r$, $\forall \alpha \in K, \forall x \in TM$; then, $(TM, +, \tilde{\ast})$ is a K-almost linear space: $\forall \alpha, \beta \in K$, $\forall x, y \in TM$: $\alpha \tilde{\ast} (x + y) = -z_r + \alpha \ast (x + y) + z_r = -z_r + \alpha \ast x + \alpha \ast y + z_r = -z_r + \alpha \ast x + z_r - z_r + \alpha \ast y + z_r = \alpha \tilde{\ast} x + \alpha \tilde{\ast} y$, and the axiom II_1 is verified; $(\alpha + \beta) \tilde{\ast} x = -z_r + (\alpha + \beta) \ast x + z_r = -z_r + \alpha \ast x + \beta \ast x + z_r = -z_r + \alpha \ast x + z_r - z_r + \beta \ast x + z_r = \alpha \tilde{\ast} x + \beta \tilde{\ast} x$, which is axiom II_2; $\alpha \tilde{\ast} (\beta \tilde{\ast} x) = -z_r + \alpha \ast (\beta \tilde{\ast} x) + z_r = -z_r + \alpha \ast (-z_r + \beta \ast x + z_r) + z_r = -z_r + \alpha \ast (\beta \ast x) + z_r = -z_r + (\alpha \cdot \beta) \ast x + z_r = (\alpha \cdot \beta) \tilde{\ast} x$, and the axiom II_3 is verified, too.

By theorem 11, the set of the vectors of $D(z_l, z_r)$ is $V = -z + uTM + z_r$; we have $1 \tilde{\ast} x = x \Leftrightarrow -z_r + 1 \ast x + z_r = x \Leftrightarrow x \in V$; this means that, for the new structure, $\widetilde{uTM} = \{x : 1 \tilde{\ast} x = x\} = -z_r + uTM + z_r$, and $D(z_l, z_r) \in \tilde{\mathcal{F}}_1$, where $\tilde{\mathcal{F}}_1$ is determined by $\widetilde{uTM}$.

All these properties allow us to call the sets of points of the form $D(z_l, z)$ *u-lines*, and open the interest for their closer study.

4. Conditions of compatibility. Generally, we can construct a K-almost affine space M starting from any group zTM and any K-vector space uTM (see F. Radó's theorem [Th. 3.8, 10]). But, in order to have some natural properties, as „two distinct points determine a unique line passing through them", and „any two non-parallel lines meet in exactly one point", we must impose some restrictions upon zTM and uTM. This is the aim of the present section.

Theorem 14. *Two distinct u-lines* $D(z_l, z_r)$ *and* $D(z_1, z_r)$, $z_l' \neq z_r'$ *or* $z_l' \neq z_r'$ *meet in at most one point iff* TM *has the property*: $u + z = z + u \Rightarrow z = 0$ *or* $u = 0$.

Proof. Let us assume that $u_0 + z_0 = z_0 + u_0$, with $u_0 \neq 0$ and $z_0 \neq 0$. Then the points $P_0 \neq P_0^*$ given by $\overrightarrow{OP_0} = 0 + z_0 = z_0 + 0$, and $\overrightarrow{OP_0} = u_0 + z_0 = z_0 + u_0$ belong both to the intersection $D(0, z_0) \cap D(z_0, 0)$. Conversely, let us assume that the condition $u + z = z + u \Rightarrow z = 0$ or $u = 0$ is satisfied. If there exist two u-lines $D(z_l, z_r)$ and $D(z_l', z_r')$, having two common points $P_1 \neq P_2$, we have $\overrightarrow{OP_1} = z_l + \lambda_1 \ast u + z_r \ast = z_l' + \lambda_1 \ast u + z_r'$ and $\overrightarrow{OP_2} = z_l + \lambda_2 \ast u + z_r = z_l' + \lambda_2 \ast u + z_r'$; then, $\overrightarrow{P_1P_2} = -\overrightarrow{OP_1} + \overrightarrow{OP_2} = -z_r + (-\lambda_1 + \lambda_2) \ast u + z_r = -z_r' + (-\lambda_1 + \lambda_2) \ast u + z_r' \Rightarrow z_r' - z_r + (-\lambda_1 + \lambda_2) \ast u = (-\lambda_1 + \lambda_2) \ast u + z_r' - z_r \Rightarrow \lambda_1 = \lambda_2$ or $z_r = z_r'$. But $\lambda_1 = \lambda_2 \Rightarrow \overrightarrow{OP_1} = \overrightarrow{OP_2}$, that is $P_1 = P_2$; $z_r = z_r'$ means that $D(z_l, z_r)$ and $D(z_l', z_r')$ belong both to $\mathcal{F}(z_r)$ and have no common points.

Theorem 15. *Two u-lines* $D(z_l, z_r)$ *and* $D(z_l', z_r')$, *with* $z_l \neq z_l'$ *and* $z_r \neq z_r'$ *have at least one common point iff the equation* $u + z = \tilde{z} + u$ *has a solution* $u \in uTM$, $\forall z, \tilde{z} \in zTM$.

Proof. Let us consider the equation $u + z = \tilde{z} + u$, $z, \tilde{z} \in zTM$; if $z = 0 \Rightarrow \tilde{z} = 0$, and the equation reduces to $u = u$. If $z \neq 0 \Rightarrow \tilde{z} \neq 0$; the u-lines $D(0, z)$, $D(\tilde{z}, 0)$ meet in at least one point P_0; $\overrightarrow{OP_0} = u_0 + z = \tilde{z} + u_0 \Rightarrow$ the given equation has at least the solution u_0. Conversely, let $D(z_l, z_r)$ and $D(z_l', z_r')$ be two u-lines with $z_l \neq z_l'$ and $z_r \neq z_r'$; the equation $-z_l' + z_l + u = u + z_r' - z_r$ has a solution u_0; this means that the point P_0 given by $\overrightarrow{OP_0} = z_l + u_0 + z_r = z_l' + u_0 + z_r'$ is a common point of the two u-lines. Combining theorems 14 and 15, one obtains:

Theorem 16. *Two u-lines* $D(z_l, z_r)$ *and* $D(z_l', z_r')$ *with* $z_l \neq z_l'$ *and* $z_r \neq$ *f*z_r' *meet in a unique point iff the equation* $u + z = \tilde{z} + u$ *has a unique solution* $u \in uTM$, $\forall \tilde{z}, z \in \tilde{z}TM$.

In the conditions of theorem 16, two u-lines $D(z_l, z_r)$ and $D(z_l', z_r')$ are in one of the following four relative positions:

(1) If $z_l = z_l'$ and $z_r = z_r'$, they are *identical*.

(2) If $z_l \neq z_l'$ and $z_r = z_r'$, they have no common point (being elements of $\mathcal{F}(z_r)$), and we call them *parallel*, denoted by $D(z_l, z_r) \| D(z_l', z_r')$.

(3) If $z_l = z'_l$ and $z_r \neq z'_r$, they have no common points, as elements of $\mathcal{F}(z_1)$; we call two such u-lines *co-parallel*, and denote this by $D(z_1, z_r) \# \# D(z'_l, z'_r)$.

(4) If $z_l \neq z'_l$ and $z_r \neq z'_r$, they meet in exactly one point ; we call them *secant*.

We shall call almost affine line, in short, *line*, any z-line or u-line.

Theorem 17. *Two distinct points determine a unique common line in the conditions of theorem* 16 *iff* $\forall x \in TM \setminus zTM$, $\exists z \in zTM$: $x = -z + 1 \divideontimes x + z$.

Proof. Let $P_1 \neq P_2$ be two arbitrary distinct points. If $\overrightarrow{P_1P_2} \in zTM$, there exists a unique z-line passing through P_1, P_2. Otherwise, we have $\overrightarrow{OP_1} = x_1$ and $\overrightarrow{OP_2} = x_2$; $\overrightarrow{P_1P_2} = -x_1 + x_2 \Rightarrow \exists z_0 \in zTM$ with $-x_1 + x_2 = -z_0 + 1 \divideontimes (-x_1 + x_2) + z_0 = -z_0 + (-u_1 + u_2) + z_0$; but, by the decomposition theorem, $x_1 - z_0 = z_1 + u_1$ and $x_2 - z_0 = z_2 + u_2 \Rightarrow x_1 = z_1 + u_1 + z_0$ and $x_2 = z_2 + u_2 + z_0 \Rightarrow -x_1 + x_2 = -z_0 - u_1 - z_1 + z_2 + u_2 + z_0$. Equating the two relations, we have $-z_1 + z_2 = 0 \Rightarrow z_1 = z_2 \Rightarrow P_1, P_2 \in D(z_1, z_0)$; this line is unique by theorem 16. Conversely, let $x \in TM \setminus zTM$ be an arbitrary vector, and let us consider the equation $x = -z + u + z$, where $u = 1 \divideontimes x$. Let O, A be the points defined by $\overrightarrow{OO} = 0$, $\overrightarrow{OA} = x$; there exists a unique line $D(z_l, z_r)$ through $O, A \Rightarrow \overrightarrow{OO} = 0 = z_l + z_r$ and $\overrightarrow{OA} = x = z_l + u + z_r \Rightarrow z_l = -z_r$ and z_r is a unique solution of the above equation.

BIBLIOGRAPHY

1. Brânzei, D. — *On some categories of Miron Spaces*. Lucr. col. național de geom. și topologie, Timișoara, 1977.
2. Cruceanu, V. — *Sur les espaces G-affines*, An. Univ. Timișoara, s. șt. mat., Vol. XVIII f. 1 (1980), 25—36.
3. Dembowski, P. — *Finite geometries*, Springer, Berlin, 1968.
4. Gheorghiev, Gh. — *Despre grupuri geometrice și evoluția ideii de spațiu*. Probleme actuale de matematică, E.D.P., București, 1963.
5. Gheorghiu, N. — *Projecteurs sur des espaces linéaires non-commutatifs et sans la propriété* $1 \cdot x = x$, An. șt. Univ. Iași, T. XXIII, s. I. a, f. 2, 1977, 263—265.
6. Golab, S. et Miron, R. — *Sur quelques généralisations des espaces vectoriels*, Rev. Roum. Math. Pures et Appl., 16 (1971), 1477—1496.
7. Hartshorne, R. — *Foundations of projective geometry*, Benjamin, New York, 1967.
8. Hughes, R. D., Piper, C. F. — *Projective planes*, Springer, New York, 1973.
9. Liebeck, H. — *The vector space axiom* $1 \cdot v = v$, The Math. Gazette, LVI, Nr. 395, 1972, 30—33.
10. Miron, R. — *Almost affine spaces over a field*, An. șt. Univ. Iași, T. XXIV, s. I. a, f. 2, 1978, 327—344.
11. Miron, R. — *On the almost linear spaces*, Mathematica, Univ. Cluj-Napoca, t. 18 (41), 2, 1976, 187—190.
12. Miron, R. — *Sur l'indépendence des axiomes d'un espace linéaire*, An. șt. Univ. Iași, s. I. mat., 10 (1964), 281—286.
13. Miron, R. — *Sur quelques structures affines*, Demostratio mathematica, 6, 1973, 289—294.
14. Miron, R. — *The minimality of Weyl's system of axioms for the affine geometry over an unitary ring*, An. șt. Univ. Iași, T. XXVI, s. I. a, f. 1, 1978, 15—19.
15. Miron, R. et Opaiț, Gh. — *Sur l'axiomatique de l'espace affine à n dimensions*, An. șt. Univ. Iași, 10 (1964), 115—119.

16. Miron, R. et Opaiț, Gh. — *Sur les espaces affines généralisés*, Rev. Roum. Math. Pures et Appl., 14 (1969), 633—660.
17. Miron, R. and Radó, F. — *On the definition of a module and a vector space by independent axioms*, Mathematica, Univ. Cluj-Napoca, 18 (41), 2, 1976, 179—186.
18. Miron, R. and Ștefănescu, M. — *Near-modules over special near-rings*. An. șt. Univ. Iași, T. XXIII, s. I. a, f. 1, 1977, 29—32.
19. Radó, F. — *On R-linear spaces* (to appear).
20. Raschewsky, P. K. — *Riemannische Geometrie und Tensoranalysis*, Berlin, 1969.
21. Rugină, A. — *A geometrical almost affine model*, An. șt. Univ. Iași, T. XXVIII, s. I. a, f. 1, 1982, 79—89.
22. Ștefănescu, M. and Miron, R. — *On distributive near-rings with a finite number of central idempotents*, An. șt. Univ. Iași, T. XXIII, s. I. a, f. 2, 1977, 235—240.
23. Weyl, H. — *Raum, Zeit, Materie*, Springer, Berlin, 1921.

Faculty of Mathematics
University of Iași
6600—Iași, R. S. Romania