$$(\omega + 2\pi)\, a_1^{(\rho)} + \omega a_2^{(\rho)} + \omega a_3^{(\rho)} + (\alpha + 2\beta)\, b_1^{(\rho)} + \alpha b_2^{(\rho)} + \alpha b_3^{(\rho)} = 0,$$

$$\omega a_1^{(\rho)} + (\omega + 2\pi)\, a_2^{(\rho)} + \omega a_3^{(\rho)} + \alpha b_1^{(\rho)} + (\alpha + 2\beta)\, b_2^{(\rho)} + \alpha b_3^{(\rho)} = 0,$$

$$\omega a_1^{(\rho)} + \omega a_2^{(\rho)} + (\omega + 2\pi)\, a_3^{(\rho)} + \alpha b_1^{(\rho)} + \alpha b_2^{(\rho)} + (\alpha + 2\beta)\, b_3^{(\rho)} = H_\rho,$$

where H_ρ are arbitrary constants.

For these displacements and rotations we have $R_i = M_i = N_i = 0$ and consequently they escape to the control of the classical conditions of equilibrium. The rising of these new effects may create the posibility for instability in the behaviour of the beam. They may be amang the sources of the buckling. The problem of Saint-Venant for this model has a unique solution which satisfies the end conditions (24)—(27). It seems that last model may be adequate to study the granular media. In order to derive the general equations of a model we may use the known procedure by starting from a principle of energy. First, we must precize the kinetic energy and the neutral motions of the model.

REFERENCES

1. Borş C. I. — *The uniqueness of solution in Saint-Venant problem*. An. şt. Univ. Iaşi, Mat. t. XXVII, 2, 1981, 421—428.
2. Borş C. I. — *Deformable solids with microstructure having a symmetric stress tensor*. An. şt. Univ. Iaşi, Mat. t. XXVII, 1, 1981, 179—184.
3. Eringen A. C. — *Theory of micropolar elasticity*. Fracture, 2, Academic Press, New York, 1960.
4. Ieşan D. — *Torsion of micropolar elastic beams*. Int. J. Eng. Sci., 9, 1971, 1047—1060.
5. Ieşan D. — *On Saint-Venant's problem in micropolar elasticity*. Int. J. Eng. Sci., 9, 1971, 879—888.
6. Nowaki W. — *Theory of micropolar elasticity*. Springer, Wien, 1970.
7. Nowacki P., Nowacki W. — *Some problems of hemitropic micropolar continuum*. Bull. Acad. Polon. Sci., Sér. Sci. techn., 25, 1977, 297—305.

Faculty of Mathematics
University of Iaşi
6600—Iaşi, R. S. România

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Supliment la tomul XXVII, s. I a, 1981

CONTINUOUS DEPENDENCE RESULTS IN THERMODYNAMICS OF MATERIALS WITH HEAT CONDUCTION AND VISCOSITY

BY

S. CHIRIŢĂ

To the memory of Professor Mendel Haimovici

1. Introduction. We consider the theory of thermodynamics of elastic materials with heat conduction and viscosity [1, 2]. The main purpose of this paper is to establish the continuous dependence of thermodynamic processes upon initial state and supply terms. The results are local and they are established under certain assumptions on material response. In fact, we prove the continuous dependence results for smooth admissible processes residing in a neighborhood of a positive smooth admissible thermodynamic process. When the class of initial-boundary value problems is restricted to the displacement-boundary value problems, the continuous dependence results are established under a weaker assumption : the smooth admissible processes reside in a neighborhood of a smooth admissible process where a strong ellipticity condition is satisfied.

2. Basic equations. We consider a body that occupies, at time $t=0$, a properly regular region R of Euclidean three-dimensional space $\mathfrak{R}^3$ and is bounded by the piecewise smooth surface ∂R [3]. The configuration of the body at time $t=0$ is taken as the reference configuration. In this paper we use the referential description of motion. We identify the material point X with its position $\mathbf{X}$ in the fixed reference configuration. Let us denote by $X_K (K=1, 2, 3)$ the coordinates of the particle X relative to a fixed system of rectangular Cartesian axes. The coordinates of the particle in the position $\mathbf{x}$ at time t are denoted by $x_i (i=1, 2, 3)$, and

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t), \qquad (\mathbf{X}, t) \in \bar{R} \times [0, t_0].$$

According to the theory developed in [2], a thermodynamic process in R is described by eight functions, of $\mathbf{X}$ and t, whose values have the following physical interpretations : $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ is the spatial position in the motion, $\mathbf{T} = \mathbf{T}(\mathbf{X}, t)$ is the first Piola-Kirchhoff stress tensor, $\mathbf{b} = \mathbf{b}(\mathbf{X}, t)$ is the body force per unit mass, $\psi = \psi(\mathbf{X}, t)$ is the Helmholtz free energy per unit mass, $\mathbf{Q} = \mathbf{Q}(\mathbf{X}, t)$ is the heat flux vector, $r = r(\mathbf{X}, t)$ is the heat supply per unit mass and unit time, $\eta = \eta(\mathbf{X}, t)$ is the specific entropy per unit mass, $\theta = \theta(\mathbf{X}, t)$ is the absolute temperature.

Throughout this paper we shall use the following notations : a superposed dot denotes the material time derivative ; Latin indices have range 1, 2, 3 ; summation over repeated subscripts is implied ; the symbol $|\cdot|$ denotes a norm either in Euclidean vector space or in a tensor space, while

$\|\cdot\|$ denotes L_2-norm ; *tr* is the trace operator and the operators *grad* and *div* refer to material derivatives ; $\mathbf{v}=\dot{\mathbf{x}}$; $\mathbf{F}=grad\ \mathbf{x}$; $\mathbf{L}=\dot{\mathbf{F}}$ and $\mathbf{g}=grad\ \theta$.

The above set of eight functions is called a *thermodynamic process* if it is compatible with the law of balance of linear momentum and the law of balance of energy. The integral forms of these laws are equivalent to the following differential equations :

$$\rho\dot{\mathbf{v}}=div\,\mathbf{T}+\rho\mathbf{b}, \tag{1}$$

$$\frac{d}{dt}\{\rho(\psi+\theta\eta)\}=tr(\mathbf{T}\mathbf{L}^T)+\rho r-div\mathbf{Q}. \tag{2}$$

where the reference mass density $\rho(\mathbf{X})$ is smooth and strictly positive,

$$\rho(\mathbf{X})\geqslant\rho_0>0, \qquad \mathbf{X}\in\overline{R}.$$

The constitutive equations for materials with heat conduction and viscosity are given by [2]

$$\begin{aligned}&\psi=\psi^*(\mathbf{F},\theta,\mathbf{g},\mathbf{L};\mathbf{X}), \qquad \eta=\eta^*(\mathbf{F},\theta,\mathbf{g},\mathbf{L};\mathbf{X}),\\ &\mathbf{T}=\mathbf{T}^*(\mathbf{F},\theta,\mathbf{g},\mathbf{L};\mathbf{X}), \qquad \mathbf{Q}=\mathbf{Q}^*(\mathbf{F},\theta,\mathbf{g},\mathbf{L};\mathbf{X}).\end{aligned} \tag{3}$$

We suppose that the functions ψ^*, $\mathbf{T}^*$, η^* and $\mathbf{Q}^*$ are smooth functions defined for $\mathbf{F}$ in the set M^+ of 3×3 matrices with positive determinant, θ in the set of positive real numbers $\mathfrak{R}^+$, $\mathbf{g}$ in $\mathfrak{R}^3$, $\mathbf{L}$ in the set M of 3×3 matrices and $\mathbf{X}$ in $\overline{R}$. In particular, we assume that the partial derivatives of ψ^*, $\mathbf{T}^*$, η^* and $\mathbf{Q}^*$, at any fixed state $(\mathbf{F},\theta,\mathbf{g},\mathbf{L})$ in state space $M^+\times\mathfrak{R}^+\times\times\mathfrak{R}^3\times M$, are bounded functions of $(\mathbf{X},t)$ on $\overline{R}\times[0,t_0]$.

We will say that a thermodynamic process is *admissible* if it is compatible with the constitutive equations (3). Therefore, in order to specify an admissible thermodynamic process it suffices to prescribe the functions $\mathbf{x}$ and θ.

We will say that an admissible thermodynamic process $(\mathbf{x},\theta)(\mathbf{X},t)$ is *smooth* in R, if it is such that $(v,\mathbf{F},\theta,\mathbf{g},\mathbf{L})(\mathbf{X},t)$ are Lipschitz continuous, uniformly on bounded subsets of their domain.

In what follows, we shall consider only smooth admissible thermodynamic processes. Every smooth admissible thermodynamic process in R must obey the Clausius-Duhem inequality

$$\rho\dot{\eta}-\rho\frac{r}{\theta}+div\left(\frac{\mathbf{Q}}{\theta}\right)\geqslant0, \tag{4}$$

for all t and all material points $\mathbf{X}$.

Let us denote by $\mathbf{T}^{(0)}(\mathbf{F},\theta;\mathbf{X})$ the equilibrium stress tensor $\mathbf{T}^{(0)}$ corresponding to $\mathbf{F}$ and θ ; i.e.,

$$\mathbf{T}^{(0)}(\mathbf{F},\theta;\mathbf{X})=\mathbf{T}^*(\mathbf{F},\theta,0,0;\mathbf{X}).$$

The value $\mathbf{S}$ of the function $\mathbf{T}^{(e)}$ defined by

$$\mathbf{T}^{(e)}(\mathbf{X},t)=\mathbf{S}(\mathbf{F},\theta,\mathbf{g},\mathbf{L};\mathbf{X})=\mathbf{T}^*(\mathbf{F},\theta\,\mathbf{g},\mathbf{L};\mathbf{X})-\mathbf{T}^{(0)}(\mathbf{F},\theta;\mathbf{X})$$

is called the extra stress ; it is the contribution to $\mathbf{T}$ caused by the gradients of temperature and velocity. Obviously, we have

$$\mathbf{S}(\mathbf{F},\theta,0,0;\mathbf{X})=0, \text{ for all } (\mathbf{F},\theta;\mathbf{X}). \tag{5}$$

In order that the Clausius-Duhem inequality (4) holds for all smooth admissible thermodynamic processes in R, it is necessary and sufficient that [2]

$$\psi=\psi^*(\mathbf{F},\theta;\mathbf{X}), \quad \mathbf{T}^{(0)}=\rho\frac{\partial\psi^*}{\partial\mathbf{F}}, \quad \eta^*=-\frac{\partial\psi^*}{\partial\theta}, \tag{6}$$

$$tr\{\mathbf{S}(\mathbf{F},\theta,\mathbf{g}\,\mathbf{L};\mathbf{X})\mathbf{L}^T\}-\frac{1}{\theta}\mathbf{Q}^*(\mathbf{F},\theta,\mathbf{g},\mathbf{L};\mathbf{X})\cdot\mathbf{g}\geqslant0. \tag{7}$$

A direct consequence of the inequality (7) is that

$$\mathbf{Q}^*(\mathbf{F},\theta,0,0;\mathbf{X})=0, \text{ for all } (\mathbf{F},\theta;\mathbf{X}). \tag{8}$$

If we use the relations (5) and (8), from the inequality (7), we deduce

$$H_{AiBj}L_{iA}L_{jB}+2P_{AiB}L_{iA}g_B+\frac{1}{\theta}K_{AB}g_Ag_B\geqslant0, \tag{9}$$

for infinitesimal gradients of temperature and velocity. Here

$$H_{AiBj}=\frac{\partial S_{Ai}}{\partial L_{jB}}(\mathbf{F},\theta,0,0;\mathbf{X}), \quad G_{AiB}=\frac{\partial S_{Ai}}{\partial g_B}(\mathbf{F},\theta,0,0;\mathbf{X}),$$

$$M_{AiB}=-\frac{\partial Q^*_A}{\partial L_{iB}}(\mathbf{F},\theta,0,0;\mathbf{X}), \qquad K_{AB}=-\frac{\partial Q^*_A}{\partial g_B}(\mathbf{F},\theta,0,0;\mathbf{X})$$

$$2P_{AiB}=G_{AiB}+\frac{1}{\theta}M_{BiA}.$$

In view of the relation (6), the equation (2) can be written in a reduced form

$$\rho\theta\dot{\eta}=tr(\mathbf{T}^{(e)}\mathbf{L}^T)+\rho r-div\mathbf{Q}. \tag{10}$$

3. A fundamental identity. Let $(\mathbf{x},\theta)(\mathbf{X},t)$ and $(\overline{\mathbf{x}},\overline{\theta})(\mathbf{X},t)$ be two thermodynamic processes corresponding to the supply terms $(\mathbf{b},r)(\mathbf{X},t)$ and $(\overline{\mathbf{b}},\overline{r})(\mathbf{X},t)$, respectively.

By using the assumed differentiability of the functions ψ^*, $\mathbf{T}^*$ and $\mathbf{Q}^*$, we obtain, by Taylor's expansion,

$$T^{(0)}_{Ai}-\overline{T}^{(0)}_{Ai}=\overline{A}_{AiBj}(F_{jB}-\overline{F}_{jB})+\overline{B}_{Ai}(\theta-\overline{\theta})+U^{(0)}_{Ai}, \tag{11}$$

$$\rho(\eta-\overline{\eta})=-\overline{B}_{Ai}(F_{iA}-\overline{F}_{iA})+\overline{C}(\theta-\overline{\theta})+\chi, \tag{12}$$

$$\begin{aligned}T^{(e)}_{Ai}-\overline{T}^{(e)}_{Ai}=&\overline{D}_{AiBj}(F_{jB}-\overline{F}_{jB})+\overline{E}_{Ai}(\theta-\overline{\theta})+\overline{G}_{AiB}(g_B-\overline{g}_B)+\\&+\overline{H}_{AiBj}(L_{jB}-\overline{L}_{jB})+U^{(e)}_{Ai}.\end{aligned} \tag{13}$$

$$Q_A-\overline{Q}_A=-\overline{I}_{AiB}(F_{iB}-\overline{F}_{iB})-\overline{J}_A(\theta-\overline{\theta})-\overline{K}_{AB}(g_B-\overline{g}_B)-\\ -\overline{M}_{AiB}(L_{iB}-\overline{L}_{iB})+V_A \tag{14}$$

where

$$\overline{A}_{AiBj}=\rho\frac{\partial^2\psi^*}{\partial F_{iA}\partial F_{jB}}(\overline{\mathbf{F}},\overline{\theta};\mathbf{X}),\qquad \overline{B}_{Ai}=\rho\frac{\partial^2\psi^*}{\partial F_{iA}\partial\theta}(\overline{\mathbf{F}},\overline{\theta};\mathbf{X}),$$

$$\overline{C}=-\rho\frac{\partial^2\psi^*}{\partial\theta^2}(\overline{\mathbf{F}},\overline{\theta};\mathbf{X}),\qquad \overline{D}_{AiBj}=\frac{\partial S_{Ai}}{\partial F_{jB}}(\overline{\mathbf{F}},\overline{\theta},\overline{\mathbf{g}},\overline{\mathbf{L}};\mathbf{X}),$$

$$\overline{E}_{Ai}=\frac{\partial S_{Ai}}{\partial\theta}(\overline{\mathbf{F}},\overline{\theta},\overline{\mathbf{g}},\overline{\mathbf{L}};\mathbf{X}),\qquad G_{AiB}=\frac{\partial S_{Ai}}{\partial g_B}(\overline{\mathbf{F}},\overline{\theta},\overline{\mathbf{g}},\overline{\mathbf{L}};\mathbf{X}),$$

$$\overline{H}_{AiBj}=\frac{\partial S_{Ai}}{\partial L_{jB}}(\overline{\mathbf{F}},\overline{\theta},\overline{\mathbf{g}},\overline{\mathbf{L}};\mathbf{X}),\qquad \overline{I}_{AiB}=-\frac{\partial Q_A^*}{\partial F_{iB}}(\overline{\mathbf{F}},\overline{\theta},\overline{\mathbf{g}},\overline{\mathbf{L}};\mathbf{X}),$$

$$\overline{J}_A=-\frac{\partial Q_A^*}{\partial\theta}(\overline{\mathbf{F}},\overline{\theta},\overline{\mathbf{g}},\overline{\mathbf{L}};\mathbf{X}),\qquad \overline{K}_{AB}=-\frac{\partial Q_A^*}{\partial g_B}(\overline{\mathbf{F}},\overline{\theta},\overline{\mathbf{g}},\overline{\mathbf{L}};\mathbf{X}),$$

$$\overline{M}_{AiB}=-\frac{\partial Q_A^*}{\partial L_{iB}}(\overline{\mathbf{F}},\overline{\theta},\overline{\mathbf{g}},\overline{\mathbf{L}};\mathbf{X}).$$

On the basis of the equations (1) and (10), and by using the relations (11)—(14), as in [4], we obtain the following identity

$$\frac{1}{2}\int_R[\rho(v_i-\overline{v}_i)(v_i-\overline{v}_i)+\overline{C}(\theta-\overline{\theta})^2](\mathbf{X},\tau)dX-$$

$$-\frac{1}{2}\int_R[\rho(v_i-\overline{v}_i)(v_i-\overline{v}_i)+\overline{C}(\theta-\overline{\theta})^2](\mathbf{X},0)dX=$$

$$=\int_0^\tau\int_{\partial R}[(v_i-\overline{v}_i)(T_{Ai}-\overline{T}_{Ai})N_A-$$

$$-\frac{1}{\overline{\theta}}(\theta-\overline{\theta})(Q_A-\overline{Q}_A)N_A](\mathbf{X},t)dSdt+\int_0^\tau\int_R[\rho(b_i-\overline{b}_i)(v_i-\overline{v}_i)+ \tag{15}$$

$$+\frac{\rho}{\overline{\theta}}(r-\overline{r})(\theta-\overline{\theta})](\mathbf{X},t)dXdt-\int_0^\tau\int_R\{\overline{H}_{AiBj}(L_{iA}-\overline{L}_{iA})(L_{jB}-\overline{L}_{jB})+$$

$$+2\overline{P}_{AiB}(L_{iA}-\overline{L}_{iA})(g_B-\overline{g}_B)+\frac{1}{\overline{\theta}}\overline{K}_{AB}(g_A-\overline{g}_A)(g_B-\overline{g}_B)\}(\mathbf{X},t)dXdt-$$

$$-\int_0^\tau\int_R\{\overline{\mathfrak{A}}(\theta-\overline{\theta})^2+\overline{\mathfrak{B}}_{Ai}(F_{iA}-\overline{F}_{iA})(\theta-\overline{\theta})\}(\mathbf{X},t)dXdt-$$

$$-\int_0^\tau\int_R\{\overline{\mathfrak{C}}_{AiBj}(F_{jB}-\overline{F}_{jB})(L_{iA}-\overline{L}_{iA})+\overline{\mathfrak{D}}_{Ai}(L_{iA}-\overline{L}_{iA})(\theta-\overline{\theta})+$$

$$+\overline{\mathfrak{E}}_A(g_A-\overline{g}_A)(\theta-\overline{\theta})+\overline{\mathfrak{G}}_{AiB}(g_A-\overline{g}_A)(F_{iB}-\overline{F}_{iB})\}(\mathbf{X},t)dXdt-$$

$$-\int_0^\tau\int_R\Big\{(U^{(c)}_{Ai}+U^{(0)}_{Ai})(L_{iA}-\overline{L}_{iA})+\dot{\chi}(\theta-\overline{\theta})-\frac{1}{\overline{\theta}}L_{iA}U^{(e)}_{Ai}(\theta-\overline{\theta})+$$

$$+\frac{1}{\overline{\theta}^2}\overline{g}_AV_A(\theta-\overline{\theta})-\frac{1}{\overline{\theta}}V_A(g_A-\overline{g}_A)\Big\}(\mathbf{X},t)dXdt,\quad \tau\in[0,t_0],$$

where N_A are the direction cosines of the outward normal to surface ∂R. In the above identity we have used the following notations

$$2\overline{P}_{AiB}=\overline{G}_{AiB}+\frac{1}{\overline{\theta}}\overline{M}_{BiA},\quad \overline{\mathfrak{A}}=\frac{1}{2}\dot{\overline{C}}+\frac{\rho}{\overline{\theta}}\dot{\overline{\eta}}-\frac{1}{\overline{\theta}}\overline{E}_{Ai}L_{iA}-\frac{1}{\overline{\theta}^2}\overline{J}_A\overline{g}_A,$$

$$\overline{\mathfrak{B}}_{Ai}=-\dot{\overline{B}}_{Ai}-\frac{1}{\overline{\theta}}\overline{D}_{BjAi}L_{jB}-\frac{1}{\overline{\theta}^2}\overline{I}_{BiA}\overline{g}_B,\quad \overline{\mathfrak{C}}_{AiBj}=\overline{A}_{AiBj}+\overline{D}_{AiBj},$$

$$\overline{\mathfrak{D}}_{Ai}=\overline{E}_{Ai}-\frac{1}{\overline{\theta}}\overline{H}_{BjAi}L_{jB}-\frac{1}{\overline{\theta}}\overline{T}^{(e)}_{Ai}-\frac{1}{\overline{\theta}^2}\overline{M}_{BiA}\overline{g}_B,$$

$$\overline{\mathfrak{E}}_A=-\frac{1}{\overline{\theta}}\overline{G}_{BiA}L_{iB}+\frac{1}{\overline{\theta}}\overline{J}_A-\frac{1}{\overline{\theta}^2}\overline{K}_{BA}\overline{g}_B,\quad \overline{\mathfrak{G}}_{AiB}=\frac{1}{\overline{\theta}}\overline{I}_{AiB}.$$

4. Continuous dependence results. We will say that the thermodynamic process $(\overline{\mathbf{x}},\overline{\theta})(\mathbf{x},t)$ is *positive* if there is a positive constant λ such that

$$\overline{H}_{AiBj}\xi_{iA}\xi_{jB}+2\overline{P}_{AiB}\xi_{iA}\zeta_B+\frac{1}{\overline{\theta}}\overline{K}_{AB}\zeta_A\zeta_B\geq\lambda(|\xi|^2+|\zeta|^2), \tag{16}$$

for all second order tensors ξ and all vectors ζ, and the instantaneous specific heat $\overline{C}$ is strictly positive, i.e.,

$$\overline{C}\geq c_0>0. \tag{17}$$

It is worth noting that our inequality (16) is in accordance with the consequence (9) of the Clausius-Duhem inequality.

Theorem 4.1. *Let $(\overline{\mathbf{x}},\overline{\theta})(\mathbf{X},t)$ be a a positive thermodynamic process corresponding to the supply terms $(\overline{\mathbf{b}},\overline{r})(\mathbf{X},t)$ in $L^\infty(R\times[0,t_0])$. Then there are positive constants δ, σ, M and N with the following property:*

If $(\mathbf{x}, \theta)(\mathbf{X}, t)$ *is any thermodynamic process defined on* $R \times [0, t_0]$, *with supply terms* $(\mathbf{b}, r)(\mathbf{X}, t) \in L^{\infty}(R \times [0, t_0])$, *such that*

$$|\mathbf{F}(\mathbf{X}, t) - \overline{\mathbf{F}}(\mathbf{X}, t)| + |\theta(\mathbf{X}, t) - \overline{\theta}(\mathbf{X}, t)| < \delta, \quad (\mathbf{X}, t) \in R \times [0, t_0], \tag{18}$$

$$(v_i - \overline{v}_i)(T_{Ai} - \overline{T}_{Ai})N_A - \frac{1}{\theta}(\theta - \overline{\theta})(Q_A - \overline{Q}_A)N_A \leqslant 0, \ \text{on} \ \partial R \times [0, t_0], \tag{19}$$

then we have the following estimate, for all $s \in [0, t_0]$,

$$\begin{aligned} \|(\mathbf{v} - \overline{\mathbf{v}}, \ \mathbf{F} - \overline{\mathbf{F}}, \ \theta - \overline{\theta})(\cdot, s)\|_{L^2(R)} \leqslant M e^{\sigma s} \|(\mathbf{v} - \overline{\mathbf{v}}, \ \mathbf{F} - \overline{\mathbf{F}}, \\ \theta - \overline{\theta})(\cdot, 0)\|_{L^2(R)} + N e^{\sigma s} \int_0^s \|(\mathbf{b} - \overline{\mathbf{b}}, \ r - \overline{r})(\cdot, t)\|_{L^2(R)} dt. \end{aligned} \tag{20}$$

Proof. Taking into account the relations (11)–(14) and (16), (19), from the identity (15) it follows that there is a positive constant δ with the property that, when (18) is satisfied, we have

$$\begin{aligned} &\int_R [\rho(v_i - \overline{v}_i)(v_i - \overline{v}_i) + \overline{C}(\theta - \overline{\theta})^2](\mathbf{X}, \tau) dX \leqslant \int_R [\rho(v_i - \overline{v}_i)(v_i - \overline{v}_i) + \\ &+ \overline{C}(\theta - \overline{\theta})^2](\mathbf{X}, 0) dX + 2\int_0^{\tau}\int_R [\rho(b_i - \overline{b}_i)(v_i - \overline{v}_i) + \\ &+ \frac{\rho}{\theta}(r - \overline{r})(\theta - \overline{\theta})](\mathbf{X}, t) dX dt - 2\lambda \int_0^{\tau} \|(\mathbf{L} - \overline{\mathbf{L}}, \ \mathbf{g} - \overline{\mathbf{g}})(\cdot, t)\|^2_{L^2(R)} dt - \\ &- 2\int_0^{\tau}\int_R \{\overline{\mathfrak{A}}(\theta - \overline{\theta})^2 + \overline{\mathfrak{B}}_{Ai}(F_{iA} - \overline{F}_{iA})(\theta - \overline{\theta})\}(\mathbf{X}, t) dX dt - \\ &- 2\int_0^{\tau}\int_R \{\overline{\mathfrak{C}}_{AiBj}(F_{jB} - \overline{F}_{jB})(L_{iA} - L_{iA}) + \overline{\mathfrak{D}}_{Ai}(L_{iA} - \overline{L}_{iA})(\theta - \overline{\theta}) + \\ &+ \overline{\mathfrak{E}}_A(g_A - \overline{g}_A)(\theta - \overline{\theta}) + \overline{\mathfrak{G}}_{AiB}(g_A - \overline{g}_A)(F_{iB} - \overline{F}_{iB})\}(\mathbf{X}, t) dX dt. \end{aligned} \tag{21}$$

By using the Schwarz inequality and the arithmetic-geometric mean inequality

$$2ab \leqslant \frac{a^2}{\pi^2} + b^2\pi^2, \tag{22}$$

we deduce

$$\begin{aligned} 2\left|\int_R \{\overline{\mathfrak{A}}(\theta - \overline{\theta})^2 + \overline{\mathfrak{B}}_{Ai}(F_{iA} - \overline{F}_{iA})(\theta - \overline{\theta})\}(\mathbf{X}, t) dX\right| \leqslant \\ \leqslant m_1 \|(\mathbf{F} - \overline{\mathbf{F}}, \ \theta - \overline{\theta})(\cdot, t)\|^2_{L^2(R)}, \end{aligned} \tag{23}$$

where

$$m_1 = \max(M_1^2 + 1, \ M_2^2),$$

$$M_1^2 = 2 \max |\overline{\mathfrak{A}}(\mathbf{X}, t)|, \quad M_2 = \max |\overline{\mathfrak{B}}(\mathbf{X}, t)| \quad \text{on} \quad \overline{R} \times [0, t_0].$$

Further, an application of the Schwarz inequality and the inequality (22) to the last term in (21) yields, for arbitrary constants π_1, π_2, π_3 and π_4,

$$\begin{aligned} &2\left|\int_R \{\overline{\mathfrak{C}}_{AiBj}(F_{jB} - \overline{F}_{jB})(L_{iA} - \overline{L}_{Ai}) + \overline{\mathfrak{D}}_{Ai}(L_{iA} - \overline{L}_{iA})(\theta - \overline{\theta}) + \right. \\ &\left. + \overline{\mathfrak{E}}_A(g_A - \overline{g}_A)(\theta - \overline{\theta}) + \overline{\mathfrak{G}}_{AiB}(g_A - \overline{g}_A)(F_{iB} - \overline{F}_{iB})\}(\mathbf{X}, t) dX\right| \leqslant \\ &\leqslant m_2 \|(\mathbf{F} - \overline{\mathbf{F}}, \ \theta - \overline{\theta})(\cdot, t)\|^2_{L^2(R)} + (\pi_1^2 + \pi_2^2)\|(\mathbf{L} - \overline{\mathbf{L}})(\cdot, t)\|^2_{L^2(R)} + \\ &+ (\pi_3^2 + \pi_4^2)\|(\mathbf{g} - \overline{\mathbf{g}})(\cdot, t)\|^2_{L^2(R)}, \end{aligned} \tag{24}$$

where

$$m_2 = \max\left(\frac{M_3^2}{\pi_1^2} + \frac{M_6^2}{\pi_4^2}, \ \frac{M_4^2}{\pi_2^2} + \frac{M_5^2}{\pi_3^2}\right),$$

$$M_3 = \max |\overline{\mathfrak{C}}(\mathbf{X}, t)|, \quad M_4 = \max |\overline{\mathfrak{D}}(\mathbf{X}, t)|,$$

$$M_5 = \max |\overline{\mathfrak{E}}(\mathbf{X}, t)|, \quad M_6 = \max |\overline{\mathfrak{G}}(\mathbf{X}, t)|, \quad \text{on} \quad \overline{R} \times [0, t_0].$$

Similar, we have

$$\begin{aligned} 2\left|\int_R [\rho(b_i - \overline{b}_i)(v_i - \overline{v}_i) + \frac{\rho}{\theta}(r - \overline{r})(\theta - \overline{\theta})](\mathbf{X}, t) dX\right| \leqslant \\ \leqslant m_3 \|(\mathbf{b} - \overline{\mathbf{b}}, \ r - \overline{r})(\cdot, t)\|_{L^2(R)} \|(\mathbf{v} - \overline{\mathbf{v}}, \ \theta - \overline{\theta})(\cdot, t)\|_{L^2(R)}. \end{aligned} \tag{25}$$

In view of the estimates (23)–(25), from the relation (21), we get

$$\begin{aligned} &\int_R [\rho(v_i - \overline{v}_i)(v_i - \overline{v}_i) + \overline{C}(\theta - \overline{\theta})^2](\mathbf{X}, \tau) dX \leqslant m_4 \|(\mathbf{v} - \overline{\mathbf{v}}, \ \theta - \overline{\theta})(\cdot, 0)\|^2_{L^2(R)} - \\ &- [2\lambda - (\pi_1^2 + \pi_2^2)]\int_0^{\tau} \|(\mathbf{L} - \overline{\mathbf{L}})(\cdot, t)\|^2_{L^2(R)} dt - [2\lambda - (\pi_3^2 + \pi_4^2)] \cdot \\ &\cdot \int_0^{\tau} \|(\mathbf{g} - \overline{\mathbf{g}})(\cdot, t)\|^2_{L^2(R)} dt + (m_1 + m_2)\int_0^{\tau} \|(\mathbf{F} - \overline{\mathbf{F}}, \ \theta - \overline{\theta})(\cdot, t)\|^2_{L^2(R)} dt + \end{aligned} \tag{26}$$

$$+m_3\int_0^\tau \|(b-\bar{b},\ r-\bar{r})(\cdot,t)\|_{L^2(R)}\cdot\|(\mathbf{v}-\bar{\mathbf{v}},\ \mathbf{F}-\bar{\mathbf{F}},\ \theta-\bar{\theta})(\cdot,t)\|_{L^2(R)}dt,$$

$$\tau\in[0,t_0],$$

where

$$m_4=\max(\rho(\mathbf{X}),\ \bar{C}(\mathbf{X},t)),\quad \text{on}\quad \bar{R}\times[0,t_0].$$

On the other hand, we have

$$\|(\mathbf{F}-\bar{\mathbf{F}})(\cdot,\tau)\|^2_{L^2(R)}=\|(\mathbf{F}-\bar{\mathbf{F}})(\cdot,0)\|^2_{L^2(R)}+$$

(27)
$$+2\int_0^\tau\int_R(F_{iA}-\bar{F}_{iA})(L_{iA}-\bar{L}_{iA})dX\,dt\leqslant\|(\mathbf{F}-\bar{\mathbf{F}})(\cdot,0)\|^2_{L^2(R)}+$$

$$+\frac{1}{\pi_5^2}\int_0^\tau\|(\mathbf{F}-\bar{\mathbf{F}})(\cdot,t)\|^2_{L^2(R)}dt+\pi_5^2\int_0^\tau\|(\mathbf{L}-\bar{\mathbf{L}})(\cdot,t)\|^2_{L^2(R)}dt,\quad \tau\in[0,t_0],$$

for arbitrary constant π_5.

From the relations (26) and (27), we deduce

$$m_0\|(\mathbf{v}-\bar{\mathbf{v}},\ \mathbf{F}-\bar{\mathbf{F}},\ \theta-\bar{\theta})(\cdot,\tau)\|^2_{L^2(R)}\leqslant\int_R[\rho(v_i-\bar{v}_i)(v_i-\bar{v}_i)+$$

$$+(F_{iA}-\bar{F}_{iA})(F_{iA}-\bar{F}_{iA})+\bar{C}(\theta-\bar{\theta})^2](\mathbf{X},\tau)dX\leqslant$$

(28)
$$\leqslant m^*\|(\mathbf{v}-\bar{\mathbf{v}},\ \mathbf{F}-\bar{\mathbf{F}},\ \theta-\bar{\theta})(\cdot,0)\|^2_{L^2(R)}-c_1\int_0^\tau\|(\mathbf{L}-\bar{\mathbf{L}})(\cdot,t)\|^2_{L^2(R)}dt-$$

$$-c_2\int_0^\tau\|(\mathbf{g}-\bar{\mathbf{g}})(\cdot,t)\|^2_{L^2(R)}dt+c_3\int_0^\tau\|(\mathbf{v}-\bar{\mathbf{v}},\ \mathbf{F}-\bar{\mathbf{F}},\ \theta-\bar{\theta})(\cdot,t)\|^2_{L^2(R)}dt+$$

$$+c_4\int_0^\tau\|(\mathbf{b}-\bar{\mathbf{b}},\ r-\bar{r})(\cdot,t)\|_{L^2(R)}\cdot\|(\mathbf{v}-\bar{\mathbf{v}},\ \mathbf{F}-\bar{\mathbf{F}},\ \theta-\bar{\theta})(\cdot,t)\|_{L^2(R)}dt,$$

where

$$m_0=\min(\rho_0,\ c_0,\ 1),\quad m^*=\max(m_4,\ 1),$$

$$c_1=2\lambda-(\pi_1^2+\pi_2^2+\pi_5^2),\quad c_2=2\lambda-(\pi_3^2+\pi_4^2),$$

$$c_3=m_1+m_2+\frac{1}{\pi_5^2},\quad c_4=m_3.$$

Now, we choose the arbitrary constants $\pi_1-\pi_5$, so that $c_1>0$ and $c_2>0$. Thus, from the relation (28), we obtain

$$\|(\mathbf{v}-\bar{\mathbf{v}},\ \mathbf{F}-\bar{\mathbf{F}},\ \theta-\bar{\theta})(\cdot,\tau)\|^2_{L^2(R)}\leqslant M^2\|(\mathbf{v}-\bar{\mathbf{v}},\ \mathbf{F}-\bar{\mathbf{F}},\ \theta-\bar{\theta})(\cdot,0)\|^2_{L^2(R)}+$$

(29)
$$+2\sigma\int_0^\tau\|(\mathbf{v}-\bar{\mathbf{v}},\ \mathbf{F}-\bar{\mathbf{F}},\ \theta-\bar{\theta})(\cdot,t)\|^2_{L^2(R)}dt+$$

$$+2N\int_0^\tau\|(\mathbf{b}-\bar{\mathbf{b}},\ r-\bar{r})(\cdot,t)\|_{L^2(R)}\cdot\|(\mathbf{v}-\bar{\mathbf{v}},\ \mathbf{F}-\bar{\mathbf{F}},\ \theta-\bar{\theta})(\cdot,t)\|_{L^2(R)}dt,$$

where

(30)
$$M^2=\frac{m^*}{m_0},\quad \sigma=\frac{c_3}{2m_0},\quad N=\frac{c_4}{2m_0}.$$

From a Gronwall type lemma [5], from (29) it follows the inequality (20) and the proof is complete.

If the class of initial-boundary value problems is restricted to the displacement-boundary value problems, then, instead of the condition (16), we may assume the weaker assumption that the thermodynamic process $(\bar{\mathbf{x}},\bar{\theta})(\mathbf{X},t)$ satisfies *a strong ellipticity condition*, namely,

(31)
$$\bar{H}_{AiBj}\omega_i\omega_j\varkappa_A\varkappa_B+2\bar{P}_{AiB}\omega_i\varkappa_A\zeta_B+\frac{1}{\bar{\theta}}\bar{K}_{AB}\zeta_A\zeta_B\geqslant$$

$$\geqslant\mu(|\omega|^2|\varkappa|^2+|\zeta|^2),$$

for all vectors ω, $\varkappa$ and ζ.

Lemma 4.1. *Let $(\bar{\mathbf{x}},\bar{\theta})(\mathbf{X},t)$ be a thermodynamic process defined on $\bar{R}\times[0,t_0]$, satisfying the strong ellipticity condition (31). Let $(\mathbf{x},\theta)(\mathbf{X},t)$ be any thermodynamic process defined on $\bar{R}\times[0,t_0]$ such that*

$$\mathbf{x}(\mathbf{X},t)=\bar{\mathbf{x}}(\mathbf{X},t),\quad \text{on}\quad \partial R\times[0,t_0].$$

Then, there are constants $\nu>0$ and Ω with the property

(32)
$$\int_R[\bar{H}_{AiBj}(L_{iA}-\bar{L}_{iA})(L_{jB}-\bar{L}_{jB})+2\bar{P}_{AiB}(L_{iA}-\bar{L}_{iA})(g_B-\bar{g}_B)+$$

$$+\frac{1}{\bar{\theta}}\bar{K}_{AB}(g_A-\bar{g}_A)(g_B-\bar{g}_B)](\mathbf{X},t)dX\geqslant\nu\|(\mathbf{L}-\bar{\mathbf{L}},\ \mathbf{g}-\bar{\mathbf{g}})(\cdot,t)\|^2_{L^2(R)}$$

$$-\Omega\|(\mathbf{v}-\bar{\mathbf{v}})(\cdot,t)\|^2_{L^2(R)}.$$

The proof of Lemma 4.1 is a straightforward generalization of the proof of the standard Gårding inequality [3].

Theorem 4.2. *Let $(\bar{\mathbf{x}},\bar{\theta})(\mathbf{X},t)$ be a thermodynamic process corresponding to the supply terms $(\bar{\mathbf{b}},\bar{r})(\mathbf{X},t)$ in $L^\infty(R\times[0,t_0])$. We assume that the ther-*

modynamic process $(\bar{\mathbf{x}}, \bar{\theta})(\mathbf{X}, t)$ *satisfies the conditions* (17) *and* (31). *Then there are positive constants* δ, σ, M *and* N *with the following property*:

If $(\mathbf{x}, \theta)(\mathbf{X}, t)$ *is any thermodynamic process defined on* $\bar{R} \times [0, t_0]$, *with supply terms* $(\mathbf{b}, r)(\mathbf{X}, t) \in L^{\infty}(R \times [0, t_0])$, *satisfying* (18), *and*

$$x_i(\mathbf{X}, t) = \bar{x}_i(\mathbf{X}, t), \quad -\frac{1}{\bar{\theta}}(\theta - \bar{\theta})(Q_A - \bar{Q}_A)N_A \leqslant 0, \quad \text{on} \quad \partial R \times [0, t_0], \tag{33}$$

then

$$\|(\mathbf{v} - \bar{\mathbf{v}}, \mathbf{F} - \bar{\mathbf{F}}, \bar{\theta} - \theta)(\cdot, s)\|_{L^2(R)} \leqslant M e^{\sigma s} \|(\mathbf{v} - \bar{\mathbf{v}}, \mathbf{F} - \bar{\mathbf{F}}, \theta - \bar{\theta})(\cdot, 0)\|_{L^2(R)} + N e^{\sigma s} \int_0^s \|(\mathbf{b} - \bar{\mathbf{b}}, r - \bar{r})(\cdot, t)\|_{L^2(R)} dt. \tag{34}$$

The proof of Theorem 4.2 is based on the Lemma 4.1 and it is similar to that in Theorem 4.1.

REFERENCES

1. Coleman, B. D. and Noll, W. — *The thermodynamics of elastic materials with heat conduction and viscosity*, Arch. Rat. Mech. Anal., 13 (1963), 167–178.
2. Coleman, B. D. and Mizel, V. J. — *Existence of caloric equations of state in thermodynamics*, J. Chem. Phys., 40 (1964), 1 116–1 125.
3. Fichera, G. — *Existence theorems in elasticity*, in Handbuch der Physik VIa/2, Berlin Heidelberg, New York: Springer, 1972.
4. Chiriţă S. — *On the uniqueness of solutions in thermodynamics of elastic materials with heat conduction and viscosity*, (to appear).
5. Dafermos, C. M. — *The second law of thermodynamics and stability*, Arch. Rat. Mech. Anal., 70 (1979), 167–179
6. Wheeler, L. and Nachlinger R. R. — *On the determinacy of motions for the displacement problem in the dynamics of elastic bodies with viscosity*, Z. Ang. Math. und Phisik, 24 (1973), 601–608.

Seminarul Matematic
Universitatea Iaşi
6600—Iaşi, R. S. Romania

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Supliment la tomul XXVII, s. I a, 1981

ON THE STABILITY OF INSTANTANEOUSLY ELASTIC MATERIALS

BY

GH. GR. CIOBANU

To the memory of Professor Mendel Haimovici

1. Introduction. The general theory of dynamical systems developed by Gurtin in [7] is applied to the class of Instantaneously Elastic Materials (I.E.M.) [5], [3]. The main part of the paper consist in construction of dynamical system of I.E.M. in Gurtin's sense [7]. The results regarding the stability of general dynamical systems obtained in [7] hold for dynamical system of I.E.M. Some of these results are rendered evident for bodies with instantaneously elastic response. These results are given without proofs because they are straightforward consequences of [7].

2. General considerations on the behaviour of a continuous body [7]. We denote by $\mathbf{E}_3$ the three-dimensional Euclidean affine space and with $\mathbf{V}_3$ the associated vector space. Let $\mathcal{B}$ be a continuous body identified with the regular region of $\mathbf{E}_3$ [6, p. 13] it occupies in a fixed reference configuration and $\mathbf{X}$ its material points.

The *configurations* of $\mathcal{B}$ are bijective continuous and piecewise continuously differentiable mappings

$$\mathbf{x} : \mathcal{B} \to \mathbf{E}_3 \tag{2.1}$$

with $\mathbf{x}(\mathcal{B})$ regular regions and with $\det \mathbf{F} > 0$, where $\mathbf{F} = \nabla \mathbf{x}$ is the deformation gradient.

The *velocity field* of $\mathcal{B}$ is a continuous and piecewise continuously differentiable mapping

$$\mathbf{v} : \mathcal{B} \to \mathbf{V}_3. \tag{2.2}$$

The *temperature field* for $\mathcal{B}$ is a piecewise continuously differentiable mapping

$$\theta : \mathcal{B} \to (0, \infty), \tag{2.3}$$

and

$$\mathbf{g} = \operatorname{Grad} \theta : \mathcal{B} \to \mathbf{V}_3 \tag{2.4}$$

is the *material gradient* of the temperature field.

If

$$\mathcal{C} = \{\mathbf{x} : \mathbf{x} \text{ is a configuration of } \mathcal{B}\},$$