modynamic process $(\bar{\mathbf{x}}, \bar{\theta})(\mathbf{X}, t)$ *satisfies the conditions* (17) *and* (31). *Then there are positive constants* δ, σ, M *and* N *with the following property* :

If $(\mathbf{x}, \theta)(\mathbf{X}, t)$ *is any thermodynamic process defined on* $R \times [0, t_0]$, *with supply terms* $(\mathbf{b}, r)(\mathbf{X}, t) \in L^\infty(R \times [0, t_0])$, *satisfying* (18), *and*

$$(33) \quad x_i(\mathbf{X}, t) = \bar{x}_i(\mathbf{X}, t), \quad -\frac{1}{\bar{\theta}}(\theta - \bar{\theta})(Q_A - \bar{Q}_A)N_A \leqslant 0, \quad on \quad \partial R \times [0, t_0],$$

then

$$(34) \quad \begin{aligned} &\|(\mathbf{v}-\bar{\mathbf{v}}, \mathbf{F}-\bar{\mathbf{F}}, \bar{\theta}-\theta)(\cdot, s)\|_{L^2(R)} \leqslant Me^{\sigma s}\|(\mathbf{v}-\bar{\mathbf{v}}, \mathbf{F}-\bar{\mathbf{F}}, \theta-\bar{\theta})(\cdot, \\ &0)\|_{L^2(R)} + Ne^{\sigma s}\int_0^s \|(\mathbf{b}-\bar{\mathbf{b}}, r-\bar{r})(\cdot, t)\|_{L^2(R)} dt. \end{aligned}$$

The proof of Theorem 4.2 is based on the Lemma 4.1 and it is similar to that in Theorem 4.1.

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ON THE STABILITY OF INSTANTANEOUSLY ELASTIC MATERIALS

BY

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To the memory of Professor Mendel Haimovici

1. Introduction. The general theory of dynamical systems developed by Gurtin in [7] is applied to the class of Instantaneously Elastic Materials (I.E.M.) [5], [3]. The main part of the paper consist in construction of dynamical system of I.E.M. in Gurtin's sense [7]. The results regarding the stability of general dynamical systems obtained in [7] hold for dynamical system of I.E.M. Some of these results are rendered evident for bodies with instantaneously elastic response. These results are given without proofs because they are straightforward consequences of [7].

2. General considerations on the behaviour of a continuous body [7] We denote by $\mathbf{E}_3$ the three-dimensional Euclidean affine space and with $\mathbf{V}_3$ the associated vector space. Let $\mathcal{B}$ be a continuous body identified with the regular region of $\mathbf{E}_3$ [6, p. 13] it occupies in a fixed reference configuration and $\mathbf{X}$ its material points.

The *configurations* of $\mathcal{B}$ are bijective continuous and piecewise continuously differentiable mappings

$$(2.1) \qquad \mathbf{x} : \mathcal{B} \to \mathbf{E}_3$$

with $\mathbf{x}(\mathcal{B})$ regular regions and with $\det \mathbf{F} > 0$, where $\mathbf{F} = \nabla \mathbf{x}$ is the deformation gradient.

The *velocity field* of $\mathcal{B}$ is a continuous and piecewise continuously differentiable mapping

$$(2.2) \qquad \mathbf{v} : \mathcal{B} \to \mathbf{V}_3.$$

The *temperature field* for $\mathcal{B}$ is a piecewise continuously differentiable mapping

$$(2.3) \qquad \theta : \mathcal{B} \to (0, \infty),$$

and

$$(2.4) \qquad \mathbf{g} = \operatorname{Grad} \theta : \mathcal{B} \to \mathbf{V}_3$$

is the *material gradient* of the temperature field.

If

$$\mathcal{C} = \{\mathbf{x} : \mathbf{x} \text{ is a configuration of } \mathcal{B}\},$$

$\mathcal{V}=\{\mathbf{v} : \mathbf{v}$ is a velocity field of $\mathcal{B}\}$,

$\mathcal{T}=\{\theta : \theta$ is a temperature field of $\mathcal{B}\}$,

$\mathcal{G}=\{\mathbf{g} : \mathbf{g}$ is the gradient of the temperature field of $\mathcal{B}\}$,

then $\mathcal{C}^*=\{\mathbf{x} : \mathbf{x}^* : (0,\infty)\to\mathcal{C}\}$, $\mathcal{T}^*=\{\theta^* : \theta^* : (0,\infty)\to\mathcal{T}\}$, and $\mathcal{G}^*=\{\mathbf{g} : \mathbf{g} : (0,\infty)\to\mathcal{G}\}$ are the *past histories* of the current configuration, temperature, and material gradient of temperature, respectively.

The past history $\mathbf{x}^*\in\mathcal{C}^*$ represents a possible configuration history for $\mathcal{B}$ and $\mathbf{x}^*(s)$, $s>0$, is the configuration occupied by $\mathcal{B}$ s time units into the past. The mapping $\mathbf{x}^t : (0,\infty)\to\mathcal{C}$ defined by $\mathbf{x}^t(s)=\mathbf{x}(t-s)$, $s>0$, belongs to $\mathcal{C}^*$ and represents the configuration occupied by $\mathcal{B}$ s time units prior to t. The same interpretations apply to θ^*, θ^t, and $\mathbf{g}^*$, $\mathbf{g}^t$. $\mathbf{x}^t$, θ^t, and $\mathbf{g}^t$ are called the histories of $\mathbf{x}$, θ, and $\mathbf{g}$ *up to time* t.

The historics

(2.5) $$\mathbf{x}^c(s)=\mathbf{x},\ \theta(s)=\theta,\ \mathbf{g}^c(s)=\mathbf{g},\quad s\in(0,\infty)$$

are the *constant histories* of $\mathbf{x}$, θ, and $\mathbf{g}$.

The *state function* of $\mathcal{B}$ is the set

(2.6) $$\Sigma=\{\sigma : \sigma=(\mathbf{x},\theta,\mathbf{g},\mathbf{v},\mathbf{x}^*,\theta,\mathbf{g}^*)$$

and a point $\sigma\in\Sigma$ is a *state* of $\mathcal{B}$.

A mapping

(2.7) $$\pi : [0,d_\pi)\to\Sigma,$$

is a *process* of $\mathcal{B}$ and d_π is the *duration* of π. The constant process is called the *rest process*.

Let $\mathcal{P}$ be the family of processes of $\mathcal{B}$ satisfying the condition that each $\pi\in\mathcal{P}$ is *maximal*, i.e. it has no continuation in $\mathcal{P}$. Such a family of processes is referred to as a *process class* for $\mathcal{B}$.

π_t denotes the value of the processes π at time $t\in[0,d_\pi)$, i.e. $\pi_t=\pi(t)$.

A mapping $f:\Sigma\to\mathbf{R}$, is referred to as a *state function* and $f(\pi)$ stands for $f\circ\pi$ so that $f(\pi) : [0,d_\pi)\to\mathbf{R}$ is the function

$$t\to f(\pi(t))=f(\pi_t),\ t\in[0,d_\pi).$$

The state function f has *trajectory derivative* over $\mathcal{P}$ if for each $\pi\in\mathcal{P}$, $f(\pi)$ is differentiable on $[0,d_\pi)$. In this case the function

$$t\to\frac{d}{dt}f(\pi_t)$$

is denoted by $\dot f$.

A map $\varphi:\Sigma\times\mathbf{R}\to\mathbf{W}$, where $\mathbf{W}$ is a topological space, is called *local state function* with values in $\mathbf{W}$ if for each $\sigma\in\Sigma$ the mapping $\varphi_\sigma:\mathcal{B}\to\mathbf{W}$ defined by

$$\varphi_\sigma(\mathbf{X})=\varphi(\sigma,\mathbf{X}),$$

is piecewise continuous.

The behaviour of $\mathcal{B}$ is determined by local state functions $\mathbf{x}$, θ, $\mathbf{g}$, $\mathbf{v}$, ε, η, $\mathbf{S}$, $\mathbf{q}$, and γ where $\mathbf{x}_\sigma\in\mathcal{C}$, $\theta_\sigma\in\mathcal{T}$, $\mathbf{g}_\sigma\in\mathcal{G}$, $\varepsilon_\sigma:\mathcal{B}\to\mathbf{R}$, $\eta_\sigma:\mathcal{B}\to\mathbf{R}$, $\mathbf{S}:\mathcal{B}\to\mathcal{L}(\mathbf{V}_3,\mathbf{V}_3)$ (the space of linear transformations of $\mathbf{V}_3$), $\mathbf{q}_\sigma:\mathcal{B}\to\mathbf{V}_3$, and $\gamma_\sigma:\mathcal{B}\to[0,\infty)$ are the configuration, the temperature field, the gradient of temperature field, the specific internal energy, the specific entropy, the Piola-Kirchhoff stress tensor, the heat flux, and the specific entropy production of $\mathcal{B}$ in the state σ.

The local state functions $\mathbf{x}_\sigma$, θ_σ, $\mathbf{g}_\sigma$, and $\mathbf{v}_\sigma$ are given by

(2.8) $$\mathbf{x}_\sigma=\mathbf{x},\ \theta_\sigma=\theta,\ \mathbf{g}_\sigma=\mathbf{g},\ \mathbf{v}_\sigma=\mathbf{v},$$

for every $\sigma\in\Sigma$.

The configurations $\mathbf{x}_\sigma$, $\sigma\in\Sigma$, are called admissible configurations and

(2.9) $$\mathcal{C}_a=\{\mathbf{x}\in\mathcal{C} : \mathbf{x}\ \text{is admissible}\}$$

is the set of admissible configurations. It is supposed that undiformed configuration $\mathbf{x}(\mathbf{X})=\mathbf{X}$ is an admissible configuration.

It is supposed that each (maximal) process π in the process class $\mathcal{P}$ defined by

(2.10) $$t\to\pi_t=\sigma(t)=(\mathbf{x}(t)\ \ \theta(t),\mathbf{g}(t),\mathbf{v}(t),\mathbf{x}^t,\theta^t,\mathbf{g}^t)$$

where $\mathbf{x}^t\in\mathcal{C}^*$, $\theta^t\in\mathcal{T}^*$, and $\mathbf{g}^t\in\mathcal{G}^*$ are the histories of $\mathbf{x}$, θ, and $\mathbf{g}$ up to time t, satisfies the following compatibility conditions

(2.11) $$\frac{\partial}{\partial t}\mathbf{x}_{\pi_t}(\mathbf{X})=\mathbf{v}_{\pi_t},\quad \operatorname{Grad}\ \theta_{\pi_t}(\mathbf{X})=\mathbf{g}_{\pi_t}(\mathbf{X}),$$

(2.12) $$f^t(s)=\begin{cases} f(t-s), & 0<s\leqslant t\\ f^c(s-t), & s>t,\end{cases}$$

where f stands for $\mathbf{x}$, θ, or $\mathbf{g}$.

The local state function of $\mathcal{B}$ determine the *global state functions* E, S, P, Q, J, and G where, for each $\sigma\in\Sigma$, $E(\sigma)$ is the *internal and kinetic energy* of $\mathcal{B}$ in σ, $S(\sigma)$ is the *entropy* of $\mathcal{B}$ in σ, $P(\sigma)$ is the *power expended* on $\mathcal{B}$ in σ, $Q(\sigma)$ is the *heat flow* into $\mathcal{B}$ in σ, $J(\sigma)$ is the entropy flow into $\mathcal{B}$ in σ, and G is the *entropy produced* by $\mathcal{B}$ in σ.

In the absence of heat supply and body force these functions are given by

(2.13) $$E(\sigma)=\int_{\mathcal{B}}\left(\varepsilon_\sigma+\frac{1}{2}\mathbf{v}^2\right)dm,\ S(\sigma)=\int_{\mathcal{B}}\eta_\sigma\,dm,$$

$$P(\sigma)=\int_{\partial\mathcal{B}}\mathbf{v}_\sigma\cdot\mathbf{S}_\sigma da,\ Q(\sigma)=-\int_{\partial\mathcal{B}}\mathbf{q}_\sigma\cdot\mathbf{n}\,da,$$

$$J(\sigma)=\int_{\partial\mathcal{B}}\frac{1}{\theta}\mathbf{q}_\sigma\cdot\mathbf{n}\,da,\ G(\sigma)=\int_{\mathcal{B}}\gamma_\sigma dm.$$

It is assumed that k and S have trajectory derivatives over the process class $\mathcal{P}$ and that each $\pi \in \mathcal{P}$ is consistent with *balance of energy* and *balance of entropy* [7], and that the entropy produced in each state $\sigma \in \Sigma$ is non-negative, i.e.

$$G(\sigma) = \int_{\mathcal{B}} \gamma_\sigma dm \geq 0. \tag{2.14}$$

The environment of $\mathcal{B}$ is described by the number $\theta_e > 0$ and the state functions U_e, P_e and G_e, where θ_e is the *environmental temperature*, $U_e(\sigma)$ is the *potential energy of the conservative external loads* in state σ, $P_e(\sigma)$ is the *power expended* on $\mathcal{B}$ by *non-conservative external loads* in σ, and $G_e(\sigma)$ is the entropy produced at the interface between the body and the environment in state σ. For each $\sigma \in \Sigma$ the non conservative loads are *dissipative* and the interfacial entropy production is non-negative, i.e.

$$P_e(\sigma) \leq 0, \quad G_e(\sigma) \geq 0. \tag{2.15}$$

Each $\pi \in \mathcal{D}$ is consistent with *interface conditions*

$$\begin{gathered} P(\pi) = -\dot{U}_e(\pi) + P_e(\pi), \\ J(\pi) = \frac{1}{\theta_e} Q(\pi) + G_e(\pi). \end{gathered} \tag{2.16}$$

relating the environment and the body [7].

It follows that state functions

$$\begin{gathered} V(\sigma) = U_e(\sigma) + k(\sigma) - \theta_e S(\sigma), \\ D(\sigma) = \theta_e [G(\sigma) + G_e(\sigma)] - P_e(\sigma) \end{gathered} \tag{2.17}$$

satisfy the conditions

$$\begin{gathered} \dot{V}(\pi) = -D(\pi), \quad \pi \in \mathcal{P}, \\ D(\sigma) \geq 0, \quad \sigma \in \Sigma. \end{gathered} \tag{2.18}$$

These conditions show that V is a natural Lyapunov function of $\mathcal{B}$ and D is the maximal dissipation function of $\mathcal{B}$ consistent with V, that is

$$V(\pi_a) - V(\pi_b) = \int_a^b D(\pi_t) dt \tag{2.19}$$

for every $[a, b] \in [0, d_\pi)$.

It is supposed that for all environment of $\mathcal{B}$ the potential energy U_e depends only on the admissible configurations, that is holds the implication

$$x_{\sigma_1} = x_{\sigma_2} \Rightarrow U_e(\sigma_1) = U_e(\sigma_2). \tag{2.20}$$

Therefore we have

$$U_e(\sigma) = U_e(\mathbf{x}_\sigma) = U_e(\mathbf{x}). \tag{2.21}$$

3. Some basic results in the theory of I.E.M. [5], [3], [1]. If

$$\sigma = (\mathbf{x}, \theta, \mathbf{g}, \mathbf{v}, \mathbf{x}^*, \theta^*, \mathbf{g}^*)$$

and $\mathbf{F} = \nabla \mathbf{x}$, $\mathbf{F}^* = \nabla \mathbf{x}^*$ we denote

$$\Lambda_\sigma = (\mathbf{F}, \theta, \mathbf{g}), \quad \Lambda^*_\sigma = (\mathbf{F}^*, \theta^*, \mathbf{g}^*) \tag{3.1}$$

Λ_σ is called the *site* [1] of the state σ and Λ^*_σ is the history of the site Λ_σ.

The process $\pi \in \mathcal{D}$ *occupies* the site $(\mathbf{F}, \theta, \mathbf{g})$ at time $t \in [0, d_\pi)$ if

$$\Lambda_{\pi_t} = (\mathbf{F}, \theta, \mathbf{g}). \tag{3.2}$$

An *I.E.M.* is defined by following constitutive functionals

$$\begin{gathered} \varepsilon_\sigma(\mathbf{X}) = \tilde{\varepsilon}(\Lambda_\sigma : \Lambda^*_\sigma), \quad \eta_\sigma(\mathbf{X}) = \tilde{\eta}(\Lambda_\sigma ; \Lambda^*_\sigma), \\ \mathbf{S}_\sigma(\mathbf{X}) = \tilde{\mathbf{S}}(\Lambda_\sigma ; \Lambda^*_\sigma), \quad \mathbf{q}_\sigma(\mathbf{X}) = \tilde{\mathbf{q}}(\Lambda_\sigma, \Lambda^*_\sigma), \end{gathered} \tag{3.3}$$

so that the free energy functional $\tilde{\psi}$ is

$$\psi_\sigma(\mathbf{X}) = \tilde{\psi}(\Lambda_\sigma ; \Lambda^*_\sigma) = \tilde{\varepsilon}(\Lambda_\sigma ; \Lambda^*_\sigma) - \theta \tilde{\eta}(\Lambda_\sigma ; \lambda^*_\sigma). \tag{3.4}$$

The argument $\mathbf{X}$ have been upressed in the second hand side of functionals in (3.3) so that the explicit dependence of $\mathbf{X}$ is implied.

The axiom defining the class of I.E.M. refer to the common domain $\mathcal{D}$ of functionals (3.3).

If the topology on $\mathcal{D}$ is given as in [2], [3], the smoothness properties of functionals (3.3) are referred to this topology and the assumption the material obey the fading memory hypothesis is adopted it follows that in any process $\pi \in \mathcal{D}$ the mapping

$$t \to \psi_{\pi_t}(\mathbf{X})$$

is continuously differentiable and

$$\dot{\psi}_{\pi_t}(\mathbf{X}) = \partial_\Lambda \tilde{\psi}(\Lambda(t) ; \Lambda^t) . \dot{\Lambda}(t) + \delta_\Lambda \tilde{\psi}(\Lambda(t) ; \Lambda^t \mid \dot{\Lambda}^t) \tag{3.5}$$

where

$$\begin{gathered} \partial_\Lambda \tilde{\psi}(\Lambda(t) ; \Lambda^t) . \dot{\Lambda}(t) = \delta_{\mathbf{F}} \tilde{\psi} . \dot{\mathbf{F}} + \partial_{\mathbf{F}} \tilde{\psi} . \dot{\theta} + \partial_{\mathbf{g}} \tilde{\psi} . \dot{\mathbf{g}} \\ \delta_\Lambda \tilde{\psi}(\Lambda(t) ; \Lambda^t \mid \dot{\Lambda}^t) = \delta_{\mathbf{F}} \tilde{\psi}(\Lambda(t), \Lambda^t \mid \dot{\mathbf{F}}^t) + \delta_\theta \tilde{\psi}(\Lambda(t) ; \Lambda^t \mid \dot{\theta}^t) + \\ + \delta_{\mathbf{g}} \tilde{\psi}(\Lambda(t) ; \Lambda^t \mid \mathbf{g}^t). \end{gathered} \tag{3.6}$$

Here $\partial_F \tilde{\psi}$, $\partial_\theta \tilde{\psi}$, $\partial_g \tilde{\psi}$ are the partial derivatives of $\tilde{\psi}$ with respect to $\mathbf{F}$, θ, $\mathbf{g}$ and $\delta_{\mathbf{F}}$, δ_θ, $\delta_{\mathbf{g}}$ are the Frechét derivatives with respect to $\mathbf{F}^t$, θ^t, $\mathbf{g}^t$ [2], [3], [1].

The Clausius-Duhem inequality holds if and only if [2], [3]:

1°. The free energy is independent of the current value of temperature gradient, i.e.

$$\partial_{g}\tilde{\psi}=0. \tag{3.7}$$

2°. The Piola-Kirchhoff stress tensor is determined by free energy through

$$\tilde{S}=\partial_{F}\tilde{\psi}. \tag{3.8}$$

3°. The entropy is determined by free energy by

$$\tilde{\eta}=-\partial_{\theta}\tilde{\psi}, \tag{3.9}$$

4°. The generalized dissipation inequality

$$\gamma(\pi\cdot)=\frac{1}{\theta(t)}\tilde{q}(\Lambda(t);\Lambda^{t})\cdot g-\delta_{\Lambda}\psi(\lambda(t):\Lambda^{t}\mid\Lambda^{t}) \tag{3.10}$$

holds in any process $\pi\in\mathcal{D}$.

The relation (3.7) shows that $\tilde{\psi}$ and therefore S and $\tilde{\eta}$ are independent of $g(t)$ though they may be affected by the past history of g. Therefore we have

$$\psi_{\sigma}(X)=\tilde{\psi}(F,\theta;\Lambda^{t}). \tag{3.11}$$

Definition. *The site* $\Lambda=(F,\theta,g)$ *is an equilibrium site* [1] *if for all processes* $\pi\in\mathcal{D}$ *occupying this site at time t we have*

$$q(\Lambda(t);\Lambda^{t})\cdot g=0. \tag{3.12}$$

It results that every site $(F,\theta,0)$ is an equilibrium site if there exists a process $\pi\in\mathcal{D}$ occupying this site at some time t.

We suppose that $(F,\theta,g;F^{c},\theta^{c},g^{c})\in\mathcal{D}$ whenever $(F,\theta,g;\Lambda^{*})\in\mathcal{D}$.

If $\Lambda=(F,\theta,g)$ is an equilibrium site the function

$$(F,\theta)\mapsto\hat{\psi}(F,\theta)$$

defined by

$$\hat{\psi}(F,\theta)=\tilde{\psi}(F,\theta;F^{c},\theta^{c},g^{c}) \tag{3.13}$$

is called the *equilibrium free energy* corresponding to state

$$\sigma=(x,\theta,g,v,x^{c},\theta^{c},g^{c})\in\Sigma \text{ if } F=\nabla x,\ g=\mathrm{Grad}\ \theta.$$

It is known that [2], [5], [3]

$$\hat{\psi}(F,\theta)\leq\tilde{\psi}(F,\theta;F^{*},\theta^{*},g^{*}). \tag{3.14}$$

for any $\sigma=(x,\theta,g,v,v^{*},\theta^{*},g^{*})\in\Sigma$.

We assume that the mapping

$$\theta\to\psi(F,\theta;\Lambda^{*})$$

is twice continuously differentiable and the *instantaneously specific heat* is strictly piostive that is

$$C(F,\theta;\Lambda^{*})=-\frac{\partial^{2}}{\partial\theta^{2}}\tilde{\psi}(F,\theta;\Lambda^{*})>0. \tag{3.15}$$

In these conditions it follows

$$\begin{aligned}\tilde{\varepsilon}(F,\theta;\Lambda^{*})-\theta_{e}\tilde{\eta}(F,\theta;\Lambda^{*})=\tilde{\psi}(F,\theta_{e};\Lambda^{*})+\\+K(F,\theta;\Lambda^{*})(\theta-\theta_{e})^{2},\end{aligned} \tag{3.16}$$

where the function

$$\begin{aligned}\theta\to K(F,\theta;\Lambda^{*})=\tfrac{1}{2}C(F,\theta_{e}+\tau(\theta-\theta_{e});\Lambda^{*}),\\ \theta\in(0,\infty),\ \tau\in(0,1),\end{aligned} \tag{3.17}$$

is strictly positive.

This result is known as Ericksen's lemma [4], [7].

From $(2.17)_1$, $(2.13)_{1,2}$, and (3.16) we obtain the Lyapunov function for an I.E.M.,

$$V(\sigma)=U_{e}(x)+\int_{\mathcal{B}}\left[\tilde{\psi}(F,\theta_{e};\Lambda^{*})+K(F,\theta;\Lambda^{*})(\theta-\theta_{e})^{2}+\frac{1}{2}v^{2}\right]dm. \tag{3.18}$$

4. The equilibrium function. We assume that the states

$$(x,\theta_{e},0,v,x^{*},\theta^{*},g^{*}),\quad(x,\theta,g,v,x^{c},\theta^{c},g^{c}),$$
$$(x,\theta_{e},0,v,x^{c},\theta^{c}_{e},\theta^{c}),\quad(x,\theta_{e},0,0,x^{c},\theta^{c}_{e},0^{c})$$

belong to Σ whenever $(x,\theta,g,v,x^{*},\theta^{*},g^{*})$.

Definition. *The state* $\sigma_{0}=(x,\theta_{e},0,0,x^{c},\theta^{c}_{e},0^{c})$ *is called the equilibrium state corresponding to state* $\sigma=(x,\theta,g,v,x^{*},\theta^{*},g^{*})$.

The mapping $\mathcal{E}:\Sigma\to\Sigma$ defined by

$$\sigma=(x,\theta,g,v,x^{*},\theta^{*},g^{*})\to\mathcal{E}(\sigma)=\sigma_{0}=(x,\theta_{e},0,0,x^{c},\theta^{c}_{e},0^{c}) \tag{4.1}$$

is the *equilibrium function* of an I.E.M. and the set

$$\Sigma_{0}=\mathcal{E}(\Sigma) \tag{4.2}$$

is called the *equilibrium state space* of an I.E.M.

The equilibrium function $\mathcal{E}$ is idempotent, i.e. $\mathcal{E}(\mathcal{E}(\sigma))=\mathcal{E}(\sigma)$ for every $\sigma\in\Sigma$.

On the basis of (4.1) the equilibrium state space Σ_0 is identified with the set $\mathcal{C}_a$ of admissible configurations and $(x,\theta_{e},0,0,x^{c},\theta^{c}_{e},0^{c})$ is the equilibrium state corresponding to the admissible configuration $x\in\mathcal{C}_a$.

The process $\pi\in\mathcal{D}$ *starts from equilibrium* if

$$\pi_{0}=\pi(0)=(x,\theta_{e},0,0,x^{0},\theta^{0},0^{c}). \tag{4.3}$$

where

$$x^{0}=x(0)^{c},\quad\theta^{0}=\theta^{c}_{e}. \tag{4.4}$$

To end this section we prove the main theorem of the paper.

Theorem 1. *The function* $V:\Sigma\to\mathbf{R}$ *defined by* (3.18) *is satisfying the following properties:*

1°. *For each* $\pi \in \mathcal{D}$ *the function* $t \to V(\pi_t)$ *is monotone decreasing.*
2°. *The function* $T : \Sigma \to \mathbf{R}$ *defined by*

$$T(\sigma) = V(\sigma) - V(\mathcal{E}(\sigma)) =$$

$$(4.5) \qquad = \int_{\mathcal{B}} \left[\bar{\psi}(\mathbf{F}, \theta_e ; \Lambda^t) - \hat{\psi}(\mathbf{F}, \theta_e) + K(\mathbf{F}, \theta ; \Lambda^t)(\theta - \theta_e)^2 + \frac{1}{2} \mathbf{v}^2 \right] dm$$

is non-negative in each proces $\pi \in \mathcal{D}$, *that is*,

$$T(\pi_t) \geq 0, \ t \in [0, d_\pi).$$

3°. $T(\sigma) = 0$ *if and only if* $\sigma \in \Sigma_0$.
4°. *If* $\pi \in \mathcal{D}$ *starts from equilibrium, and* $T(\pi_t) = 0$ *for all* $t \in [0, d_\pi)$ *then* π *is a rest process.*

Proof. The assertion 1° follows from the general properties (2.18) of functions (2.17), while 2° and 3° are consequences of (3.14), (3.17), and (4.1). Part 3° of theorem shows that $T(\pi_t) = 0$ for all $t \in [0, d_\pi)$ implies $\mathbf{v}(\mathbf{x}, t) = 0$, $\theta(\mathbf{x}, t) = \theta_e$ for all $t \in [0, d_\pi)$ so that $\mathbf{x}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X})$ and $\mathbf{g}(\mathbf{X}, t) = 0$ for all $t \in [0, d_\pi)$, $\mathbf{X} \in B$. Since π starts from equilibrium, i.e. (4.3) and (4.4) hold, the compatibility conditions (2.13) entail that

$$\mathbf{x}^t = \mathbf{x}(\mathbf{X}, 0)^c, \ \theta^t = \theta_e, \ \mathbf{g}^t = 0_c.$$

Therefore π is a rest process.

Let us consider the triplet $(\Sigma, \mathcal{D}, \mathcal{E})$ defined as above. Theorem 1 shows that $(\Sigma, \mathcal{D}, \mathcal{E})$ is a dynamical system in Gurtin's sense and the function V given by (3.18) is the Lyapunov function of this system. We call $(\Sigma, \mathcal{D}, \mathcal{E})$ the *dynamical system* of I.E.M.

The function $U : \Sigma_0 \to \mathbf{R}$ defined as the restriction of V to Σ_0 is called the *potential energy* of the dynamical system $(\Sigma, \mathcal{D}, \mathcal{E})$.

It results that

$$(4.6) \qquad U(\mathbf{x}) = U_e(\mathbf{x}) + \int_{\mathcal{B}} \hat{\psi}(\mathbf{F}, \theta_e) dm,$$

and therefore

$$(4.7) \qquad V(\sigma) = U(\mathbf{x}) + T(\sigma).$$

The potential energy U of an I.E.M. coincides with the potential energy of an elastic material provided the stored energy is interpreted as the equilibrium free energy.

A function $D : \Sigma \to [0, \infty)$ is a *dissipation function* for the dynamical system $(\Sigma, \mathcal{D}, E)$ consistent with V [7] if for each $\pi \in \mathcal{D}$ the mapping $t \to D(\pi_t)$ is integrable over any interval $[a, b] \subset [0, d_\pi)$ and

$$(4.8) \qquad V(\pi_a) - V(\pi_b) \geq \int_a^b D(\pi_t)\, dt.$$

If (4.8) holds with $\geq$ replaced by $=$, D is a maximal dissipation function of the system.

From (2.19) it follows the function $(2.17)_2$ is a maximal dissipation function.

From (2.14), (2.15), and (2.17) it results that the functions $\theta_e G$, $\theta_e G_e$, and $-P_e$ are dissipation functions for dynamical system of I.E.M. consistent with function (3.18). In our case

$$(4.9) \qquad G(\pi_t) = \int_{\mathcal{B}} \gamma(\pi_t) dm,$$

where $\gamma(\pi_t)$ is given by (3.10).

5. Some results regarding the stability of I.E.M.

In view of Theorem 1 all results established in [7] for a general dynamical system may be applied to the dynamical system of I.E.M. In what follows we give without proofs three results on the stability of I.E.M. These results are analogous to Theorems (10.4), (10.5), and (10.6) of [7].

Theorem 2. *Suppose that the potential energy* (4.6) *has a global minimum* [7] *at an admissible configuration* $\mathbf{x}_0 \in \mathcal{C}_a$. *In this case for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that any process* $\pi \in \mathcal{D}$ *satisfies*

$$\int_{\mathcal{B}} \mathbf{v}^2(t) dm < \varepsilon, \quad \int_{\mathcal{B}} K(\mathbf{F}(t), \theta(t) ; \Lambda^t) [\theta(t) - \theta_e]^2 dm < \varepsilon,$$

$$(5.1) \qquad \int_{\mathcal{B}} [\bar{\psi}(\mathbf{F}(t), \theta_e ; \Lambda^t) - \hat{\psi}(\mathbf{F}(t), \theta_e)]\, dm < \varepsilon$$

$$U(\mathbf{x}(t)) - U(\mathbf{x}_0) < \varepsilon, \qquad \int_0^{d_\pi} G(\pi) d\tau < \varepsilon.$$

for all $t \in [0, d\pi)$, *as soon as*

$$\int_{\mathcal{B}} \mathbf{v}^2(0) dm < \delta, \ \int_{\mathcal{B}} K(\mathbf{F}(0), \theta(0) ; \Lambda^0) [\theta(0) - \theta_e]^2 dm < \delta$$

$$(5.2) \qquad \int_{\mathcal{B}} [\bar{\psi}(\mathbf{F}(0), \theta_e ; \Lambda^0) - \hat{\psi}(\mathbf{F}(0), \theta_e)] dm < \delta$$

$$U(\mathbf{x}(0)) - U(\mathbf{x}_0) < \delta.$$

Assuming that the admissible configurations set $\mathcal{C}_a$ is endowed with the topology induced by the topology of $\mathcal{D}$ and that the mapping $t \to \mathbf{x}(t)$ is continuous for each $\pi \in \mathcal{D}$ we have

Theorem 3. *If* $\Omega_0 \subset C_a$ *is a potential well for* U *relative to an admissible configuration* $\mathbf{x}_0$ [7], *then for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that any process* $\pi \in \mathcal{D}$ *satisfying* (5.2) *and having* $\mathbf{x}(0) \in \Omega_0$ *satisfies* (5.1) *and has* $\mathbf{x}(t) \in \Omega_0$ *for all* $t \in [0, d_\pi)$.

If the topology on $\mathcal{C}_a$ is generated by a pseudo-metric ρ it holds

Theorem 4. *If U has a strong local minimum* [7] *at an admissible configuration* $x_0 \in \mathcal{C}_a$ *then given any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that any process* $\pi \in \mathcal{D}$ *satisfying* (5.2) *and*

$$\rho(x(0), x_0) < \delta,$$

satisfies (5.1) *and*

$$\rho(x(t), x_0) < \varepsilon$$

for all $t \in [0, d_\pi)$.

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A QUASI-VARIATIONAL INEQUALITY OCCURRING IN FRICTION PROBLEMS

BY

MARIUS COCU

To the memory of Professor Mendel Haimovici.

An important and very useful class of non-linear problems arising from mechanics, physics, etc., consists of the so-called quasi-variational inequalities ([4], [7]).

In this paper we shall restrict our attention to the study of the existence und uniqueness of the solution of some quasi-variational inequality which may be considered as a direct generalisation of elliptic variational inequalities of the second kind (Lions-Stampacchia [6]). Some examples and applications to friction problems are considered or indicated.

1. Introduction. The abstract problem, denoted by P, consists in finding a vector $u \in K \cap H$ that satisfies the following quasi-variational inequality

(1.1) $$a(u, v-u) + j(u, v) - j(u, u) \geq f(v-u) \quad \forall v \in K,$$

where

(1.2) $(H, (\cdot,\cdot))$ and $(V, (\cdot,\cdot)_v)$ are two real Hilbert spaces,

H densely imbedded in V, $H \overset{d}{\hookrightarrow} V$, $\| v \|_v \leq \| v \| \quad \forall v \in H$,

K is a closed, convex subset of V such that $\mathring{K} \neq \emptyset$; it follows from above that $K \cap H$ is convex and closed in H. We observe that $K \cap H$ is non-empty. $a : V \times V \to R$ is a bilinear, continuous and V-elliptic form, that is

(1.3) $| a(u, v) | \leq \| \bar{A} \|_{(v)} \| u \|_v \| v \|_v$ where $\bar{A} : V \to V'$ denotes the operator $(\bar{A}u, v) = a(u, v)$ and α is a positive constant such that

(1.4) $$a(v, v) \geq \alpha \| v \|_v^2 \quad \forall u, v \in V;$$

f is a linear and continuous form on V; $j : H \times K \to R$ is such that for each $u \in H$, $j(u,.)$ is a convex and continuous functional on K and

(1.5) $$| j(u_1, v_2) + j(u_2, v_1) - j(u_1, v_1) - j(u_2, v_2) | \leq C \| u_1 - u_2 \| \|v_1 - v_2\|_v, \quad \forall u_1, u_2 \in H, \ \forall v_1, v_2 \in K \text{ with}$$

(1.6) $$2\alpha > C \text{ and } \| \bar{A} \| > C$$

Remark 1.1. In general, we do not assume $a(\cdot,\cdot)$ to be symmetric since in some applications non-symmetric bilinear forms may occur naturally.

Remark 1.2. If $j = 0$, we obtain a variational inequality of the first kind; if $j(u, w) = j(v, w)$, $\forall u, v \in H$, $\forall w \in K$, we obtain a variational inequality of the second kind.