If the topology on $\mathcal{C}_a$ is generated by a pseudo-metric $\rho$ it holds

**Theorem 4.** *If $U$ has a strong local minimum [7] at an admissible configuration $\mathbf{x}_0 \in \mathcal{C}_a$ then given any $\varepsilon > 0$ there exists $\delta > 0$ such that any process $\pi \in \mathcal{D}$ satisfying (5.2) and*

$$\rho(\mathbf{x}(0), \mathbf{x}_0) < \delta,$$

*satisfies (5.1) and*

$$\rho(\mathbf{x}(t), \mathbf{x}_0) < \varepsilon$$

*for all $t \in [0, d_\pi)$.*

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*Department of Mathematics*
*Polytechnical Institute of Iaşi*
*6600—Iaşi, R.S. Romania*



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# A QUASI-VARIATIONAL INEQUALITY OCCURRING IN FRICTION PROBLEMS

BY


MARIUS COCU


*To the memory of Professor Mendel Haimovici.*

An important and very useful class of non-linear problems arising from mechanics, physics, etc., consists of the so-called quasi-variational inequalities ([4], [7]).

In this paper we shall restrict our attention to the study of the existence und uniqueness of the solution of some quasi-variational inequality which may be considered as a direct generalisation of elliptic variational inequalities of the second kind (L i o n s-S t a m p a c c h i a [6]). Some examples and applications to friction problems are considered or indicated.

**1. Introduction.** The abstract problem, denoted by $P$, consists in finding a vector $u \in K \cap H$ that satisfies the following quasi-variational inequality

(1.1) $$a(u, v-u) + j(u, v) - j(u, u) \geqslant f(v-u) \quad \forall v \in K,$$

where

(1.2) $(H, (\cdot,\cdot))$ and $(V, (\cdot,\cdot)_v)$ are two real Hilbert spaces, $H$ densely imbedded in $V$, $H \hookrightarrow V$, $\| v \|_v \leqslant \| v \| \quad \forall v \in H$,

$K$ is a closed, convex subset of $V$ such that $\mathring{K} \neq \emptyset$; it follows from above that $K \cap H$ is convex and closed in $H$. We observe that $K \cap H$ is non-empty. $a : V \times V \to R$ is a bilinear, continuous and $V$-elliptic form, that is

(1.3) $| a(u, v) | \leqslant \| \overline{A} \|_{(v)} \| u \|_v \| v \|_v$ where $\overline{A} : V \to V'$ denotes the operator $(\overline{A}u, v) = a(u, v)$ and $\alpha$ is a positive constant such that

(1.4) $$a(v, v) \geqslant \alpha \| v \|_v^2 \quad \forall u, v \in V;$$

$f$ is a linear and continuous form on $V$; $j : H \times K \to R$ is such that for each $u \in H$, $j(u,.)$ is a convex and continuous functional on $K$ and

(1.5) $$| j(u_1, v_2) + j(u_2, v_1) - j(u_1, v_1) - j(u_2, v_2) | \leqslant C \| u_1 - u_2 \| \| v_1 - v_2 \|_v, \quad \forall u_1, u_2 \in H, \ \forall v_1, v_2 \in K \text{ with}$$

(1.6) $$2\alpha > C \text{ and } \| \overline{A} \| > C$$

*Remark* 1.1. In general, we do not assume $a(\cdot,\cdot)$ to be symmetric since in some applications non-symmetric bilinear forms may occur naturally.

*Remark* 1.2. If $j = 0$, we obtain a variational inequality of the first kind; if $j(u, w) = j(v, w)$, $\forall u, v \in H$, $\forall w \in K$, we obtain a variational inequality of the second kind.

*Remark* 1.3. We may also consider the inequality (1.1) where $K = V$.

**2. An existence and uniqueness result for problem** $P$. We shall prove the following theorem

**Theorem 2.1.** *Let $H, V, K, a, f$ and $j$ be given satisfying* (1.2)–(1.6) *respectively. Then problem $P$ (1.1) has one and only one solution.*

*Proof.* For each $u \in K \cap H$ and $\rho > 0$, we associate a problem $P(u, \rho)$, namely the following variational inequality of the second kind

$$\text{to find } w \in K \cap H \text{ such that, } \forall v \in H \cap K, \tag{2.1}$$
$$(w, v-w) + \rho j(u, v) - \rho j(u, w) \geq (u, v-w) + \rho f(v-w) - \rho a(u, v-w).$$

From the properties (1.2)–(1.6) and standard results (Lions-Stampacchia [6] or Duvaut-Lions [3], etc.) it follows that the problem $P(u, \rho)$ has one and only one solution $w$, denoted by $S_\rho(u)$.

In this case it is easy to show that $u$ is a solution of $P$ if and only if $u$ is a fixed point of the map $S_\rho$ $K \cap H \to K \cap H$ for some $\rho > 0$. If $S_\rho$ has a unique fixed point $u$ for some $\rho > 0$, this $u$ turns out to be the unique solution of problem $P$. In order to show the uniqueness of $u$, observe that if $u'$ is a solution of $P$, then $u'$ is a fixed point of $S_\rho$ for every $\delta > 0$ and hence for $\rho$ chosen as above, $S_\rho$ has two distinct fixed points, $u$ and $u'$, which contradicts the choice of $\rho$.

We shall prove that $S_\rho$ is a strict contracting mapping for suitably chosen $\rho$ and hence has a unique fixed point $u$.

Let $u_i \in K \cap H$ and $w_i = S_\rho(u_i)$, $i = 1, 2$. Therefore we have

$$(w_1, w_2 - w_1) + \rho j(u_1, w_2) - \rho j(u_1, w_1) \geq (u_1, w_2 - w_1) + \rho f(w_2 - w_1) - \tag{2.2}$$
$$- \rho a(u_1, w_2 - w_1), \text{ and}$$

$$(w_2, w_1 - w_2) + \rho\, j(u_2, w_1) - j(u_2, w_2) \geq (u_2, w_1 - w_2) + \rho\, f(w_1 - w_2) - \tag{2.3}$$
$$- \rho\, a(u_2, w_1 - w_2).$$

Adding these two inequalities we obtain

$$-\|w_1 - w_2\|^2 + \rho[j(u_1 w_2) - j(u_1, w_1) + j(u_2, w_1) - j(u_2, w_2)] \geq -(u_1 - u_2, w_1 - \tag{2.4}$$
$$- w_2) + \rho a(u_1 - u_2, w_1 - w_2) = - <(I - \rho\bar{A})(u_1 - u_2),\ w_1 - w_2>.$$

Hence

$$\| S_\rho(u_1) - S_\rho(u_2) \|^2 = \| w_1 - w_2 \|^2 \leq \rho[j(u_1, w_2) + j(u_2, w_1) - j(u_1, w_1) -$$
$$- j(u_2, w_2)] + + <(I - \rho\bar{A})(u_1 - u_2),\ w_1 - w_2> \leq C\rho \| u_1 - \tag{2.5}$$
$$- u_2 \| \, \| w_1 - w_2 \|_v + \| (I - \rho A)\,(\bar{u_1} - u_2)\, v \| \, w_1 - w_2 \|_v$$

where we used (1.5) and (1.3). With (1.4) we obtain the estimate

$$\| (I - \rho\bar{A})(u) \|^2 = ((I - \rho\bar{A})u,\ (I - \rho\bar{A})u)_v = \| u \|_v^2 - 2\rho(\bar{A}u, u)_v +$$
$$+ \rho^2 \| Au \|_v^2 \leq \rho^2 \| \bar{A} \|_v^2 \| u \|_v^2 - 2\rho\alpha \| u \|_v^2 + \| u \|_v^2 = (\rho^2 \| \bar{A} \|_v^2 - \tag{2.6}$$
$$- 2\rho\alpha + 1) \| u \|_v^2 \leq (\rho^2 \| \bar{A} \|_v^2 - 2\alpha\rho + 1) \| u \|^2,$$

where we identified $\bar{A}u \in V'$ with an element of $V$ by Riesz's theorem. Using (2.5), (2.6) and (1.2) we have:



$$\| w_1 - w_2 \|^2 \leq (C\rho + \sqrt{\rho^2 \| \bar{A} \|_v^2 - 2\alpha\rho + 1}) \| u_1 - u_2 \| \, \| w_1 - w_2 \|. \tag{2.7}$$

Hence $\forall u_1, u_2 \in K \cap H$,

$$\| S_\rho(u_1) - S_\rho(u_2) \| = \| w_1 - w_2 \| \leq [\sqrt{\rho^2 \| \bar{A} \|_v^2 - 2\alpha\rho + 1} - C\rho] \cdot \| u_1 - u_2 \|. \tag{2.8}$$

Thus $S_\rho : K \cap H \to K \cap H$ is a strict contracting mapping for $0 < \rho < (2\alpha - C)/\| \bar{A} \|^2$ and hence has a unique fixed point $u \in K \cap H$ which is the (unique) solution of problem $P$ q.e.d.

*Remark* 2.1. The advantage of considering the problem $P(u, \rho)$ is that the bilinear form associated with $P(u, \rho)$ is the inner product of $H$, which is symmetric and important in applications.

*Remark* 2.2. For simplicity, we restricted our attention to a quasi-variational inequality involving linear operators defined on a real Hilbert space, but the results can be extended.

*Remark* 2.3. For a description, in a general framework, of different methods that have been used in the analysis of quasi-variational inequalities, see Mosco [7].

**3. Some Applications.** We shall now consider a Signorini friction problem (Duvaut [2], [1]).

Let $\Omega$ be a bounded domain of $R^3$, with a smooth boundary $\Gamma = \partial\Omega$ $g \in [L^2(\Omega)]^3$ and we define

$$V = \{v \mid v \in [H^1(\Omega)]^3,\ v = 0 \text{ on } \Gamma_u\}\ (u, v)_v = (u_i, v_i)_{L^2(\Omega)} + \tag{3.1}$$
$$+ (\partial u_i/\partial x_j,\ \partial v_i/\partial x_j)_{L^2(\Omega)}\ (Au)_i = -\partial/\partial x_j(a_{ijkh}\varepsilon_{kh}(u)) = -\partial/\partial x_j\, \sigma_{ij}(u), \text{ where}$$

$$\varepsilon_{kh} = (1/2)(\partial u_k/\partial x_h + \partial u_h/\partial x_k), \tag{3.2}$$

$$a_{ijkh} = a_{jikh} = a_{khij} \in L^\infty(\Omega) \text{ such that } \forall u_i \in C^\infty(\bar{\Omega}) \Rightarrow (Au)_i \in L^2(\Omega), \tag{3.3}$$
$$a_{ijkh}\varepsilon_{ij}\varepsilon_{kh} \geq \beta\varepsilon_{ij}\varepsilon_{ij} \quad \forall \varepsilon_{ij} \in R \quad \forall i, j, k, h = 1, 2, 3.$$

$$H = \{u \mid u \in V,\ Au \in [L^2(\Omega)]^3\}, \text{ with} \tag{3.4}$$

$$\| u \| = \{ \| u \|^2_{[H_1(\Omega)]^3} + \| Au \|^2_{[L^2(\Omega)]^3} \}^{1/2}, \tag{3.5}$$

$$\Gamma = \Gamma_U \cup \Gamma_f \cup \Gamma_f \text{ and mes } (\Gamma_U) > 0,$$

$$K = \{v \mid v \in V,\ v_{i|\Gamma_u} = 0,\ v_n \leq 0 \text{ on } \Gamma_f\}, \text{ where} \tag{3.6}$$

$$v_n = v_i n_i \quad v_{ti} = v_i - v_n n_i \text{ and } \sigma_n = \sigma_{ij} n_i n_j,\ \sigma_i = \sigma_{ij} n_j - \sigma_n n_i \tag{3.7}$$

are the normal and tangential components and denote by $a$ the bilinear form $a : V \times V \to R$:

$$a(u, v) = \int_\Omega a_{ijkh}\varepsilon_{kh}(u)\,\varepsilon_{ij}(v)\,dx = \int_\Omega \sigma_{ij}(u)\,\varepsilon_{ij}(v)\,dx \tag{3.8}$$

$a$ is $V$-elliptic (Korn's inequality)

$$a(u, u) \geq \alpha \| u \|_v^2, \quad u \in V \tag{3.9}$$

$$j : H \times V \to R_+ \text{ defined by } j(u, v) = \int_{\Gamma_f} \mu \mid \sigma_n^*(u) \mid \mid v_t \mid ds \tag{3.10}$$

where we suppose that the mapping $H^{-1/2}(\Gamma_f) \ni \tau \to \tau^* \in L^2(\Gamma_f)$ is linear and continuous (a regularization of $\tau$) and preserves the positivity.

We recall that by trace thorems (see Lions-Magenes [5]) for any $u \in H$ the normal component $\sigma_n$ is defined by duality as an element of the fractional Sobolev space $H^{-1/2}(\Gamma_f)$ and $\| \sigma_n(u) \|_{H^{-1/2}} \leqslant C_1 \| u \|_H$. Thus $j(u, v) \leqslant C \| u \| \| v \|_V$ $\forall u \in H$, $\forall v \in V$ (using Schwartz ineq.)

Let us consider the following quasi-variational inequality *to find $u \in K \cap H$ such that*

$$a(u, v-u) + j(u, v) - j(u, u) \geqslant f(v-u) \qquad \forall v \in K,$$

$$\text{where } f(v) = \int_{\Omega} g v \, dx + \int_{\Gamma_f} F v \, dx \qquad F \in [L^2(\Gamma_F)]^3. \tag{3.11}$$

**Theorem 3.1.** *There exists $\mu' > 0$ such that the q.v.i.* (3.11) *has one and only one solution for $0 \leqslant \mu \leqslant \mu'$.*

*Proof.* We shall apply Theorem 2.1; the only assumption to be verified is (1.6) and this is realised if we take $\mu'$ such that $\alpha > C$.q.e.d.

We do not show in detail that (3.11) is indeed the weak formulation of the following friction problem occuring in elasticity (see Duvaut [2], Duvaut-Lions [3])

$$\partial / \partial x_j \sigma_{ij}(u) + f_i = 0 \text{ in } \Omega (\text{or } Au = f), \quad \sigma_{ij} = a_{ijkh} \varepsilon_{kh}(u), \tag{3.12}$$

$$u = 0 \text{ on } \Gamma_U \text{ with mes } (\Gamma_U) > 0, \tag{3.13}$$

$$\sigma_{ij} n_j = F_i \text{ on } \Gamma_F, \tag{3.14}$$

$$u_n \leqslant 0, \ \sigma_n \leqslant 0, \ u_n \sigma_n = 0 \text{ on } \Gamma_f, \tag{3.15}$$

$$\begin{gathered} |\sigma_t| \leqslant \mu |\sigma_n^*| \ . \ |\sigma_t| < \mu |\sigma_n^*| \Rightarrow u_t = 0 \text{ on } \Gamma_f, \\ |\sigma_t| = \mu |\sigma_n^*| \Rightarrow \exists \lambda \geqslant 0 \text{ such that } u_t = -\lambda \sigma_t \text{ on } \Gamma_f. \end{gathered} \tag{3.16}$$

It should be observed that (3.16) represent. a non-local friction which may be obtained from the Coulomb friction law.

Other examples which can be considered by specializing the Theorem 2.1 are given in Duvaut-Lions [3], Ch. 3, 5.

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# ANELASTIC PROPERTIES OF SOLID POROUS MATERIALS

BY

M. CONSTANTINESCU and N. CRISTESCU

*To the memory of Professor Mendel Haimovici*

The aim of the present paper is to formulate the constitutive equation for porous solid materials as rocks, cement concrete, concrete etc., starting from various kinds of experiments.

In most cases where cement concrete is used relatively small mean stresses are involved and therefore of significance is the determination of the constitutive equation in unconfined tests. If however, high mean stresses are involved, the constitutive equation already established by uniaxial tests can be generalized if additional tests are done in the triaxial apparatus [1].

**1. Unconfined tests.** In order to determine the constitutive equation for cement concrete an important diagnostic test can be performed with a standard testing machine which applies an axial stress $\sigma_1$ to a cylindrical specimen whose lateral surface remains free of stress, i.e. $\sigma_2 = \sigma_3 = 0$. In such tests called unconfined, the axial strain and diameter strain are recorded with various devices. Details of the set-up experiment are given in [2].

Generally the experiments, size of specimen, ways of measuring strains etc. have followed the norms suggested by the International Society of Rock Mechanics [3]. The standard kind of specimens have been manufactured by the first of the authors [4].

Typical stress-strain curves $\sigma_1 - \varepsilon_1$, $\sigma_1 - \varepsilon_2$ and $\sigma_1 - \varepsilon_v$ are shown on fig. 1, which were obtained for $\dot{\varepsilon}_1 = 9 \times 10^{-6}$ sec$^{-1}$. Here $\varepsilon_1$ is the axial strain, $\varepsilon_2$ the diameter strain and $\varepsilon_v = \varepsilon_1 + 2\varepsilon_2$ the volumetric strain. The following conclusions yield from fig. 1.

— the beginning of the stress-strain curves are nearly straight lines;
— for higher stresses the stress-strain curves are nonlinear;
— unloading is generally a nonlinear process;
— after full unloading a significant irreversible portion of the strain is put into evidence as well as an increase of the density;
— the loading produces a certain strain-hardening which certainly has another meaning than for metals.

The so called „elastic constants" are not really constant as for most rock-like materials. These constants are generally increasing with the increasing of the maximum applied stress. There are several methods to determine these constants [3]. For instance the Young modulus can be determined by a „tangent" procedure, a „secant" procedure or during unloading:

$$E_{\text{tang}} = \frac{\Delta \sigma_1}{\Delta \varepsilon_1}, \quad E_{\text{secant}} = \frac{\sigma_1}{\varepsilon_1}, \quad E_{\text{unload}} = \frac{\sigma_1}{\varepsilon_1 - \varepsilon_{1\text{plastic}}}.$$