where we suppose that the mapping $H^{-1/2}(\Gamma_f) \ni \tau \to \tau^* \in L^2(\Gamma_f)$ is linear and continuous (a regularization of τ) and preserves the positivity.

We recall that by trace thorems (see Lions-Magenes [5]) for any $u \in H$ the normal component σ_n is defined by duality as an element of the fractional Sobolev space $H^{-1/2}(\Gamma_f)$ and $\| \sigma_n(u) \|_{H^{-1/2}} \leqslant C_1 \| u \|_H$. Thus $j(u, v) \leqslant C \| u \| \| v \|_r$ $\forall u \in H$, $\forall v \in V$ (using Schwartz ineq.)

Let us consider the following quasi-variational inequality *to find $u \in K \cap H$ such that*

$$a(u, v-u) + j(u, v) - j(u, u) \geqslant f(v-u) \qquad \forall v \in K,$$

(3.11)
$$\text{where } f(v) = \int_\Omega gv\,dx + \int_{\Gamma_f} Fv\,dx \qquad F \in [L^2(\Gamma_F)]^3.$$

Theorem 3.1. *There exists $\mu' > 0$ such that the q.v.i.* (3.11) *has one and only one solution for $0 \leqslant \mu \leqslant \mu'$.*

Proof. We shall apply Theorem 2.1; the only assumption to be verified is (1.6) and this is realised if we take μ' such that $\alpha > C$, q.e.d.

We do not show in detail that (3.11) is indeed the weak formulation of the following friction problem occuring in elasticity (see Duvaut [2], Duvaut-Lions [3])

(3.12) $$\partial/\partial x_j \sigma_{ij}(u) + f_i = 0 \text{ in } \Omega (\text{or } Au = f), \quad \sigma_{ij} = a_{ijkh} \varepsilon_{kh}(u),$$

(3.13) $$u = 0 \text{ on } \Gamma_U \text{ with mes } (\Gamma_U) > 0, \qquad (3.14) \quad \sigma_{ij} n_j = F_i \text{ on } \Gamma_F,$$

(3.15) $$u_n \leqslant 0, \ \sigma_n \leqslant 0, \ u_n \sigma_n = 0 \text{ on } \Gamma_f,$$

(3.16) $$| \sigma_t | \leqslant \mu | \sigma_n^* |, \ | \sigma_t | < \mu | \sigma_n^* | \Rightarrow u_t = 0 \text{ on } \Gamma_f,$$
$$| \sigma_t | = \mu | \sigma_n^* | \Rightarrow \exists \lambda \geqslant 0 \text{ such that } u_t = -\lambda \sigma_t \text{ on } \Gamma_f.$$

It should be observed that (3.16) represent a non-local friction which may be obtained from the Coulomb friction law.

Other examples which can be considered by specializing the Theorem 2.1 are given in Duvaut-Lions [3], Ch. 3, 5.

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ANELASTIC PROPERTIES OF SOLID POROUS MATERIALS

BY

M. CONSTANTINESCU and N. CRISTESCU

To the memory of Professor Mendel Haimovici

The aim of the present paper is to formulate the constitutive equation for porous solid materials as rocks, cement concrete, concrete etc., starting from various kinds of experiments.

In most cases where cement concrete is used relatively small mean stresses are involved and therefore of significance is the determination of the constitutive equation in unconfined tests. If however, high mean stresses are involved, the constitutive equation already established by uniaxial tests can be generalized if additional tests are done in the triaxial apparatus [1].

1. Unconfined tests. In order to determine the constitutive equation for cement concrete an important diagnostic test can be performed with a standard testing machine which applies an axial stress σ_1 to a cylindrical specimen whose lateral surface remains free of stress, i.e. $\sigma_2 = \sigma_3 = 0$. In such tests called unconfined, the axial strain and diameter strain are recorded with various devices. Details of the set-up experiment are given in [2].

Generally the experiments, size of specimen, ways of measuring strains etc. have followed the norms suggested by the International Society of Rock Mechanics [3]. The standard kind of specimens have been manufactured by the first of the authors [4].

Typical stress-strain curves $\sigma_1 - \varepsilon_1$, $\sigma_1 - \varepsilon_2$ and $\sigma_1 - \varepsilon_v$ are shown on fig. 1, which were obtained for $\dot{\varepsilon}_1 = 9 \times 10^{-6}$ sec^{-1}. Here ε_1 is the axial strain, ε_2 the diameter strain and $\varepsilon_v = \varepsilon_1 + 2\varepsilon_2$ the volumetric strain. The following conclusions yield from fig. 1.

— the beginning of the stress-strain curves are nearly straight lines;
— for higher stresses the stress-strain curves are nonlinear;
— unloading is generally a nonlinear process;
— after full unloading a significant irreversible portion of the strain is put into evidence as well as an increase of the density;
— the loading produces a certain strain-hardening which certainly has another meaning than for metals.

The so called „elastic constants" are not really constant as for most rock-like materials. These constants are generally increasing with the increasing of the maximum applied stress. There are several methods to determine these constants [3]. For instance the Young modulus can be determined by a „tangent" procedure, a „secant" procedure or during unloading:

$$E_{\text{tang}} = \frac{\Delta\sigma_1}{\Delta\varepsilon_1}, \quad E_{\text{secant}} = \frac{\sigma_1}{\varepsilon_1}, \quad E_{\text{unload}} = \frac{\sigma_1}{\varepsilon_1 - \varepsilon_{1\text{plastic}}}.$$

Similar procedures are used for Poisson ratio ν, the bulk modulus K and the shearing modulus G. Of significance for the constitutive equation are the values of the elastic constants determined during unloading or during fast loading tests.

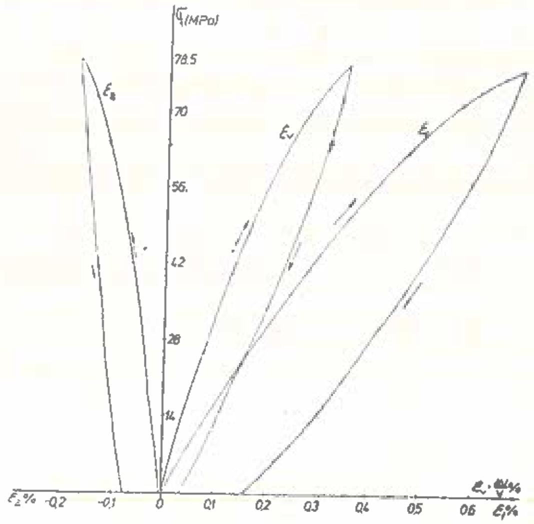

Fig. 1

2. **Hencky type constitutive equation.** A Hencky type constitutive equation is to be reformulated in order to describe the specific behaviour of the volume for rock-like materials, mainly the nonlinear and partially irreversible compressibility of the volume.

Starting from the experimental results obtained with unconfined tests [4] the following assumptions will be put forward :

— the volume is compressible and a part of the volume strain is irreversible ;

— the elastic and plastic components of the strain deviator are additive ; the elastic part of the strain satisfies the Hooke law while the plastic one is proportional with the stress deviator ;

— there is no sharp initial yield stress and it can be assumed that elastic and plastic deformation take place from the smallest applied stresses ;

— material rotations and displacements are small ; initial reference configuration is used.

Thus

(1) $$\varepsilon'_{ij} = \varepsilon'^{E}_{ij} + \varepsilon'^{P}_{ij}, \quad \varepsilon'^{E}_{ij} = \frac{\sigma'_{ij}}{2G}, \quad \varepsilon'^{P}_{ij} = \lambda\sigma'_{ij}.$$

Here G is the elastic shear modulus determined in unloading tests. Generally G will be assumed to be constant on portion. However, if necessary G can be considered to be variable with maximum applied stress according to the relationship determined by experiments

(2) $$\begin{aligned} G &= 51\,200 + 187\ \sigma_1 \ \text{if}\ \sigma_1 < 100 \\ G &= 69\,900 + 7.33\ (\sigma_1 - 100)\ \text{if}\ 100 \leqslant \sigma_1 \leqslant 530 \\ G &= 73\,000 + 64.3\ (\sigma_1 - 530)\ \text{if}\ 530 \leqslant \sigma_1 \leqslant 740 \end{aligned}$$

where all constants are given in daN cm^{-2}. The failure of the specimen takes place at the compressive stress $\sigma_{1c} = 740$ daN cm^{-2}.

A simple mathematical expression which can describe quite well the volume compressibility is for instance

(3) $$\varepsilon = -\varepsilon_* \ln\left(1 - \frac{\sigma_1}{3\sigma_*}\right)$$

where ε_* and σ_* are material constants and ε the mean strain. For relatively small stresses ($\sigma_1 < 400$ daN cm^{-2} say) the relationship (3) can be approximated by a linear relation

(4) $$\sigma = \frac{\sigma_*}{\varepsilon_*}\ \varepsilon = 3K\varepsilon$$

where σ is the mean stress and K the secant bulk modulus.

During unloading the volume variation can be approximated by a linear relation

(5) $$\varepsilon - \varepsilon_M = \frac{1}{3K_u}\ (\sigma - \sigma_M)$$

where ε_M and σ_M are the maximum values reached during loading and K_u is the bulk modulus during unloading , with $K_u > K_{\text{loading}}$.

From the experimental data results

(6) $$\varepsilon_* = 0{,}2\%, \quad K = 105\,600 \ \text{daN cm}^{-2}, \quad \sigma_* = 600 \ \text{daN cm}^{-2}.$$

The principal components of the strain obtained from (1) are now

(7) $$\begin{aligned} \varepsilon_1 &= \frac{\sigma_1}{3G} + \lambda\frac{2}{3}\sigma_1 - \varepsilon_* \ln\left(1 - \frac{\sigma_1}{3\sigma_*}\right) \\ \varepsilon_2 &= -\frac{\sigma_1}{6G} - \lambda\frac{\sigma_1}{3} - \varepsilon_* \ln\left(1 - \frac{\sigma_1}{3\sigma_*}\right) \end{aligned}$$

and from the equivalent plastic strain $\bar{\varepsilon}^P$ and equivalent stress $\bar{\sigma}$ we have

(8) $$\bar{\varepsilon}'^P = |\varepsilon'^P_1|, \qquad \bar{\sigma} = |\sigma_1|.$$

Therefore for the scalar coefficient λ we have

(9) $$\lambda = \frac{3}{2}\frac{|\varepsilon'^P_1|}{|\sigma_1|} = \frac{3}{2}\frac{\bar{\varepsilon}'^P}{\bar{\sigma}}.$$

Now it is assumed that a strain hardening relation of the form

(10) $$|\varepsilon'^P_1| = f(|\sigma_1|) \ \text{or}\ \bar{\varepsilon}'^P = f(\bar{\sigma})$$

can be established from experiments. Here f is a strictly increasing and differentiable function. The relationship between

$$\varepsilon_1'^P = \varepsilon_1 + \varepsilon_* \ln\left(1 - \frac{\sigma_1}{3\sigma_*}\right) - \frac{\sigma_1}{3G} \tag{11}$$

and σ_1 obtained from experiments can be expressed by the empirical formula

$$\sigma_1 = a\left(\frac{\varepsilon_1'^P}{b}\right)^{m/n} \qquad \text{or} \qquad \bar{\sigma} = a\left(\frac{\bar{\varepsilon}'^P}{b}\right)^{m/n} \tag{12}$$

with $a = 1\,250$ da N cm^{-2}, $b = 0.0117$ and $m/n = 0.29$.

In conclusion, a Hencky type constitutive equation adopted for rock-like materials and valid for stress states close to the uniaxial stress-states can be written as

$$\begin{aligned}
&\dot{\varepsilon}_{ij}' = \frac{\dot{\sigma}_{ij}'}{2G} + \frac{3}{2}\frac{\dot{f}(\bar{\sigma})}{\bar{\sigma}}\sigma_{ij}' \quad \text{if } \bar{\varepsilon}'^P = f(\bar{\sigma}) \text{ and } \dot{\bar{\sigma}} > 0\\
&\dot{\varepsilon}_{ij}' = \frac{\dot{\sigma}_{ij}'}{2G} \text{ if } \begin{cases} \bar{\varepsilon}'^P = f(\bar{\sigma}) \text{ and } \dot{\bar{\sigma}} \leqslant 0\\ \text{or}\\ \bar{\varepsilon}'^P < f(\bar{\sigma})\end{cases}\\
&\varepsilon = -\varepsilon_* \ln\left(1 - \frac{\sigma}{\sigma_*}\right) \quad \text{if } \sigma = \sigma_M \text{ and } \dot{\sigma} > 0\\
&\varepsilon - \varepsilon_M = \frac{1}{3K_u}(\sigma - \sigma_M) \text{ if } \begin{cases} \sigma = \sigma_M \text{ and } \dot{\sigma} \leqslant 0\\ \text{or}\\ \sigma < \sigma_M\end{cases}
\end{aligned} \tag{13}$$

with $f(\bar{\sigma})$ obtained from (10) and G from (2).

If much higher mean stresses are involved, as in problems with rock mechanics, the strain hardening law (10) may depend on the mean stress as well and the volume compressibility law (3) may depend on the equivalent stress $\bar{\sigma}$ as well. One possible generalization of the constitutive equation is given in [5].

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COMPRESSIBILITY AND/OR DILATANCY OF ROCKS

BY

N. CRISTESCU

To the memory of Professor Mendel Haimovici

1. Introduction. What distinguishes the mechanical behaviour of rocks from those of most other materials is the behaviour of the volume. In the standard tests for rocks a cylindrical specimen is subjected to an axial stress σ_1 and to a lateral pressure $\sigma_2 = \sigma_3 = p$. Both axial strain ε_1 and diameter strain ε_2 are measured during experiment. In uniaxial testes $\sigma_2 = 0$ while in the so called triaxial tests generally $\sigma_2 = p = \text{const}$. In the present paper one will refer to these two kinds of tests. Thus in these tests the $\sigma_1 - \varepsilon_1$ and $\sigma_1 - \varepsilon_2$ relationships are obtained either for $\sigma_2 = p = \text{const.} \neq 0$ or for $\sigma_2 = 0$ respectively. The behaviour of the volume $\sigma_1 - \varepsilon_v$ then follows since $\varepsilon_v = \varepsilon_1 + 2\varepsilon_2$.

For most rocks the volume is decreasing when stress is increasing. First the law is nonlinear due to the closing of pores and of microcracks. This first part will be disregarded here. Further the $\sigma_1 - \varepsilon_v$ relation is linear and can be approximated by $\sigma_1 = 3K\varepsilon_v$ with K the elastic bulk modulus (fig. 1). Further there are two possibilities. If the volume decreases more than described by the linear relationship, we say that the rock is nonlinear *compressible*. If however after a period in which it had decreased the volume starts to increase we say that the volume is *dilatant*. Sometimes by dilatancy the final value at failure of the volume may be even higher than the one

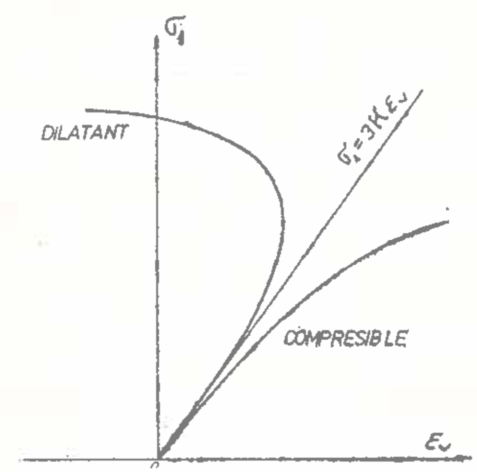

Fig. 1