can be established from experiments. Here f is a strictly increasing and differentiable function. The relationship between

$$\varepsilon_1'^P = \varepsilon_1 + \varepsilon_* \ln\left(1 - \frac{\sigma_1}{3\sigma_*}\right) - \frac{\sigma_1}{3G} \tag{11}$$

and σ_1 obtained from experiments can be expressed by the empirical formula

$$\sigma_1 = a\left(\frac{\varepsilon_1'^P}{b}\right)^{m/n} \quad \text{or} \quad \bar{\sigma} = a\left(\frac{\bar{\varepsilon}'^P}{b}\right)^{m/n} \tag{12}$$

with $a = 1\,250$ da N cm^{-2}, $b = 0.0117$ and $m/n = 0.29$.

In conclusion, a Hencky type constitutive equation adopted for rock-like materials and valid for stress states close to the uniaxial stress-states can be written as

$$\begin{aligned}
&\dot{\varepsilon}_{ij}' = \frac{\dot{\sigma}_{ij}'}{2G} + \frac{3}{2}\frac{f(\bar{\sigma})}{\bar{\sigma}}\sigma_{ij}' \ \text{ if } \bar{\varepsilon}'^P = f(\bar{\sigma}) \text{ and } \dot{\bar{\sigma}} > 0\\
&\dot{\varepsilon}_{ij}' = \frac{\dot{\sigma}_{ij}'}{2G} \text{ if } \begin{cases} \bar{\varepsilon}'^P = f(\bar{\sigma}) \text{ and } \dot{\bar{\sigma}} \leqslant 0\\ \text{or}\\ \bar{\varepsilon}'^P < f(\bar{\sigma}) \end{cases}\\
&\varepsilon = -\varepsilon_* \ln\left(1 - \frac{\sigma}{\sigma_*}\right) \ \text{ if } \sigma = \sigma_M \text{ and } \dot{\sigma} > 0\\
&\varepsilon - \varepsilon_M = \frac{1}{3K_u}(\sigma - \sigma_M) \text{ if } \begin{cases} \sigma = \sigma_M \text{ and } \dot{\sigma} \leqslant 0\\ \text{or}\\ \sigma < \sigma_M \end{cases}
\end{aligned} \tag{13}$$

with $f(\bar{\sigma})$ obtained from (10) and G from (2).

If much higher mean stresses are involved, as in problems with rock mechanics, the strain hardening law (10) may depend on the mean stress as well and the volume compressibility law (3) may depend on the equivalent stress $\bar{\sigma}$ as well. One possible generalization of the constitutive equation is given in [5].

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COMPRESSIBILITY AND/OR DILATANCY OF ROCKS

BY

N. CRISTESCU

To the memory of Professor Mendel Haimovici

1. Introduction. What distinguishes the mechanical behaviour of rocks from those of most other materials is the behaviour of the volume. In the standard tests for rocks a cylindrical specimen is subjected to an axial stress σ_1 and to a lateral pressure $\sigma_2 = \sigma_3 = p$. Both axial strain ε_1 and diameter strain ε_2 are measured during experiment. In uniaxial testes $\sigma_2 = 0$ while in the so called triaxial tests generally $\sigma_2 = p = \text{const}$. In the present paper one will refer to these two kinds of tests. Thus in these tests the $\sigma_1 - \varepsilon_1$ and $\sigma_1 - \varepsilon_2$ relationships are obtained either for $\sigma_2 = p = \text{const.} \neq 0$ or for $\sigma_2 = 0$ respectively. The behaviour of the volume $\sigma_1 - \varepsilon_v$ then follows since $\varepsilon_v = \varepsilon_1 + 2\varepsilon_2$.

For most rocks the volume is decreasing when stress is increasing. First the law is nonlinear due to the closing of pores and of microcracks. This first part will be disregarded here. Further the $\sigma_1 - \varepsilon_v$ relation is linear and can be approximated by $\sigma_1 = 3K\varepsilon_v$ with K the elastic bulk modulus (fig. 1). Further there are two possibilities. If the volume decreases more than described by the linear relationship, we say that the rock is nonlinear *compressible*. If however after a period in which it had decreased the volume starts to increase we say that the volume is *dilatant*. Sometimes by dilatancy the final value at failure of the volume may be even higher than the one

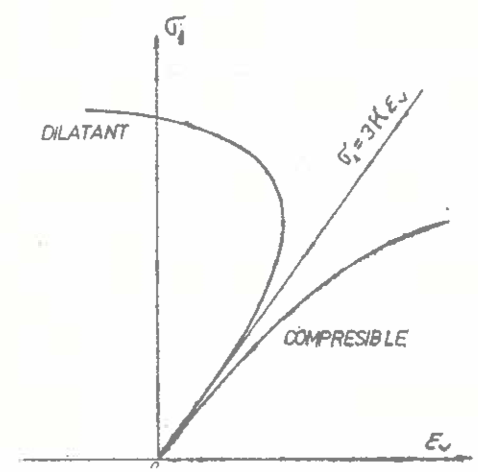

Fig. 1

at the beginning of the experiment. Both phenomena have been studied experimentally by several authors. It is B r a c e 's opinion [1] that both phenomena are simultaneously present in every rock but that only one at a time is dominant; generally rocks of significant porosity (over 0.05) are mainly compressible, while rocks of smaller porosity are dilatant.

2. Time independent constitutive equation. A mathematical model to describe these two phenomena can be formulated starting from the results obtained in the so called triaxial tests. In these tests, at various constant levels of the lateral pressure $\sigma_2 = \sigma_3 = p$ at which a cylindrical rock specimen is subjected, the axial stress σ_1 is increased starting from the value $\sigma_1 = p$. Generally in the neighbourhood of the hydrostatic stress states $\sigma_1 = \sigma_2 = \sigma_3$ the behaviour of the rock can be approximated by a linear constitutive equation. Thus the experimental data can be approximated by constitutive equations of the form

$$
\begin{aligned}
\varepsilon_1 &= \left(\frac{1}{3G} + \frac{1}{9K}\right)\sigma_1 + \left(-\frac{1}{3G} + \frac{2}{9K}\right)\sigma_2 + f_1(\sigma_1, \sigma_2),\\
\varepsilon_2 &= \left(-\frac{1}{6G} + \frac{1}{9K}\right)\sigma_1 + \left(\frac{1}{6G} + \frac{2}{9K}\right)\sigma_2 + f_2(\sigma_1, \sigma_2),
\end{aligned} \tag{1}
$$

where the elastic constants G and K are determined in unloading tests and functions f_1 and f_2 during loading. If now we postulate for loading a general constitutive equation of the form

$$
\begin{aligned}
\varepsilon &= \frac{\sigma}{3K} + g(\sigma, \bar{\sigma})\\
\varepsilon' &= \frac{\sigma'}{2G} + f(\sigma, \bar{\sigma})\sigma',
\end{aligned} \tag{2}
$$

then it is easy to show that

$$
\begin{aligned}
f(\sigma, \bar{\sigma}) &= \frac{f_1\left(\bar{\sigma}, \sigma - \frac{\bar{\sigma}}{3}\right) - f_2\left(\bar{\sigma}, \sigma - \frac{\bar{\sigma}}{3}\right)}{\bar{\sigma}}\\
g(\sigma, \bar{\sigma}) &= \frac{1}{3}\left[f_1\left(\bar{\sigma}, \sigma - \frac{\bar{\sigma}}{3}\right) + 2f_2\left(\bar{\sigma}, \sigma - \frac{\bar{\sigma}}{3}\right)\right].
\end{aligned} \tag{3}
$$

Here „prime" stands for deviator, σ is the mean stress, ε the mean strain $\bar{\sigma} = \sqrt{3}\sqrt{II_{T'}}$, is the equivalent stress with $II_{T'}$, the second invariant of the stress deviator.

Depending of the two functions f_1 and f_2, if during triaxial tests there are stress states satisfying one of the conditions

$$
\begin{aligned}
&g(\sigma, \bar{\sigma}) < 0\text{-dilatancy}\\
&g(\sigma, \bar{\sigma}) = 0\text{-elastic}\\
&g(\sigma, \bar{\sigma}) > 0\text{-compressible}
\end{aligned} \tag{4}
$$

then the volume of the corresponding rock is either dilatant, or elastic or finally nonlinear compressible. Model (2) can be adapted to all three cases provided that no time effects are of significance for the particular rock considered. For instance for granite type rock which is dilatant, functions f_1 and f_2 are of the form (Cristescu [2])

$$
f\left(\bar{\sigma}, \sigma - \frac{\bar{\sigma}}{3}\right) = -\frac{1}{b_1\left(\sigma - \frac{\bar{\sigma}}{3}\right)} \ln\left[1 - \frac{\bar{\sigma}}{a_1\left(\sigma - \frac{\bar{\sigma}}{3}\right)}\right] \tag{5}
$$

where a_1 and b_1 are power functions of the arguments shown.

3. Time dependent constitutive equations. For most rocks, mainly for the soft ones, the time effects cannot be disregarded. In this case the experiments needed for the formulation of the mathematical model are the creep experiments. In such experiments a cylindrical rock specimen is loaded axially with a stress which is held constant for quite a long time interval. During this time interval the variation in time of both diameter strain and axial strain are recorded. If the stress is not surpassing a certain limit (Cristescu [3], Cristescu and Suliciu [4] Ch. 9) then after a certain time interval the deformation by creep stops at a certain limit strain which depends on the applied stress. All these limit stress-strain states form the so called stabilization boundary for creep. For instance on fig. 2 are given for cement concrete (Constantinescu [5]) the stair-case looking $\sigma_1 - \varepsilon_1$, $\sigma_1 - \varepsilon_2$ and $\sigma_1 - \varepsilon_v$ curves obtained for three instantaneous loadings followed by three periods of creep obtained for three levels of the applied stress. Thus it is obvious that a volume creep is also present and a stabilization boundary for the volume creep exists. If the stabilization boundary for the volume is at the right side of the elastic instantaneous response of the volume $\sigma_1 = 3K\varepsilon_v$ then we say that the rock is *compressible*. If this stabilization boundary is at the left side of this line then we say that the rock is *dilatant*.

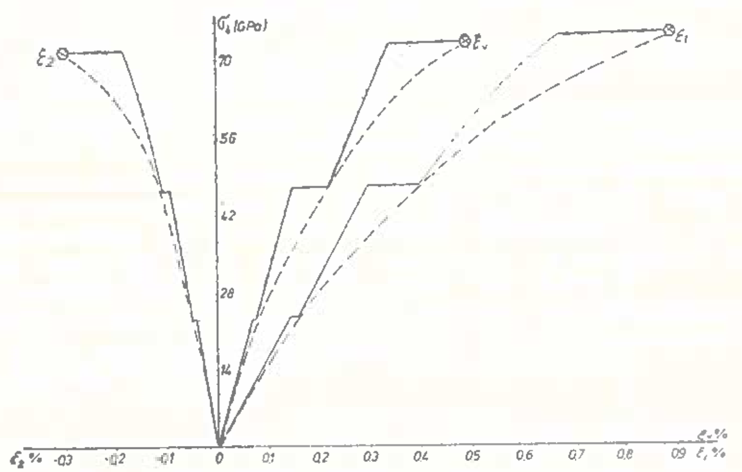

Fig. 2

It can be shown that time effects on dilatancy or compressibility can be described for instance by a constitutive equation of the form

$$
(6)\qquad
\begin{aligned}
\dot{\varepsilon}' &= \frac{\dot{\sigma}'}{2G} + \frac{3k}{2E}\left\langle 1 - \frac{P(\bar{\varepsilon}, \varepsilon)}{\bar{\sigma}} \right\rangle \sigma' \\
\dot{\varepsilon} &= \frac{\dot{\sigma}}{3K} + \frac{\eta}{3K} \langle \sigma - Q(\bar{\varepsilon}, \varepsilon) \rangle
\end{aligned}
$$

where elastic constants are determined from the slope of instantaneous response curves, $\bar{\sigma} = P(\bar{\varepsilon}, \varepsilon)$ is the stabilization boundary for the deviatoric strains, $\sigma = Q(\bar{\varepsilon}, \varepsilon)$ is the stabilization boundary for the volume strain,, E/k is a viscosity coefficient for the deviatoric strain (measured in Poise), K/η is a viscosity coefficient for the volume (given also in Poise), and $\langle\ \rangle$ stands for the Heaviside function

$$
\langle Z \rangle = \begin{cases} Z & \text{if } \; Z > 0 \\ 0 & \text{if } \; Z \leqslant 0. \end{cases}
$$

From creep tests for ε_1 and ε_2 obtained for various levels of σ_1 and $\sigma_3 = 0$ or $\sigma_2 = p$ the one-dimensional viscosity coefficients E/k_1 and E/k_2 can be obtained. Then yields

$$
(7)\qquad k = 2\,\frac{k_1 - k_2}{3} \quad \text{and} \quad \frac{\eta}{3K} = \frac{k_1 + 2k_2}{E}.
$$

From the same experiments if $\sigma_1 = p(\varepsilon_1)$ and $\sigma_1 = q(\varepsilon_2)$ are the stabilization boundaries for ε_1 and ε_2 respectively obtained in uniaxial tests then

$$
(8)\qquad
\begin{aligned}
P(\bar{\varepsilon}, \varepsilon) &= \frac{k_1 p(\bar{\varepsilon} + \varepsilon) - k_2 q\left(\varepsilon - \frac{\bar{\varepsilon}}{3}\right)}{k_1 - k_2} \\
Q(\bar{\varepsilon}, \varepsilon) &= \frac{k_1 p(\bar{\varepsilon} + \varepsilon) + 2k_2 q\left(\varepsilon - \frac{\bar{\varepsilon}}{2}\right)}{3(k_1 + 2k_2)}.
\end{aligned}
$$

Thus the model is obtained for the uniaxial tests, but a generalization for triaxial stress states can be obtained easily.

For instance for a typical hard rock we have

$$
\begin{gathered}
G = 27.3\,\text{GPa}, \quad K = 48.3\ \text{GPa} \\
\frac{E}{k_1} = 4.3 \times 10^{11}\ \text{Poise}, \qquad \frac{E}{k_2} = -1.7 \times 10^{11}\ \text{Poise} \\
\frac{E}{k} = 1.9 \times 10^{11}\ \text{Poise}, \qquad \frac{3K}{\eta} = -1.05 \times 10^{11}\ \text{Poise},
\end{gathered}
$$

and the stabilization boundaries $\sigma_1 = p(\varepsilon_1)$ and $\sigma_1 = q(\varepsilon_2)$ can be approximated by

$$
\sigma_1 = a_i[1 - \exp(-b_i \varepsilon)] \quad (i = 1,2)
$$

with $a_1 = 0.55$ GPa, $b_1 = 120$, $a_2 = 0.25$ GPa, $b_2 = -900$. These stabilization boundaries are shown on fig. 3 by full lines. The stabilization boundary for the volume is the dotted line labelled $\dot{\sigma}_{1a}$. Thus this is a dilatant rock.

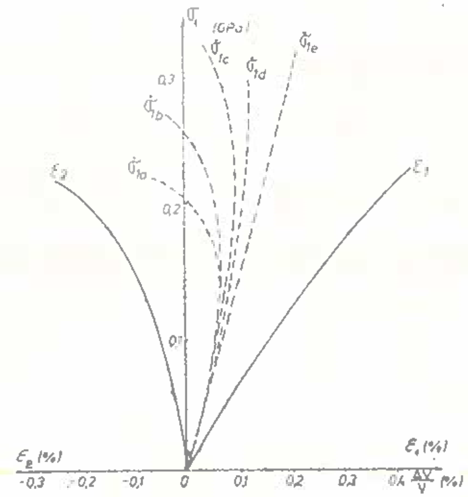

Fig. 3

The figure 3 is showing the influence of the rate of loading on the rock dilatancy as described by model (6) with the assumption that the instantaneous response is elastic and that the viscosity coefficients are constant (mean value determined on the whole period of creep). Figure 3 was plotted for the following rates of loading : $\dot{\sigma}_{1a} \to 0$, $\dot{\sigma}_{1b} = 1.39 \times 10^{-7}$ GPa sec^{-1}, $\dot{\sigma}_{1c} = 3.33 \times 10^{-7}$ GPa sec^{-1}, $\dot{\sigma}_{1d} = 5.56 \times 10^{-7}$ GPa sec^{-1} and $\dot{\sigma}_{1e} \to \infty$. Thus the first curve is obtained in creep tests and the last one is the instantaneous response.

It must be observed that for the rock given as an example here the viscosity coefficient of the volume is negative.

The experiments are however showing a very interesting phenomenon what concerns the creep of the volume (Cristescu [6]). During creep the volume strain oscillates around its final stable value before reaching it. This very interesting phenomenon which shows that during creep dilatancy and compressibility are successive (in time) phenomena, can also be described by model (6) if the viscosity coefficients are determined more carefully from the $\varepsilon_1 - t$ and $\varepsilon_2 - t$ data.

Model (6) which is able to describe dilatancy and/or nonlinear compressibility of the volume can be generalized for triaxial stress states if experimental data are available.

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TORSION OF ORTHOTROPIC BARS IN THE PLASTIC DOMAIN

BY

VASILE DIACONIŢA

To the memory of Professor Mendel Haimovici

1. Introduction. The problem of the torsion of isotropic bar composed of materials with different yield limits has been studied in [1],[2], [3].

In this paper we consider the torsion of an orthotropic bar composed of two different materials. The results are applied to the bar with rectangular cross-section.

2. Basic equations. We consider a bar composed of two different orthotropic materials bounded by planes perpendicular to the generators. We suppose that the body forces are absent and the lateral surface is free of applied forces. Throughout this paper the Oz axis of our coordinate system will be directed parallel to the generators of the bar. The bar is assumed to be of length l; one of the bases is taken to lie in the $x0y$ — plane, while the other is in the plane $z=l$. We suppose the bar to be kept in the plastic domain when the end $z=l$ is twisted by a moment M. The anisotropic axis coincide with coordinate axis.

The cross-section $\Omega=\Omega_0\cup\Omega_1$ is bounded by the curve Γ with continuous tangent. The contact line between Ω_0 and Ω_1 will be denoted by L. The materials which occupe the domains Ω_0 and Ω_1 are characterized by yield limits $T_0, S_0 (T_0>S_0)$ and $T_1, S_1 (T_1>S_1)$, respectively. We assume that $S_1>T_0$ and then the material who occupies the domain Ω_0 passes first in the plastic domain. Denoting with T_{01}, S_{01} the yield limits on L, we assume $T_{01}, \ S_{01}>T_0$.

The stress components must satisfy the equilibrium equation

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}=0, \tag{2.1}$$

the yield condition

$$\frac{1}{T_i^2}\tau_{xz}^2+\frac{1}{S_i^2}\tau_{yz}^2=1,\ i=0,1, \tag{2.2}$$

the boundary condition

$$\tau_{xz}\cos(n,x)+\tau_{yz}\cos(n,y)=0 \text{ on } \Gamma, \tag{2.3}$$

the continuity condition of the normal component on L