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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Supliment la tomul XXVII, s. I a, 1981

TORSION OF ORTHOTROPIC BARS IN THE PLASTIC DOMAIN

BY

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To the memory of Professor Mendel Haimovici

1. Introduction. The problem of the torsion of isotropic bar composed of materials with different yield limits has been studied in [1],[2], [3].

In this paper we consider the torsion of an orthotropic bar composed of two different materials. The results are applied to the bar with rectangular cross-section.

2. Basic equations. We consider a bar composed of two different orthotropic materials bounded by planes perpendicular to the generators. We suppose that the body forces are absent and the lateral surface is free of applied forces. Throughout this paper the Oz axis of our coordinate system will be directed parallel to the generators of the bar. The bar is assumed to be of length l; one of the bases is taken to lie in the $x0y$ — plane, while the other is in the plane $z=l$. We suppose the bar to be kept in the plastic domain when the end $z=l$ is twisted by a moment M. The anisotropic axis coincide with coordinate axis.

The cross-section $\Omega=\Omega_0\cup\Omega_1$ is bounded by the curve Γ with continuous tangent. The contact line between Ω_0 and Ω_1 will be denoted by L. The materials which occupe the domains Ω_0 and Ω_1 are characterized by yield limits $T_0, S_0(T_0>S_0)$ and $T_1, S_1(T_1>S_1)$, respectively. We assume that $S_1>T_0$ and then the material who occupies the domain Ω_0 passes first in the plastic domain. Denoting with T_{01}, S_{01} the yield limits on L, we assume $T_{01},\ S_{01}>T_0$.

The stress components must satisfy the equilibrium equation

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}=0, \tag{2.1}$$

the yield condition

$$\frac{1}{T_i^2}\tau_{xz}^2+\frac{1}{S_i^2}\tau_{yz}^2=1,\ i=0,1, \tag{2.2}$$

the boundary condition

$$\tau_{xz}\cos(n,x)+\tau_{yz}\cos(n,y)=0 \text{ on } \Gamma, \tag{2.3}$$

the continuity condition of the normal component on L

$$(2.4)\qquad \begin{aligned}&[\tau_{xz}\cos(n,x)+\tau_{yz}\cos(n,y)]_1=\\&=[\tau_{xz}\cos(n,x)+\tau_{yz}\cos(n,y)]_0,\end{aligned}$$

where n is the outward normal to Γ and

$$(2.5)\qquad \int_\Omega (x\tau_{yz}-y\tau_{xz})d\Omega = M$$

3. Local solutions. Let us consider the case where the regular line L separating the materials intersecting a regular arc of the Γ at A, lies entirely on the right-hand side of the straight line passing through A (the side Ω_0) Fig. 1 and intercepting the Ox axis at angle $\bar{\varphi}_1 = -\dfrac{S_0}{T_0}\operatorname{ctg}\varphi_1$.

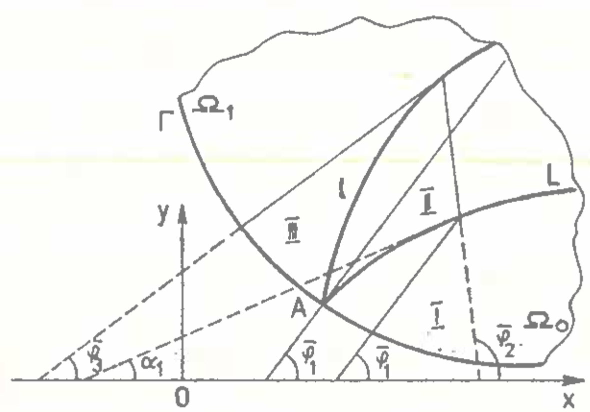

Fig. 1

In the domain I the yield condition will be satisfied if we assume

$$(3.1)\qquad \tau_{xz}=T_0\cos\varphi_1,\quad \tau_{yz}=S_0\sin\varphi_1.$$

From the equilibrium equation (2.1) we get

$$(3.2)\qquad -T_0\sin\varphi_1\frac{\partial\varphi_1}{\partial x}+S_0\cos\varphi_1\frac{\partial\varphi_1}{\partial y}=0.$$

The differential equation of the slip line will be

$$(3.3)\qquad \frac{dx}{-T_0\sin\varphi_1}=\frac{dy}{S_0\cos\varphi_1},\quad \varphi_1=\text{const}.$$

From (3.3) we get

$$(3.4)\qquad y+x\frac{S_0}{T_0}\operatorname{ctg}\varphi_1=\Phi(\varphi_1).$$

From (2.3) and (3.1) we have

$$(3.5)\qquad \operatorname{ctg}\varphi_1=\frac{S_0}{T_0}\operatorname{ctg}\alpha,$$

where α is the angle between the positive direction of Ox and the tangent at Γ curve. Assuming that

$$(3.6)\qquad x=x(\alpha),\quad y=y(\alpha),$$

from (3.4) and (3.5) we have for the function φ_1 the equation

$$(3.7)\qquad y+x\frac{S_0}{T_0}\operatorname{ctg}\varphi_1=y(\alpha)+x(\alpha)\frac{S_0^2}{T_0^2}\operatorname{ctg}\alpha.$$

This solution holis until the discontinuity lines. The slip lines, are not normal at Γ. From the regularity of the lines Γ and L, it follows that in a neighbourhood of A, of which the size depends on the forms of Γ and L, the slip lines intersect the contact line at sharp angle. This determines in a unique manner the function φ_1 at every point of L.

In the region II we get

$$(3.8)\qquad \tau_{xz}=T_1\cos\varphi_2,\quad \tau_{yz}=S_1\sin\varphi_2,$$

and

$$(3.9)\qquad y+x\frac{S_1}{T_1}\operatorname{ctg}\varphi_2=\Phi(\varphi_2).$$

From (3.1), (3.8) and (2.4), we obtine

$$(3.10)\qquad \operatorname{tg}\alpha_1=\frac{S_0\sin\varphi_1-S_1\sin\varphi_2}{T_0\cos\varphi_1-T_1\cos\varphi_2},$$

where α_1 is the angle between the positive direction of Ox and the tangent of L curve.

From this condition, we must determine on L the function $\varphi_2=\varphi_2(\varphi_1,\alpha_1)$. If now we suppose that on L

$$(3.11)\qquad x=x(\alpha_1),\quad y=y(\alpha_1),$$

from (3.9) we can determine the function $\Phi(\varphi_2)$ and we get for the function $\varphi_2(x,y)$ the equation

$$(3.12)\qquad y+x\frac{S_1}{T_1}\operatorname{ctg}\varphi_2=y(\alpha_1)+x(\alpha_1)\frac{S_1^2}{T_1^2}\operatorname{ctg}\varphi_2(\varphi_1,\alpha_1).$$

The slip lines in the region II are straight lines which form with the Ox axis an angle $\bar{\varphi}_2$, where

$$(3.13)\qquad \operatorname{tg}\bar{\varphi}_2=-\frac{S_1}{T_1}\operatorname{ctg}\varphi_2.$$

In the region III we get

$$\tau_{xz}=T_1\cos\varphi_3,\quad \tau_{yz}=S_1\sin\varphi_3, \tag{3.14}$$

and

$$y+x\frac{S_1}{T_1}\operatorname{ctg}\varphi_3=\Phi(\varphi_3). \tag{3.15}$$

From (2.3) and (3.14) we have

$$\operatorname{ctg}\varphi_3=\frac{S_1}{T_1}\operatorname{ctg}\alpha. \tag{3.16}$$

For the function φ_3 we have

$$y+x\frac{S_1}{T_1}\operatorname{ctg}\varphi_3=y(\alpha)+x(\alpha)\frac{S_1^2}{T_1^2}\operatorname{ctg}\alpha. \tag{3.17}$$

The slip lines in the region III forme with the Ox axis an angle $\bar{\varphi}_3$ where

$$\operatorname{tg}\bar{\varphi}_3=-\frac{S_1}{T_1}\operatorname{ctg}\varphi_3. \tag{3.18}$$

The continuity of the normal component of the vector of shear stress on the discontinuity line l gives:

$$\operatorname{tg}\alpha_l=\frac{S_1}{T_1}\operatorname{tg}\frac{1}{2}(\varphi_2+\varphi_3-\pi), \tag{3.19}$$

where α_e is the angle between the positive direction of Ox and the tangent at the line l of discontinuity.

The differential equation of the discontinuity line is

$$\frac{dy}{dx}=-\frac{S_1}{T_1}\operatorname{ctg}\frac{1}{2}(\varphi_2+\varphi_3). \tag{3.20}$$

If the contour Γ and the separation line L are straight lines, the quantities φ_2, φ_3, S_1/T_1 are constants and the discontinuity line is also a straight line.

For $S_0=T_0$, $S_1=T_1$, we obtain the results from [1], [3]. On the discontinuity line l the jump of the tangential component is

$$\begin{aligned}\tau_{lz}^{II}-\tau_{lz}^{III}=2\sin\frac{1}{2}(\varphi_3-\varphi_2)\Big[S_1\cos^2\frac{1}{2}(\varphi_2+\varphi_3)+\\ +T_1\sin^2\frac{1}{2}(\varphi_2+\varphi_3)\Big]^{1/2}.\end{aligned} \tag{3.21}$$

On the line L the jump of the tangential component is

$$\tau_{lz}^{II}-\tau_{lz}^{I}=[(S_0\sin\varphi_1-S_1\sin\varphi_2)^2+(T_0\cos\varphi_1-T_1\cos\varphi_2)^2]^{1/2}$$

4. Application. Let us consider here the bar with cross-section as shown in Fig. 2. The rectangle $2a\times 2b$ is divided by the contact line L, normal to the edge $2b$ into two regions Ω_0, Ω_1 having constant but different yield limits T_0, S_0, T_1, $S_1(T_1>S_1>T_0>S_0)$.

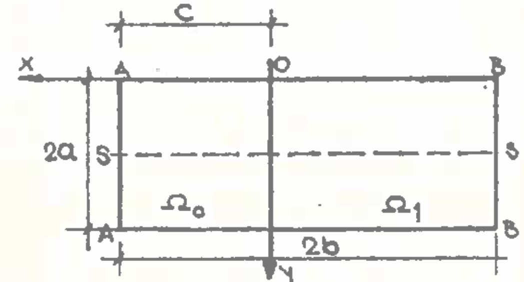

FIG. 2

In what follows we assume that the adhesion is perfect and that $T_{01}>S_{01}>T_0$.

The problem is characterized by five independent parameters

$$\begin{gathered}0<\lambda_0=\frac{S_0}{T_0}<1,\quad 0<\lambda_1=\frac{S_1}{T_1}<1,\quad 0<\lambda_{01}=\frac{T_0}{T_1}<1,\\ 0<\frac{a}{b}<\infty,\quad 0<\frac{c}{b}<2.\end{gathered} \tag{4.1}$$

The properties of the parameters λ_0, λ_1, λ_{01}, a/b, c/b, lead to nine different solutions of the problem of limit load of the bar.

Solution 1, Figure 3.* The solution of the equations (2.1), (2.2) with the conditions (2.3) in region I, has the form

$$\tau_{xz}=T_0,\quad \tau_{yz}=0. \tag{4.2}$$

The continuity condition (2.4) must be satisfied at the contact line L, therefore the solution in the region III has the form

$$\begin{gathered}\tau_{xz}=T_1\cos\varphi,\quad \tau_{yz}=S_1\sin\varphi,\\ \varphi=\arc\cos\frac{T_0}{T_1},\quad \varphi\in\left(0,\ \frac{\pi}{2}\right).\end{gathered} \tag{4.3}$$

In the region II the solution has the form

$$\tau_{xz}=0,\quad \tau_{yz}=S_0. \tag{4.4}$$

In the region IV

$$\tau_{xz}=0,\quad \tau_{yz}=S_1. \tag{4.5}$$

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* All the figures are drawn for $\lambda_0=\frac{\sqrt{3}}{3}$, $\lambda_1=0{,}84$, $\lambda_{01}=\frac{1}{2}$

In the region V

(4.6) $$\tau_{xz}=T_1,\quad \tau_{yz}=0.$$

In the region VI

(4.7) $$\tau_{xz}=0,\quad \tau_{yz}=-S_1.$$

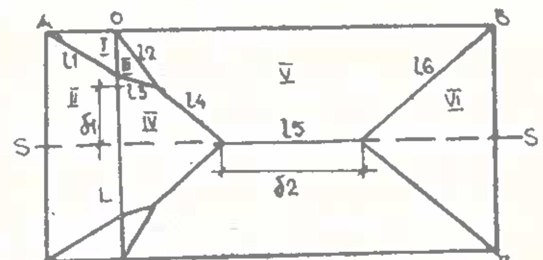

FIG.3

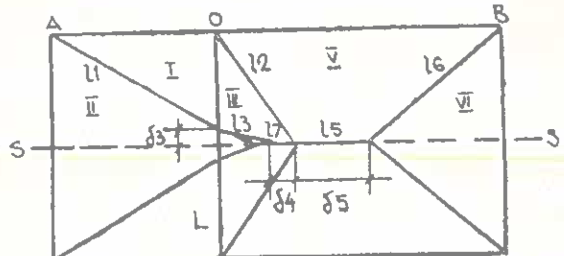

FIG.4

From the corner A the discontinuity line l_1 divides the weaker zone into the regions I, II in which slip lines are normal to AO and AS, respectively. In view of the maximum condition of the limit moment it is assumed that there are no other discontinuity lines in the weaker region and that l_1 intersects the contact line L. The discontinuity line l_1 is inclined to Ox at an angle $\beta_1(\mathrm{tg}\,\beta_1=S_0/T_0)$. The discontinuity line l_2 is inclined to Ox at an angle $\beta_2\left(\mathrm{tg}\,\beta_2=\frac{S_1}{T_1}\,\frac{\sin\varphi}{\cos\varphi-1}\right)$. The discontinuity line l_3 is inclined to Ox at an angle $\beta_3\left(\mathrm{tg}\,\beta_3=\frac{S_1}{T_1}\,\frac{\sin\varphi-1}{\cos\varphi}\right)$. The discontinuity lines l_2, l_3 have an intersection point. In the region V the slip lines are normal to the edge OB and in the region IV they are normal to L and the discontinuity line line l_4 is inclined to Ox an angle β_4 ($\mathrm{tg}\,\beta_4=-S_1/T_1$). The discontinuity line l_6 is inclined to Ox an angle $\beta_6(\mathrm{tg}\,\beta_6=S_1/T_1)$. The discontinuity line l_5 requires no comment. The solution obtained is valid if $\delta_1\geq 0$, $\delta_2\geq 0$. From these conditions we obtain the inequalities

$$\delta_1\geq 0,\quad \frac{a}{b}-\frac{1}{2}\lambda_0(1+\sin\varphi+\cos\varphi)\frac{c}{b}\geq 0,\quad (d_1),$$

(4.8)

$$\delta_2\geq 0,\quad -\frac{a}{b}-\frac{1}{2}(\lambda_1-\lambda_0\cos\varphi)\frac{c}{b}+\lambda_1\geq 0,\quad (d_2).$$

For $\lambda_0, \lambda_1, \lambda_{01}=$ const., in the plane $(a/b, c/b)$ this corresponds to the region 1 between the axis a/b and the straight lines d_1, d_2 (Fig. 12) intersecting at the point P_1, of which the coordonater are

(4.9) $$P_1:\frac{a}{b}=\frac{\lambda_0\lambda_1(1+\sin\varphi+\cos\varphi)}{\lambda_1+\lambda_0(1+\sin\varphi)}\ :\ \frac{c}{b}=\frac{2\lambda_1}{\lambda_1+\lambda_0(1+\sin\varphi)}$$

Solution 2. Figure 4. With the increasing parameter c/b, for fixed a/b, λ_0, λ_1 and λ_{01}, the conditions (4.8) may no longer be satisfied. Let us consider first case where the condition $\delta_1\geq 0$ is not satisfied. We obtain a modification of the previous solution. Instead of the discontinuity line l_4 we have a new discontinuity line l_7. By calculating on the basis of Fig. 4 the quantities δ_3, δ_4, δ_5 we obtain, for the validity of solution, the following inequalities

$$\delta_3\geq 0,\ \frac{a}{b}-\lambda_0\frac{c}{b}\geq 0,\quad (d_3),$$

(4.10) $$\delta_4\geq 0,\ \frac{a}{b}-\frac{1}{2}\lambda_0(1+\sin\varphi+\cos\varphi)\frac{c}{b}\leq 0\quad (d_1),$$

$$\delta_5\geq 0,\ 2\lambda_1-\frac{1+\sin\varphi+\cos\varphi}{1+\cos\varphi}\,\frac{a}{b}-\lambda_1\frac{c}{b}\geq 0,\quad (d_5).$$

In the plane $(a/b, c/b)$ this corresponds to the region 2 between the straight lines d_3, d_1, d_5 (Fig. 12). The straight lines d_1, d_5 intersect, as is easily seen, at points P_1.

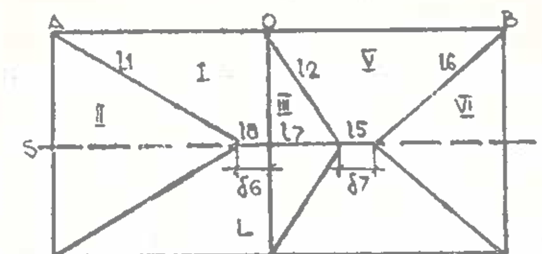

FIG.5

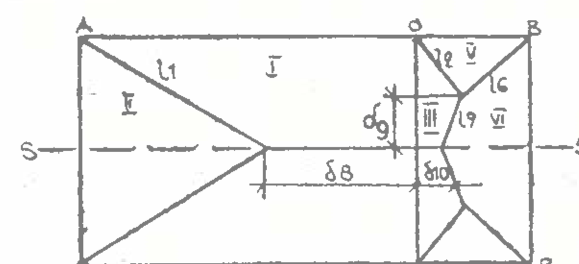

FIG 6

Solution 3. Figure 5. If, in the preceeding solution $c=a/\lambda_0$, the zone IV vanishes. From Fig. 5 it follows that for sufficiently small values of the parameter a/b this solution is valid for any λ_0, λ_1, λ_{01} and a broad range of values of c/b.
The validity region of solution is expressed by the inequalities

$$\delta_6\geq 0,\ \frac{a}{b}-\lambda_0\frac{c}{b}\leq 0,\ (d_3),$$

(4.11)

$$\delta_7\geq 0,\ 2\lambda_1-\frac{1+\sin\varphi+\cos\varphi}{1+\cos\varphi}\frac{a}{b}-\lambda_1\frac{c}{b}\geq 0,\ (d_5).$$

The corresponding straight lines d_3, d_5 in the plane $(a/b, c/b)$ have an intersection point P_2 whose coordinates are

(4.12)

$$\frac{a}{b}=\frac{2\lambda_0\lambda_1(1+\cos\varphi)}{\lambda_0+\lambda_1+\lambda_0\ \mathrm{son}\ \varphi+(\lambda_0+\lambda_1)\cos\varphi},$$

$$\frac{c}{b}=\frac{2\lambda_1(1+\cos\varphi)}{\lambda_0+\lambda_1+\lambda_0\sin\varphi+(\lambda_0+\lambda_1)\cos\varphi}.$$

Solution 4, Figure 6. With increasing parameter c/b the discontinuity lines l_2, l_6 may intersect at the point above the symmetry axis. This point is the origin of a new discontinuity line l_9 which is inclined to Ox at an angle $\beta_9\left(\operatorname{tg}\beta_9=\frac{S_1}{T_1}\frac{\sin\varphi+1}{\cos\varphi}\right)$. The validity range of the solution is expressed by the inequalities

$$\delta_8\geq 0,\ \frac{a}{b}-\lambda_0\frac{c}{b}\leq 0,\quad (d_3),$$

(4.13)

$$\delta_9\geq 0,\ 2\lambda_1-\frac{1+\sin\varphi+\cos\varphi}{1+\cos\varphi}\frac{a}{b}-\lambda_1\frac{c}{b}\leq 0\quad (d_5)$$

$$\delta_{10}\geq 0,\ \left(2-\frac{c}{b}\right)\lambda_1-\frac{a}{b}\cos\varphi\geq 0,\quad (d_{10}),$$

to which correspond the straight lines d_3, d_5, d_{10}.

Solution 5. Figure 7. Further increase of the width of the weaker zone leads to the case where the discontinuity line l_9 intersects the contact line L above the symmetry axis. From this point originates a new discontinuity line l_{10} in the weaker zone bounding a new region in which the slip lines are normal to the contact line L. The validity range of solution 5 is determined by the inequalities

$$\delta_{11}\geq 0,\ \left(2-\frac{c}{b}\right)\lambda_1-\left(2\frac{a}{b}-\lambda_0\frac{c}{b}\right)\cos\varphi\geq 0,\quad (d_{11}),$$

(4.14)

$$\delta_{12}\geq 0,\ \left(2-\frac{c}{b}\right)\lambda_1-\frac{a}{b}\cos\varphi\leq 0,\quad (d_{10}).$$

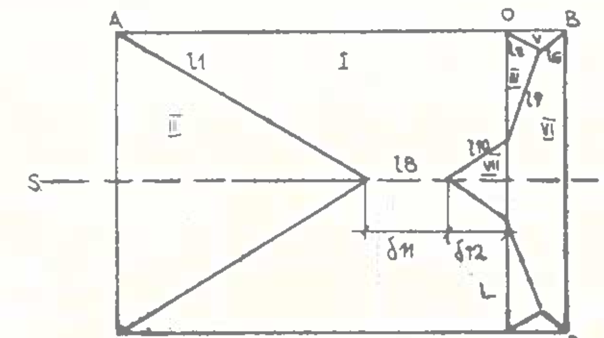

FIG. 7

The corresponding lines d_{11}, d_{10} in Fig. 12 intersect at the point P_3 whose coordinates are

(4.15)

$$P_3:\frac{a}{b}=\frac{2\lambda_1\lambda_0}{\lambda_1+\lambda_0\cos\varphi},\ \frac{c}{b}=\frac{2\lambda_1}{\lambda_1+\lambda_0\cos\varphi}.$$

Solution 6, 7, 8, 9, Figure 8, 9, 10, 11.
The validity ranges are by the following inequalities

$$\delta_{13}\geq 0,\ \frac{c}{b}\leq\frac{2\lambda_1}{\lambda_1+\lambda_0(1+\sin\varphi)},\quad (d_{13}),$$

(4.16)

$$\delta_{14}\geq 0,\ -\frac{a}{b}-\frac{1}{2}(\lambda_1-\lambda_0\cos\varphi)\frac{c}{b}+\lambda_1\leq 0,\ (d_2),$$

for solution 6,

$$\delta_{15}\geq 0,\ \frac{c}{b}\leq\frac{2\lambda_1}{\lambda_1+\lambda_0\cos\varphi},\quad (d_{15}),$$

(4.17)

$$\delta_{16}\geq 0,\left(2-\frac{c}{b}\right)\lambda_1\frac{\cos\varphi}{1+\sin\varphi}-2\frac{a}{b}+\lambda_0\frac{c}{b}(1+\sin\varphi)\leq 0,\ (d_{16}),$$

$$\delta_{17}\geq 0,\ \frac{c}{b}\geq\frac{2\lambda_1}{\lambda_1+\lambda_0(1+\sin\varphi)},\quad (d_{13}),$$

for solution 7,

$$\delta_{18}\geq 0,\ \frac{a}{b}-\lambda_0\frac{c}{b}\geq 0,\ (d_3),$$

(4.18)

$$\delta_{19}\geq 0,\ \left(2-\frac{c}{b}\right)\lambda_1\frac{\cos\varphi}{1+\sin\varphi}-2\frac{a}{b}+\lambda_0\frac{c}{b}(1+\sin\varphi)\geq 0\ (d_{16}),$$

$$\delta_{20}\geq 0\ 2\lambda_1-\frac{1+\sin\varphi+\cos\varphi}{1+\cos\varphi}\frac{a}{b}-\lambda_1\frac{c}{b}\leq 0,\quad (d_5).$$

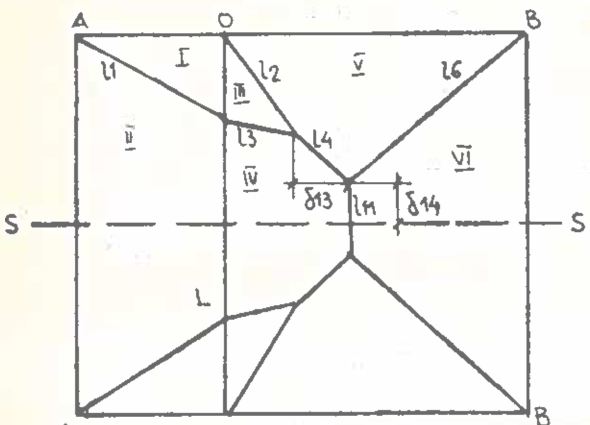

FIG. 8

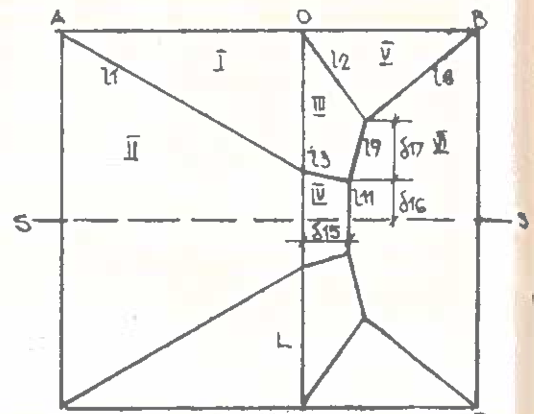

FIG. 9

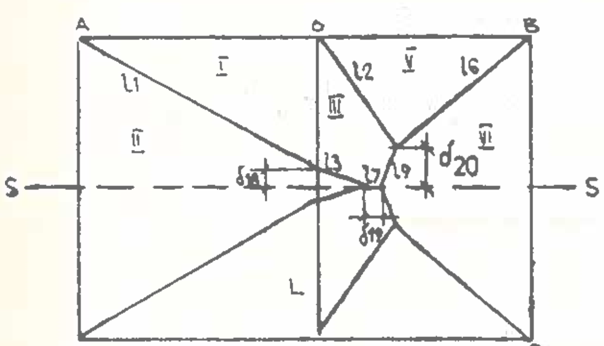

FIG. 10

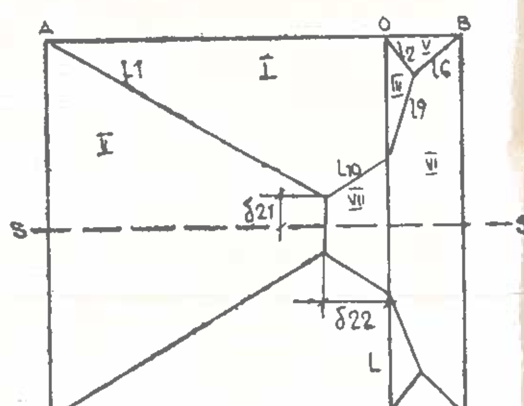

FIG. 11

for solution 8 and

$$\delta_{21} \geqslant 0, \ \left(2 - \frac{c}{b}\right) \lambda_1 - \left(2\frac{a}{b} - \lambda_0 \frac{c}{b}\right) \cos \varphi \leqslant 0, \quad (d_{11}),$$

(4.19)

$$\delta_{22} \geqslant 0, \ \frac{c}{b} \geqslant \frac{2\lambda_1}{\lambda_1 + \lambda_0 \cos \varphi}, \ (d_{15}),$$

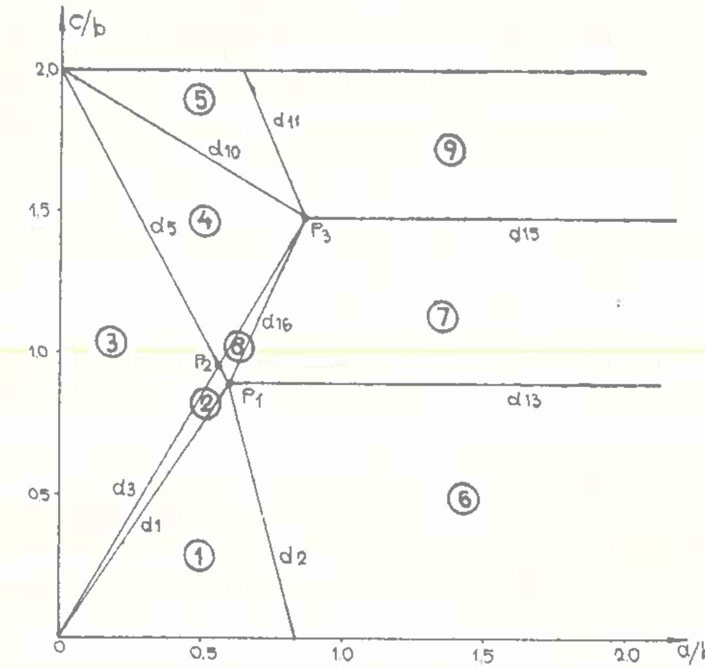

FIG 12

for solution 9.

The limit twisting moment for each particular solution is determined from equation (2.5).

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