He was able to teach various courses: mechanics, hydrodynamics, mechanics of continuous media. However, he did not neglect his scientific activity. He published a course on elasticity and organized special seminars and national conferences. He himself delivered informal reports, led the group of mechanicians of the Institute of Mathematics and organized weekly sessions for updating and improving the knowledge of his collaborators. He supported the journals „Studii și Cercetări Științifice" and „Analele Științifice ale Universității din Iași". The extension of the laboratory of mechanics and the reorganization of the astronomical observatory are his merits too.

His collaborators and the young researchers who were working for a doctoral degree under his guidance approached large themes on continuous media, hydrodynamics, magneto-hydrodynamics, isotropic and nonisotropic elasticity in homogeneous and nonhomogeneous media, termoelasticity, plasticity, continuous media with microstructure and analytical mechanics. During the same time, the construction of the first radiotelescope in our country began at the astronomical observatory. We witnessed the flourishing of new branches of mechanics. Ten dissertations were defended under his guidance. Other ten dissertations were in their final stage at the time of his death. Seven textbooks and monographs were published by his students and collaborators. Some prizes of the Academy were awarded to specialists of Iași for their valuable contributions to mechanics.

This intensive activity led to the creation of a genuine school of mechanics, which, no doubt, is the result of professor Mendel Haimovici's life time endeavour. Unfortunately, his tireless work, his total self-devotion to the advancement of science in Iași, and his permanent effort to satisfy all exigencies contributed to his premature death.

His devotion to science is materialized in two of his many achievements: the School of Mechanics of Iași and the Institute of Mathematics.

All these make us add a sign of gratitude, a symbol of our love and deep appreciation to the homage paid to the memory of the Romanian mathematician Mendel Haimovici through the organization of the National Colloquium of Mechanics and Geometry.

Radu Miron

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SOME TENSORIAL FINSLER STRUCTURES ON THE TANGENT BUNDLE

BY

MIHAI ANASTASIEI

To the memory of Professor Mendel Haimovici

In a joint paper with R. Miron (see [7]), we gave a free of coordinates definition of the fields of Finsler geometric objects. Particularly, the fields of Finsler tensor were obtained as cross-sections of a differentiable fibre bundle. In this paper, a „model" of that bundle is given (the Theorem 2.1). This means that every Finsler tensor field appears as a cross-section of a vector bundle obtained by taking the tensor product of some copies of vertical bundle V and of its dual V^*.

The prolongation of Finsler tensor fields to tensor fields on the manifold TM is usefull in various problems of Finsler geometry. Two remarkable prolongations are pointed out.

Using the first, a subalgebra of the tensorial algebra on TM isomorphic to the algebra of Finsler tensor fields is obtained.

The second allow us to find some tensorial structures (almost product, almost complex, almost Hermitian and others) on the manifold TM. Here and above TM means the total space of the tangent bundle to a differentiable manifold M.

1. Let M be a n-dimensional manifold, differentiable of class C^∞ and let $\pi: TM \to M$ be its tangent bundle. Let us denote by $d\pi: TTM \to TM$ the differential of π. The subbundle $V = \ker d\pi$ is called the vertical bundle over TM. Its fiber in the point $y \in TM$ will be denoted by V_y.

Every local chart (U, φ) on M induces a local chart $(\pi^{-1}(U), \Phi)$ where $\Phi: \pi^{-1}(U) \to \varphi(U) \times R^n \subset R^n \times R^n$ is given by $\Phi(y) = (\varphi(\pi(y)), d_x\varphi(y))$. Here $d_x\varphi$ denotes the tangent map in x of the diffeomorphism $\varphi: U \to R^n$.

If one puts $\varphi(x) = (\tilde{x}^1(x), \dots, \tilde{x}^n(x))$, then $\left(\frac{\partial}{\partial \tilde{x}^1}\right)_x, \dots, \left(\frac{\partial}{\partial \tilde{x}^n}\right)_x$ is the natural basis of T_xM defined by the local chart (U, φ). So, every $y \in T_xM$ can be expressed as $y = y^i \frac{\partial}{\partial \tilde{x}^i}$. Thus $(y^1, \dots, y^n)$ are the coordinates of $d_x(y)$ with respect to the canonical basis of R^n. It follows that $x^i = (x^i \circ \pi, y^i)$, $i = 1, \dots, n$ are the local coordinates on TM defined by the local chart $(\pi^{-1}(U), \Phi)$.

A vector field X on TM can be locally written as $X = X^i \frac{\partial}{\partial x^i} +$

$+ Y^i \frac{\partial}{\partial y^i}$. It is vertical if and only if $X^i = 0$ i.e. V_y is spanned by $\left(\frac{\partial}{\partial y^i}\right)$. The linear map $v_y : T_{\pi(y)}M \to V_y$, $y \in TM$, defined by $v_y(z) = z^i \frac{\partial}{y\partial^i}$, where $z = z^i \frac{\partial}{\partial x^i}$, is called vertical lift.

Let $\pi^* TM \to TM$ be the pull-back of TM by π. The map $j : \pi^* TM \to V$ defined by $j(z_1, z_2) = v_{z_1}(z_2)$, with $\pi(z_1) = \pi(z_2)$, is a vector bundle isomorphism. Its inverse is given by $j^{-1}(X_y) = (y, v_y^{-1}(X_y))$, $y \in TM$, $X_y \in V_y$. Let us put $\pi ! = p \times d_y\pi$, where p is the projection $TTM \to TM$.

It is obvious that $\pi !$ is onto and $\ker \pi ! = \ker d_y\pi = V_y$. Therefore, the following exact sequence of vector bundles over TM can be considered:

$$0 \to V \xrightarrow{i} TTM \xrightarrow{\pi !} \pi^* TM \to 0. \tag{1.1}$$

Here i is the inclusion map.

The fiber-preserving morphism $J : TTM \to TTM$ given by $J = i \circ j \circ \pi !$ which obviously satisfies $J^2 = 0$, is called the almost tangent structure on M. We have $\ker J_y = V_y = \operatorname{im} J_y$, $y \in TM$, therefore $J\left(\frac{\partial}{\partial y^i}\right) = 0$ and $J\left(\frac{\partial}{\partial x^i}\right) = \left(\frac{\partial}{\partial y^i}\right)$.

By taking in the above considerations $\widetilde{M} = TM \setminus 0$ with $\tilde{\pi} : \widetilde{M} \to M$, one obtains an exact sequence over $\widetilde{M}$:

$$0 \to V \xrightarrow{i} T\widetilde{M} \xrightarrow{\tilde{\pi}!} \pi^* TM \to 0. \tag{1.2}$$

A splitting of the exact sequence (1.2) is called a nonlinear connection on M. Thus, a nonlinear connection is a vector bundle morphism $w : \pi^* TM \to T\widetilde{M}$ such that $\tilde{\pi} ! \circ w =$ identity map on $\pi^* TM$. Let us put $H_y = \operatorname{im} w_y$, $y \in \widetilde{M}$. The set $H = \bigcup H_y$, where y covers $\widetilde{M}$, can be viewed as a subbundle of $T\widetilde{M}$. Further, we have $T\widetilde{M} = H \oplus V$ (the Whitney'sum).

The subbundle H is called the horizontal bundle associated to w. The horizontal lift $h_y : T_{\pi(y)}M \to H_y$ associated to w is defined as follows: $h_y(z) = w_y(y, z)$, $z \in T_{\pi(y)}M$.

Putting $\delta_i = h_y\left(\frac{\partial}{\partial \bar{x}}\right)$, it follows $d_y\pi\left(\delta_i - \frac{\partial}{\partial x^i}\right) = 0$, because of $d_y\pi \circ h_y =$ = identity and $d_y\pi\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial \tilde{x}^i}$. Since V_y is spanned by $\frac{\partial}{\partial y^i}$ we may write $\delta_i = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}$, $i, j = 1, \ldots, n$. The real functions (N_i^j) defined on M are called the local components of the nonlinear connection w.

Let (V, ψ) be another local chart on M such that $U \cap V \neq \varnothing$. Then $\psi\varphi^{-1}$ gives the coordinate transformation from the local chart (U, φ) to (V, ψ). Let us put $(\psi\varphi^{-1}) : x'^i = x'^i(x^1, \ldots, x^n)$. Then the corresponding coordinates transformation $\Psi\Phi^{-1}$ from $(\pi^{-1}(U), \Phi)$ to $(\pi^{-1}(V), \Psi)$ is given by:

$$x'^i = x'^i(x^1, \ldots, x^n), \quad y'^i = \frac{\partial x'^i}{\partial x^j} y^j, \tag{1.3}$$

and the change of the natural basis defined by the local charts $(\pi^{-1}(U), \Phi)$ and $(\pi^{-1}(V), \Psi)$ is obtained as follows

$$\left\{\begin{aligned} \frac{\partial}{\partial x^i} &= \frac{\partial x'^j}{\partial x^i} \frac{\partial}{\partial x'^j} + \frac{\partial^2 x'^j}{\partial x^i \partial x^k} y^k \frac{\partial}{\partial y'^j}, \\ \frac{\partial}{\partial y^i} &= \frac{\partial x'^j}{\partial x^i} \frac{\partial}{\partial y'^j}. \end{aligned}\right. \tag{1.4}$$

From the last equality it follows that $\left(\frac{\partial}{\partial y^i}\right)$ are Finsler vector fields i.e. each of them is defined on TM but it changes as a vector field on M, when a coordinate transformation on TM is performed. By applying h to $\frac{\partial}{\partial \tilde{x}^i} = \frac{\partial \tilde{x}'^j}{\partial \tilde{x}^i} \frac{\partial}{\partial \tilde{x}'^j}$ one obtains $\delta_i = \frac{\partial x'^j}{\partial x^i} \delta'_j$, where $\delta'_j = h\left(\frac{\partial}{\partial \tilde{x}'^j}\right)$ i.e. δ_i are Finsler vector fields, too. This facts lead to the following rule of transformation of the (N_i^j), when a change of local charts on TM is performed.

$$\frac{\partial x'^j}{\partial x^i} N'^k_j = \frac{\partial x'^k}{\partial x^j} N_i^j - \frac{\partial^2 x'^k}{\partial x^i \partial x^s} y^s. \tag{1.5}$$

Therefore (N_i^j) does not define a Finsler tensor field.

2. In the paper [7], see also [1], a free coordinate definition for the fields of Finsler geometric objects on a differentiable manifold M was given. Among the other examples, we have pointed out the Finsler tensor fields of any type on the manifold M. According to that definition, a Finsler tensor field t of type (p, q) is a differentiable section $t : TM \to FO^1$, where FO^1 is the total space of a differentiable bundle with type fiber $F = R^{n^{p+q}}$ associated to the Finsler bundle FM (the pull-back of the frames bundle, by π). Let $p : FO^1 \to TM$ be the natural projection and let $(p^{-1}(\pi^{-1}(U)), \tilde{\Phi})$ be the bundle chart of $FO^1 \to TM$ induced by the local chart $(\pi^{-1}(U), \Phi)$ on TM. The map $t_\Phi : \pi^{-1}(U) \to F$, given by $t_\Phi = \mathrm{pr}_2 \circ \tilde{\Phi} \circ t$, where pr_2 denotes the projection $\pi^{-1}(U) \times F \to F$, will be called the local reprezentative of t. Let us put $t_\Phi(z) = (t^{i_1 \ldots i_p}_{j_1 \ldots j_q}(z))$, $i_1, \ldots, i_p, j_1, \ldots, j_q = 1, \ldots, n$. The local representative $t_\Psi(z)$ of t with respect to the local chart $(\pi^{-1}(V), \Psi)$ will be given by

$$t_\Psi(z) = \left(\frac{\partial x'^{s_1}}{\partial x^{i_1}} \cdots \frac{\partial x'^{s_p}}{\partial x^{i_p}} \, t^{i_1 \ldots i_p}_{j_1 \ldots j_q} \, \frac{\partial x^{j_1}}{\partial x'^{r_1}} \cdots \frac{\partial x^{j_q}}{\partial x'^{r_q}}\right), \tag{2.1}$$

where the matrix $\left(\frac{\partial x^j}{\partial x'^k}\right)$ is the inverse of the matrix $\left(\frac{\partial x'^i}{\partial x^j}\right)$, both being calculated in the point $(x, y) = \Phi(z)$. For details, see [1].

Let now $V \to TM$ be the vertical bundle and let $V^* \to TM$ be its dual. Therefore, $V^* = \bigcup V_z^*$, where z covers TM and V_z^* is the dual of V_z. Let us denote by (θ^j), $j = 1, \ldots, n$, the dual basis of the basis $\left(\frac{\partial}{\partial y^i}\right)$ of V_z. Performing a change of local coordinates on TM one obtains:

$$(2.2)\qquad \begin{cases} \dfrac{\partial}{\partial y^i} = \dfrac{\partial x'^j}{\partial x^i}\dfrac{\partial}{\partial y'^j}, \\ \theta^j = \dfrac{\partial x^j}{\partial x'^k}\,\theta^k. \end{cases}$$

Now, we consider the bundle $\underbrace{V^*\otimes \dots \otimes V^*}_{q \text{ times}}\otimes \underbrace{V \dots \otimes V}_{p \text{ times}} \to TM$ and denote by $T_{Fq}^{\,p}$ the $\mathcal{F}(TM)$-module of its sections. Here $\mathcal{F}(TM)$ means the ring of all real functions on TM. These can be regarded as fields of a kind of Finsler geometric objects (see [1]) so we call them Finsler functions.

Theorem 2.1. *The $\mathcal{F}(TM)$-module $T_{Fq}^{\,p}$ and the $\mathcal{F}(TM)$-module of the sections of the fibre bundle $FO^1 \to TM$ are isomorphic.*

Proof. Let be $t : TM \to FO^1$ and let t_Φ be its local representative with respect to a local chart $(\pi^{-1}(U), \Phi)$ on TM. We associate with t the member of $T_{Fq}^{\,p}$ which is given, with respect to the local chart $(\pi^{-1}(U), \Phi)$, by $t^{i_1 \dots i_p}_{j_1 \dots j_q}(z)\dfrac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \dfrac{\partial}{\partial x^{i_p}} \otimes \theta^{j_1} \dots \otimes \theta^{j_q}$. Keeping in mind (2.1) and (2.2), we see that, in such a way, one obtains a well-defined map, which obviously is one-to-one and onto. Also, we see easily that this map is $\mathcal{F}(TM)$-linear.

The Theorem 2.1 shows that we can regard the bundle $V^*\otimes \dots \otimes V^*\otimes V\otimes \dots \otimes V \to TM$ as materializing the „abstract" bundle $FO \xrightarrow{1} TM$. Since the vertical bundle is isomorphic to the horizontal bundle, it follows that the bundle $H^*\otimes \dots \otimes H^*\otimes H\otimes \dots \otimes H \to TM$ can also be viewed as materializing the bundle $FO^1 \to TM$.

3. In a joint paper with I. Popovici [8], we studied a kind of geometric structures on a vector bundle $E \to N$, where N is a differentiable manifold, called by us B- and $\widetilde{B}$-structures.

Let $S(E)$ be the $\mathcal{F}(N)$-module of global sections of $E \to N$ and let $A(E)$ be the algebra of $\mathcal{F}(N)$-linear operators on $S(E)$. A B-structure is a pair $(P, R) \in A(E) \times A(E)$ such that $P^2 = I$, $R^2 = 0$, $RP = -PR = R$ and a $\widetilde{B}$-structure is a pair $(\widetilde{P}, \widetilde{Q}) \in A(E) \times A(E)$ such that $\widetilde{P}^2 = \widetilde{Q}^2 = I$ and $\widetilde{P}\widetilde{Q} + \widetilde{Q}\widetilde{P} = 0$.

A B-structure (P, R) is said to be the restriction of a $\widetilde{B}$-structure $(\widetilde{P}, \widetilde{Q})$ if and only if $P = \widetilde{P}$ and $R = (\widetilde{Q} - P\widetilde{Q})/2$. In order that a B-structure (P, R) be the restriction of a $\widetilde{B}$-structure, it is necessary and sufficient that $\ker R = \operatorname{im} R$ (see [8], [3]). The line of the proof is as follows:

The almost product structure P defines for each fiber E_x, $x \in N$ a splitting $E_x = E_x^+ \oplus E_x^-$, where E_x^+ and E_x^- are the eigenspaces which correspond to eigenvalues $+1$ and -1, respectively. Assuming N connected, $V^+ = \bigcup E_x^+$ and $V^- = \bigcup E_x^-$ when x covers N, are both subbundles of E and $E = V^+ \oplus V^-$ holds. If the B-structure (P, R) is the restriction of a $\widetilde{B}$-structure, it follows $\ker R = \operatorname{im} R = S(V^-)$ and the restriction of R to $S(V^+)$, denoted also by R, is an isomorphism of $S(V^+)$ onto $S(V^-)$. Conversely, if (P, R) is a B-structure satisfying $\ker R = \operatorname{im} R$, then the $\widetilde{B}$-structure $(P, \widetilde{Q})$, where $\widetilde{Q}$ is well-determined by $R = (\widetilde{Q} - P\widetilde{Q})/2$, prolonges the B-structure (P, R). The operator $\widetilde{Q}$ is defined by $\widetilde{Q}(X) = R(X)$ if $X \in S(V^+)$ and $\widetilde{Q}(X) = Q(X)$ if $X \in S(V^-)$, where Q is the inverse of R.

Let us apply the previous considerations to $N = \widetilde{M}$ and $E = T\widetilde{M}$. In such a case we have at our disposal an operator $J \in A(T\widetilde{M})$ which satisfies $\ker J = \operatorname{im} J$. Let us suppose there exists an operator $P \in A(T\widetilde{M})$ such that $JP = -PJ = J$. Then a theorem due to J. Grifone (see [4]) shows that $P^2 = I$ and the eigenspace of P_y corresponding to -1 is V_y, $y \in M$. Thus, the pair (P, J) is a B-structure, the restriction of a $\widetilde{B}$-structure $(P, \widetilde{Q})$, where $J = (\widetilde{Q} - P\widetilde{Q})/2$. Putting $H = QV$, one obtains $TM = V \oplus H$ and a splitting of the exact sequence (1.2) can be derived. Conversely, a splitting of the exact sequence (1.2) determines P (see [4], for details). In [8] we showed that the triad (P, J, D), where D is a connection on TM, such that $DP = D\widetilde{Q} = 0$, defines a Finsler connection (in Matsumoto'sense) on M and conversely.

4. Now we consider M endowed with a nonlinear connection or equivalently, with a B-structure (P, J).

Lemma 4.1. a) *For every vector field X on $\widetilde{M}$, there exist A, $B \in S(V)$ such that $X = A + QB$; $A = B$ if and only if $\widetilde{Q}X = X$.* b) *For every covector field U on M, there exist α, $\beta \in S(V)$ such that $U = \alpha + \beta Q$, where $\beta Q : H = QV \to R$ is given by $\beta Q(QA) = \beta(A)$ $A \in S(V)$; $\alpha = \beta$* if and only if $\widetilde{U}Q = U$.

Proof. Let $p_v : S(T\widetilde{M}) \to S(V)$ and $p_h : S(T\widetilde{M}) \to S(H)$ be the projections defined by the decomposition $T\widetilde{M} = V \oplus H$. Given $X \in S(T\widetilde{M})$, we put $A = p_v X$. It follows that $X - A \in S(H)$, therefore $B \in S(V)$ exists such that $X = A + QB$. If $\widetilde{Q}X = X$ holds, then $Q(A - B) = A - B$ and it follows $A = B$. Conversely, if $A = B$ it is easy to see that $\widetilde{Q}X = X$.

b) Every 1-form U on M can be written as $U = \alpha + U_2$, where α is a 1-form null on the horizontal vectors and U_2 is a 1-form null on the vertical vectors, therefore $\alpha \in S(V^*)$ and $U_2 \in S(H^*)$. Since $S(V)$ and $S(H)$ are isomorphic, there exists an isomorphism, let say $Q^* : S(V^*) \to S(H^*)$. So, there exists $\beta \in S(V^*)$ such that $U_2 = Q^*\beta$ i.e $U_2(X) = \beta(Q^{-1}X)$ for $X \in S(H)$ or $U_2(QY) = \beta(Y)$ for $Y \in S(V)$. Therefore we can denote $U_2 = \beta Q$ and the first part of b) is proved. The second part is clear.

The Lemma 4.1 sugests the possibility to prolonge every $A \in S(V)$ to a vector field X on TM by taking $X = A + QA$. Also, we can prolonge $\alpha \in S(V^*)$ to a 1-form U on TM by taking $U = \alpha + \alpha Q$.

A Finsler tensor field of type (p, q) can be prolonged on various ways to a tensor field on TM. We say that a tensor field t' of type (p, q) on TM is the S-prolongation of a Finsler tensor field t of the same type if it is given by

$$(4.1)\qquad \begin{aligned} t'(\alpha_1 + \beta_1 Q, \dots, \alpha_p + \beta_p Q, A_1 + QB_1, \dots, A_q + QB_q) = \\ = \Sigma t(u_1, \dots, u_p, C_1, \dots, C_p), \end{aligned}$$

where $u_i = \alpha_i$ or β_i and $C_j = A_j$ or B_j, the sum being over all possibilities, $\alpha_1, \dots, \alpha_p \in S(V^*)$, $\beta_1, \dots, \beta_p \in S(V^*)$, $A_1, \dots, A_q$, $B_1, \dots, B_q \in S(V)$.

A characterization of the tensor fields on TM which are S-prolongations of Finsler tensor fields is as follows.

Theorem 4.1. *A tensor field t' of type (p, q) on TM is the S-prolongation of a Finsler tensor field t if and only if*

$$t'(U_1, \ldots, U_r\widetilde{Q}, \ldots, U_p, X_1, \ldots, X_q) = \tag{4.2}$$
$$= t'(U_1, \ldots, U_p, X_1, \ldots, Q\widetilde{X}_s, \ldots, X_q) = t'(U_1, \ldots, U_p, X_1, \ldots, X_q)$$

hold for every 1-forms $U_1, \ldots, U_p$ and every vector fields $X_1, \ldots, X_q$; $r = 1, \ldots, p$ $s = 1, \ldots, q$.

Proof. By considering $t'(\ldots, (\alpha_r + \beta_r Q)\widetilde{Q}, \ldots)$ or $t'(\ldots, \widetilde{Q}(A_s + QB_s), \ldots)$ in the right hand of (4.1) only a change of order is produced, therefore (4.2) holds. Conversely, given t' which satisfies (4.2), it S-prolonges the Finsler tensor field t given by

$$t(\alpha_1, \ldots, \alpha_p, A_1, \ldots, A_q) = t'(\alpha_1, \ldots, \alpha_p, A_1, \ldots, A_q) \text{ for} \tag{4.3}$$
$$\alpha_1, \ldots, \alpha_p \in S(V^*),\ A_1, \ldots, A_q \in S(V).$$

Let $I_F = \bigoplus_{p,q} I_q^p$ be the subalgebra of the tensor algebra on TM made up of tensors which satisfy (4.2). The S-prolongation introduced above leads to the following theorem due to I. Popovici.

Theorem 4.2. *There exists a canonical isomorphism of algebras $\psi : I_F \to T_F = \bigoplus_{p,q} T_1{}_q^p$.*

Proof. The map ψ is defined by (4.3), while its inverse obviously is the map defined by (4.1). It is easy to see that ψ is a morphism of algebras.

5. By taking in the sum from the right hand of the formula (4.1), each term of it multiplied by a real functions on TM one obtains a different prolongation of Finsler tensor fields, which we call CS-prolongation. For example, a Finsler tensor field t of type $(1, 1)$, $t : S(V) \to S(V)$ can be prolonged to $t' : S(V) \oplus S(H) \to S(V) \oplus S(H)$ given by $t'(A + QB) = at(A) + bQt(A) + ct(B) + dQt(B)$, $A, B \in S(V)$, where a, b, c, d belong to $\mathcal{F}(TM)$.

Let us consider the CS-prolongation of the identity morphism $I : S(V) \to S(V)$, $I'(A + QB) = aA + bQA + cB + dQB$. This tensor field provides a large number of almost product and almost complex structures on TM. Firstly, we are looking for real functions a, b, c, d such that I' be an almost product structure i.e. $I'^2 = identity$. One obtains the following system of equations:

$$a^2 + bc = 1, \quad ac + cd = 0, \quad ab + bd = 0, \quad bc + d^2 = 1,$$

which, resolved, leads to the following almost product structures on TM:

$$I'(A + QB) = A + QB, \tag{5.1}$$

$$I'(A + QB) = -A - QB, \text{ which are trivial. Then:} \tag{5.2}$$

$$I'(A + QB) = aA + \frac{1 - a^2}{c} QA + cB - aQB, \quad c \neq 0, \tag{5.3}$$

$$I'(A + QB) = aA + bQA + \frac{1 - a^2}{b} B - aQB, \quad b \neq 0, \tag{5.4}$$

$$I'(A + QB) = A + cB - QB, \tag{5.5}$$

$$I'(A + QB) = -A + cB + QB, \text{ for every } a, b, c \in \mathcal{F}(TM). \tag{5.6}$$

It must be observed that the almost product structures given by (5.6) with $c = 0$, coincide with P. The formula (5.3) with $a = 0$ and $c = 1$ gives the almost product structure $\widetilde{Q}$.

Now, we are looking for real functions a, b, c, d such that I' be an almost complex structure. By imposing the condition $I'^2 = -identity$, one obtains the following system of equations:

$$a^2 + bc = -1, \quad ac + cd = 0, \quad ab + bd = 0, \quad bc + d^2 = -1,$$

which, solved leads to the following almost complex structures on TM:

$$I'(A + QB) = aA - \frac{1 + a^2}{c} QA + cB - aQB, \quad c \neq 0, \tag{5.7}$$

$$I'(A + QB) = aA + bQA - \frac{1 + a^2}{b} B - aQB, \quad b \neq 0. \tag{5.8}$$

The formula (5.7) with $a = 0$ and $c = -1$ gives the almost complex structure F, canonically associated to the nonlinear connection (P, J). It is easy to see that $F = \widetilde{P}Q$. Using the same method, we found the tensor field $\mathcal{R}$ given by

$$\mathcal{R}(A + QB) = \frac{1}{2} A + bQA + \frac{1}{4b} B + \frac{1}{2} QB, \tag{5.9}$$

which satisfies $\mathcal{R}^2 = \mathcal{R}$ and the tensor field S given by

$$S(A + QB) = -\frac{1}{2} A + bQA + \frac{1}{4b} B - \frac{1}{2} QB, \quad b \neq 0, \tag{5.10}$$

which verifies $S^3 = S$.

Remark. If M is paracompact, the manifolds TM and $\widetilde{M}$ are paracompact, therefore the sequence (1.2) is splitting i.e. M admits a nonlinear connection. Then all tensorial structures (5.1)—5.10) exist. The nonlinear connection is canonically determined if M is endowed with a fundamental Finsler function or, more generaly, with a regular metrical Finsler structure i.e. a nondegenerate, symmetrical Finsler tensor of type $(0, 2)$ satisfying the conditions given below. Let g denotes the metrical Finsler structure. Let us put $g_{ij} = g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)$. The conditions are (see [6]):

$$\frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} y^i y^j = 0, \quad \det\left(\delta_j^k + g^{ks} \frac{\partial g_{ij}}{\partial y^s} y^i\right) \neq 0. \tag{5.11}$$

6. Let us suppose M endowed with a metrical Finsler structure g and with a nonlinear connection (P, J). The existence of g is assured if M is paracompact (see [2]). The CS-prolongation of g is given by

$$G(A_1 + QB_1, A_2 + QB_2) = kg(A_1, A_2) + lg(A_1, B_2) + \tag{6.1}$$
$$+ mg(B_1, A_2) + ng(B_1, B_2).$$

We remark that $l=m$ is necessary for the symmetry of G. For $k=n=1$ and $l=m=\alpha$ one obtains the α-lift of g considered by M. Matsumoto ([5], p. 105). Generally, G with $l=m$ is quasi-Riemannian. The α-lift is Riemannian if $-1<\alpha<1$. By a direct computation it follows:

Theorem 6.1. *a) A CS-prolongation of g with $k=n$ and $l=m=\alpha$ is almost hermitian with respect to F if and only if $\alpha=0$. b) A CS-prolongation of g with $l=m=\alpha$ satisfies $G(PX, PY)=G(X, Y)$ for every $X, Y\in S(TTM)$ if and only if $\alpha=0$. c) A CS-prolongation of g with $l=m$ and $k=m$ satisfies $G(\widetilde{Q}X, \widetilde{Q}Y)=G(X, Y)$ for every $X, Y\in S(TTM)$.*

A pair (G, P) on TM is called almost Hermitian hiperbolic structure if $G(PX, PY)=-G(X, Y)$ holds for every $X, Y\in S(TTM)$. By taking G as CS-prolongation of g, one obtains, after a simple computation

Theorem 6.2. *The pair (G, P), where G is the CS-prolongation of g with $l=m=\alpha$ is an almost Hermitian hiperbolic structure if and only if $k=n=0$ i.e. G is as follows*

$$G(A_1+QB_1, A_2+QB_2)=\alpha(g(A_1, B_2)+g(A_2, B_1)). \tag{6.2}$$

Now we take G given by (6.2) and we obtain

Theorem 6.3. *a) The almost product structures given by (5.5) and (5.6) pairing with G give almost Hermitian hyperbolic structures if and only if $c=0$. b) The almost product structures given by (5.3) and (5.4) pairing with G give almost Hermitian hyperbolic structures.*

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Faculty of Mathematics,
University of Iaşi, 6600 – Iaşi, R.S. România

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ON SEMI-INVARIANT SUBMANIFOLDS OF AN ALMOST CONTACT METRIC MANIFOLD

BY

AUREL BEJANCU

To the memory of Professor Mendel Haimovici

1. Introduction. Let N be a $(2n+1)$-dimensional almost contact metric manifold with structure tensors (φ, ξ, η, g), where φ is a tensor field of type (1,1), ξ is a vector field, η is a 1-form and g is the Riemannian metric on N. These tensor fields are related by

$$\varphi^2X=-X+\eta(X)\xi\ ;\ \varphi\xi=0\ ;\quad \eta(\xi)=1\ ;\quad \eta(\varphi X)=0 \text{ and} \tag{1.1}$$

$$g(\varphi X, \varphi Y)=g(X, Y)-\eta(X)\,.\,\eta(Y)\ ;\ \eta(X)=g(X, \xi), \tag{1.2}$$

for any vector fields X, Y tangent to N.

Let M be an m-dimensional submanifold of N. We assume that the structure vector field ξ is tangent to M. Then M is called a *semi-invariant submanifold* of N if there exist on M two differentiable orthogonal distributions D and $D^\perp$ such that the following conditions are satisfied:

(i) $TM=D\oplus D^\perp\oplus\{\xi\}$, where $\{\xi\}$ is the 1-dimensional distribution spanned by ξ on M,

(ii) the distribution D is invariant by φ, that is, $\varphi(D_x)=D_x$ for each $x\in M$ and

(iii) the distribution $D^\perp$ is anti-invariant by φ, that is, $\varphi(D^\perp_x)\subset T_xM^\perp$, for each $x\in M$, where $T_xM^\perp$ is the normal space to M at x, (see [1]).

We denote by $2p$ (resp. q) the real dimension of a fiber of D (resp. $D^\perp$) and by P (resp. Q) the projection morphisms of TM to D (resp. $D^\perp$). Then we see that for $p=0$ (resp. $q=0$) we obtain an anti-invariant submanifold tangent to ξ (see [3]) (resp. an invariant submanifold (see [4])) of N. Moreover, when $q=\dim T_xM^\perp$ we obtain generic submanifolds studied by K. Yano and M. Kon in [2]. It is easy to check that each hypersurface of N which is tangent to ξ inherites a structure of a semi-invariant submanifold of N.

For any vector bundle H on M we denote by $\Gamma(H)$ the module of all differentiable sections of H on a neighborhood coordinates on M.

Now, suppose D' be another invariant distribution on M. Then we have $D'_x\subset D_x$ for any $x\in M$. In fact, if $X\in\Gamma(D')$ then there exists $Y\in\Gamma(D')$ such that $X=\varphi Y$ and we have

$$g(\varphi Y_x, Z_x)=-g(Y_x, \varphi Z_x)=0 \text{ for any } Z\in\Gamma(D^\perp) \text{ and}$$

$$g(\varphi Y_x, \xi_x)=-g(Y_x, \varphi\xi_x)=0.$$