Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
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THE FUNDAMENTAL PLANE MATRIX IN THE UNIFORM FLOW OF THE IDEAL COMPRESSIBLE FLUIDS

BY

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To the memory of Professor Mendel Haimovici

1. In the paper [1] I have determined the fundamental three-dimensional matrix for the system which described the uniform nonstationary flow of an ideal compressible fluid. This matrix is defined by the equations

$$\begin{gathered}\partial_t\rho+\partial_1\rho+\operatorname{div}\mathbf{v}=m\delta(t,\mathbf{x}),\quad \partial_t\mathbf{v}+\partial_1\mathbf{v}+\operatorname{grad}p=\mathbf{f}\delta(t,\mathbf{x}),\\ M^2(\partial_t p+\partial_1\rho)+\operatorname{div}\mathbf{v}=r\delta(t,\mathbf{x}),\quad (\rho,\mathbf{v},p)=0 \text{ for } t\leqslant 0,\quad \lim_{\infty}(\rho,\mathbf{v},p)=0,\end{gathered}\tag{1}$$

where

$$\partial_t=\partial/\partial t,\quad \partial_1=\partial/\partial x_1\quad \delta(t,\mathbf{x})=\delta(t)\delta(\mathbf{x})$$

represents the Dirac's distribution, and $M(=v_0/c_0)$ is the Mach number in the unperturbed flow (at large distances).

Note that from $(1)_1$ and $(1)_3$ we obtain

$$\partial_t(\rho-M^2p)+\partial_1(\rho-M^2p)=(m-r)\delta(t,\mathbf{x}),$$

from which, H being the Heaviside function,

$$\rho-M^2p=(m-r)H(t)\delta(x_1-t,\ x_2,\ x_3).\tag{2}$$

Therefore it is a sufficient to consider the system

$$\begin{gathered}\partial_t\mathbf{v}+\partial_1\mathbf{v}+\operatorname{grad}p=\mathbf{f}\delta(t,\mathbf{x}),\quad M^2(\partial_t p+\partial_1 p)+\operatorname{div}\mathbf{v}=r\delta(t,\mathbf{x}),\\ (\mathbf{v},p)=0,\ t\leqslant 0\ ;\ \lim_{\infty}(\mathbf{v},p)=0\end{gathered}\tag{3}$$

Let us consider now the plane problem. Introducing the matrices

$$V=\begin{pmatrix}v_1\\ v_2\\ M^2p\end{pmatrix},\qquad f=\begin{pmatrix}f_1\\ f_2\\ f_3\end{pmatrix},\quad A_1=\begin{pmatrix}1&0&M^{-2}\\0&1&0\\1&0&1\end{pmatrix},\quad A_2=\begin{pmatrix}0&0&0\\0&0&M^{-2}\\0&1&0\end{pmatrix}$$

the system (3) becomes

$$\begin{gathered}\dot V+A_k\partial_k V=f\delta(t,\mathbf{x})\ \text{ where }\ \dot V=\partial V/\partial t\\ V(t\leqslant 0,\mathbf{x})=0,\ V(t,\infty)=0\end{gathered}\tag{4}$$

Let us define the fundamental matrix L by the formula

$$V=Lf\ ;\tag{5}$$

we obtain

$$L + A_k \partial_k L = E\,\delta(t, \mathbf{x}), \quad L(t \le 0, \mathbf{x}) = 0, \quad L(t, \infty) = 0, \tag{6}$$

where E is the unit 3×3 matrix. Using Duhamel's principle, we deduce the following solution of the system (6)

$$L = H(t) K(t, \mathbf{x}), \tag{7}$$

where

$$\dot{K} + A_k \partial_k K = 0, \quad K(0, \mathbf{x}) = E\,\delta(\mathbf{x}), \quad K(t, \infty) = 0. \tag{8}$$

By applying the Fourier's transform we obtain

$$\dot{\overline{K}} = \overline{A}\,\overline{K}, \quad \overline{K}(0, \alpha) = E \tag{9}$$

where $\overline{K} = \mathcal{F}[K]$, $\overline{A} = i\alpha_k A_k$. Using the method of minimal polynom, as it was done in [1], [2], we deduce

$$\begin{aligned} \overline{K} &= E e^{i\alpha_1 t} + M(-i\alpha_1 E + \overline{A}) \frac{\sin |\alpha| M^{-1} t}{|\alpha|} e^{i\alpha_1 t} + \\ &+ M^2(-\alpha_1^2 E - 2i\alpha_1 \overline{A} + \overline{A}^2) \frac{1 - \cos |\alpha| M^{-1} t}{|\alpha|^2} e^{i\alpha_1 t} \end{aligned} \tag{10}$$

If in the formula $(r = \sqrt{x_1^2 + x_2^2})$

$$\mathcal{F}^{-1}\left(\frac{\sin a |\alpha| t}{|\alpha|}\right) = \frac{1}{2\pi} \frac{H(at - r)}{\sqrt{a^2 t^2 - r^2}}$$

proved, for instance, in [2] [3], we replace x_1 by $x_1 - t$, we have

$$\mathcal{F}^{-1}\left(\frac{\sin a |\alpha| t}{|\alpha|} e^{i\alpha_1 t}\right) = \frac{1}{2\pi} \frac{H(at - R)}{\sqrt{a^2 t^2 - R^2}}, \tag{11}$$

where $R = \sqrt{(x_1 - t)^2 + x_2^2}$. By the same substitution, from the formula [2]

$$\mathcal{F}^{-1}\left(\frac{1 - \cos a |\alpha| t}{|\alpha|^2}\right) = \frac{H(at - r)}{2\pi} \ln \frac{at + \sqrt{a^2 t^2 - r^2}}{r}$$

we obtain

$$\mathcal{F}^{-1}\left(\frac{1 - \cos a |\alpha| t}{|\alpha|^2} e^{i\alpha_1 t}\right) = \frac{H(at - R)}{2\pi} \ln \frac{at + \sqrt{a^2 t^2 - R^2}}{R} \tag{12}$$

Taking into account (11) and (12), from (10) we deduce

$$\begin{aligned} K(t, \mathbf{x}) &= E\,\delta(x_1 - t, x_2) + \frac{M^2}{2\pi}\left(E \frac{\partial}{\partial x_1} + A\right) \frac{H(t - MR)}{\sqrt{t^2 - M^2 R^2}} + \\ &+ \frac{M^2}{2\pi}\left(\frac{\partial^2}{\partial x_1^2} + 2A \frac{\partial}{\partial x_1} + A^2\right) H(t - MR) \ln \frac{t + \sqrt{t^2 - M^2 R^2}}{MR} \end{aligned} \tag{13}$$

where

$$A = -\begin{pmatrix} \partial_1 & 0 & M^{-2}\partial_1 \\ 0 & \partial_1 & M^{-2}\partial_2 \\ \partial_1 & \partial_2 & \partial_1 \end{pmatrix}.$$

2. By means of the fundamental matrix (13), it is proved that the solution of the equation

$$\dot{V} + A_k \partial_k V = F(t, \mathbf{x}), \tag{14}$$

under the conditions

$$V(0, x) = 0, \quad V(t, \infty) = 0, \tag{14'}$$

has the following form

$$\begin{aligned} V(t, \mathbf{x}) &= \int_0^t K * F d\tau = \int_0^t d\tau \iint_{-\infty}^{+\infty} K(t - \tau, \mathbf{x} - \xi) F(\tau, \xi) d\xi_1 d\xi_2 = \\ &= \int_0^t d\tau \iint_{-\infty}^{+\infty} K(\tau, \xi) F(t - \tau, \mathbf{x} - \xi) d\xi_1 d\xi_2. \end{aligned} \tag{15}$$

By (13), and taking into account the properties of the convolution, we have

$$\begin{aligned} V(t, \mathbf{x}) &= \int_0^t F(t - \tau, x_1 - \tau, x_2) d\tau + \frac{M^2}{2\pi}\left(E \frac{\partial}{\partial x_1} + A\right) \int_0^t d\tau \iint_{-\infty}^{+\infty} F(t - \tau, x - \xi) \\ &\frac{H(\tau - MS)}{\sqrt{\tau^2 - M^2 S^2}} d\xi_1 d\xi_2 + \frac{M^2}{2\pi}\left(E \frac{\partial^2}{\partial x_1^2} + 2A \frac{\partial}{\partial x_1} + A^2\right) \int_0^t d\tau \iint_{-\infty}^{+\infty} F(t - \tau, \mathbf{x} - \xi) \\ &H(\tau - MS) \ln \frac{\tau + \sqrt{\tau^2 - M^2 S^2}}{MS} d\xi_1 d\xi_2 \end{aligned} \tag{16}$$

where $S = \sqrt{(\xi_1 - \tau)^2 + \xi_2^2}$.

3. As an application let us study the perturbation which occurrs in a uniform fluid flow due to sources of intensity f which are acting at the origin permanently for $t > 0$. To this purpose, we substitute (17) in (16)

$$F(t, \mathbf{x}) = f H(t) \delta(\mathbf{x}) \tag{17}$$

When the sources are reduces to the momentum $(m = 0, r = 0)$, then (17) indicates the leading edge on the $0x_3$-axis $(-\infty < x_3 < \infty)$ of a cilindrical body of infinite average. In the general case, we obtain

$$\begin{aligned} V(t, \mathbf{x}) &= f[H(x_1) - H(x_1 - t)]\,\delta(x_2) + \\ &+ \frac{M^2}{2\pi}\left(E \frac{\partial}{\partial x_1} + A\right) f I_1 + \frac{M^2}{2\pi}\left(E \frac{\partial^2}{\partial x_1^2} + 2A \frac{\partial}{\partial x_1} + A^2\right) f I_2 \end{aligned} \tag{18}$$

where

$$I_1 = \int_0^t \frac{H(\tau - MQ)}{\sqrt{\tau^2 - M^2 Q^2}} d\tau, \quad I_2 = \int_0^t H(\tau - MQ) \ln \frac{\tau + \sqrt{\tau^2 - M^2 Q^2}}{MQ} d\tau$$

$$Q=\sqrt{(x_1-\tau)^2+x_2^2}$$

In order to determine the integrals I_1 and I_2 we have to analyse the function $h(\tau)=\tau-MQ$. The integrants are vanishing except for the domains where $h(\tau)>0$. Since $h(0)=-Mr<0$ and $h(\infty)=(1-M^2)\infty$ we shall distinguish two cases: $M<1$ and $M>1$. The equation $h(\tau)=0$ has at most two roots namely

$$\tau_{\pm}=\frac{-M^2x_1\pm M\sqrt{x_1^2+(1-M^2)x_2^2}}{1-M^2} \tag{19}$$

Hence we have the situations represented in Fig. 1 where $h_0=x_1-\sqrt{M^2-1}\,|x_2|$. As a consequence, in the cases $M<1$ and $(M>1,\ h_0>0,\ t<\tau_-)$ we have

$$I_1=H(t-\tau_+)\int_{\tau_+}^{t}\frac{d\tau}{\sqrt{\tau^2-M^2Q^2}},\quad I_2=H(t-\tau_+)\int_{\tau_+}^{t}\ln\frac{\tau+\sqrt{\tau^2-M^2Q^2}}{MQ}\,d\tau$$

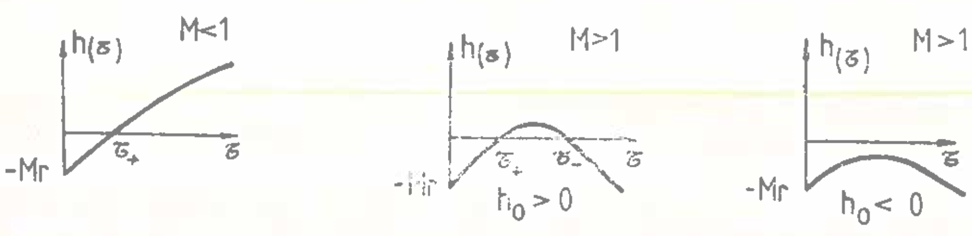

Fig. 1.

For $M>1$, $h_0>0$, $t>\tau_-$ we have

$$I_1=H(t-\tau_+)\int_{\tau_+}^{\tau_-}\frac{d\tau}{\sqrt{\tau^2-M^2Q^2}},\quad I_2=H(t-\tau_+)\int_{\tau_+}^{\tau_-}\ln\frac{\tau+\sqrt{\tau^2-M^2Q^2}}{MQ}\,d\tau$$

while in the case $M>1$, $h_0<0$ we have $I_1=I_2=0$. The calculations involved in the integrals I is elementary.

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SIMPLE SHEAR OF SOILS

BY

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To the memory of Professor Mendel Haimovici

1. Introduction. A horizontal layer of uniform soil (illustrated in Fig. 1) tends to move the vertical supporting wall.

In 1776, C. A. Coulomb tried to find the soil action upon the wall, assumed that the plane $ABA'B'$ is a slip plane. He defined the active pressure as the maximum of force F, the maximum being calculated in function of the angle between the slip and vertical plane.

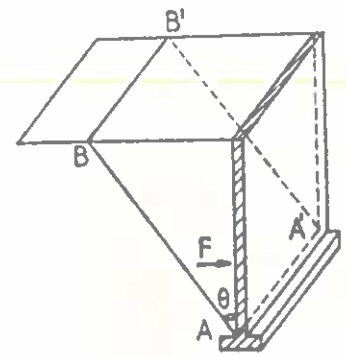

Fig. 1

According to Coulomb, the maximum value $F_{\max}=\frac{1}{2}\gamma h^2\,\mathrm{tg}^2\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)$ is attained for $\theta=\frac{\pi}{4}-\frac{\varphi}{2}$, where γ is the dry bulk density of soil, h is the wall height, φ is the friction angle of soil (see [2, cap. 9], [3, cap. 15]).

Nowadays, the Coulomb theory is applied to smooth vertical walls which retain a horizontal sand layer.

Using the constitutive equations of hypoelastic type, this paper finds the value of active pressure for some initial conditions. The plane $ABA'B'$ is regarded as a shear plane surface.

2. Simple shear of soils. We consider the case of simple shear:

$$x_1=2kx_2t,\quad x_2=x_2^0\quad x_3=x_3^0,$$

where t is the time and k is a constant value (see Fig. 2).

Because the time derivatives are:

$$\dot{x}_1=2kx_2,\quad \dot{x}_2=0,\quad \dot{x}_3=0,$$

the strain rates tensor and the spin tensor have the form:

$$D=\begin{bmatrix}0&-k&0\\-k&0&0\\0&0&0\end{bmatrix},\qquad W=\begin{bmatrix}0&k&0\\-k&0&0\\0&0&0\end{bmatrix}.$$