$$(3.4)\quad A_n=\frac{1}{Z(\tau_*,\lambda_n)}\left[-\frac{Q}{n\pi}-\frac{Q}{n\pi}\frac{Z'(\tau_1,\lambda_n)Z(\tau_*,-\lambda_n)}{W(\tau_1)}\right];\quad B_n=\frac{Q}{n\pi}\frac{Z'(\tau_1,\lambda_n)}{W(\tau_1)}.$$

Pour ce cas-là, dans la relation (3.3.) on calcule $\frac{\partial\psi_2}{\partial\tau}$ avec (3.4.) ; en intégrant par la suite (3.3) de O à $L(\mu,\tau_*)$ et après 0 de μ à O on obtient

$$(3.5)\quad \begin{aligned} L(\mu,\tau_*)&=\frac{Q}{V_*(1-\tau_*)^{\beta}}\sum_{n=1}^{\infty}\{(-1)^{n-1}\cos\mu+1\}\cdot\\ &\cdot\left[x_n(\tau_*)-x_n(\tau_1)\frac{Z(\tau_1,\lambda_n)}{Z(\tau_*,\lambda_n)}\left(\frac{1-\tau_*}{1-\tau_1}\right)^{\beta}\right]\frac{n\pi}{n^2\pi^2-\mu^2}.\end{aligned}$$

En évaluant par la suite le débit à l'infini aval où le jet est large de l_∞ on obtient de (3.5).

$$(3.6)\quad \begin{aligned} L(\mu,\tau_*)&=l_\infty\sum_{n=1}^{\infty}\frac{n\pi}{n^2\pi^2-\mu^2}[(-1)^{n-1}\cos\mu+1]\times\\ &\times\left[X(\tau_*)-X(\tau_1)\frac{Z(\tau_1,\lambda_n)}{Z(\tau^*,\lambda_n)}\left(\frac{1-\tau_*}{1-\tau_1}\right)^{\beta}\right].\end{aligned}$$

Par la mise à profit des nouvelles propriétés des functions de Tchapliguine étudiées par C. J a c o b [2] on peut faire des délimitations au cas des calculs numériques pour la ligne de passage (3.6).

Si $\mu=\frac{\pi}{2}$ alors de (3.6) on a le résultat obtenu par Aslanov et Legkova [13] pour l'ajutage de Falkovitch. Egalement, toujours de (3.6) lorsque $\tau_1\to 0$ pour l'ajutage à une seule vitesse critique de C. J a c o b [2] on a le résultat obtenu par Galina C a m e n s k i [14].

B I B L I O G R A P H I E

1. T c h a p l i g u i n e, S. A. — *Oeuvres complètes*, Moscou, 1948, 2, 19—138.
2. J a c o b, C. — *Introduction mathématique à la mécanique des fluides*, Bucarest, Paris, 1959.
3. J a c o b, C. — *Archiwum Mech. Stosow*. 14, 3/4, 603—619 (1962).
4. F a l k o v i c h, S. V. — Prikl. Mat. Mekh. 21, 459—464 (1957).
5. P o p p Simona — *Leçons de dynamique des gaz*. Bucarest 1979 (roum).
6. P o p p Simona — Rev. Roum. Math. pures appl. 9, 3—11 (1966).
7. F r a n k l, F. I. — Izv. Acad. nauk U.R.S.S. 9, 121—143 (1945).
8. B e r s, L. — *Mathematical aspects of subsonic and transonic gas dynamics* (New York, London, 1958).
9. F r a n k l, F. I. — Dokl. Acad. nauk. U.R.S.S. LVIII. 5. 1947.
10. A s l a n o v, S. K. — Prikl. Mat. Mekh. 21, 297—302 (1957).
11. G u c h i n, B. A. — Prikl. Mat. Mekh. 23, 770—776 (1959).
12. L u p u, M. — Studii şi cercetări matematice, 2, 25, 285—308, Bucureşti (1973).
13. A s l a n o v, S. K., L e g k o v a, V. A. — Prikl. Mat. Mekh. 23, 190—193 (1959).
14. C a m e n s c h i Galina — Studii şi cerc. mat. 19, 7, 957—962, Bucureşti (1967).
15. S e d o v, L. I. — *Ploskie zadaci ghidrodinamiki i aerodinamiki*. M—L 1950.
16. C h e r r y, T. M. — Proc. Royal. Soc. 191, 45—79 (1947).

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ON THE BENDING OF MICROPOLAR PLATES RESTED ON A VINKLER FOUNDATION

BY

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To the memory of Professor Mendel Haimovici

In this paper we present an introductory study concerning the linear bending of a thin micropolar plate rested on an elastic foundation of Vinkler type. In the first part we establish the form of the general solution of equilibrium equations. In the second part we consider the small deflections of an infinite plate acted upon by a concentrated force.

The field equations used in this analysis are those from the papers [2, 4].

I. 1. Statement of the problem. Consider a thin elastic micropolar plate of constant thickness $2h$ in equilibrium under a given system of external loads acting on the lateral surfaces, $x_3=\pm h$.

In the absence of extensional deformations paralel to the median plane, the differential equations governing the small deflections of an isotropic thin micropolar plate are :

$$(\mathrm{I}.1.1)\quad \begin{gathered} D\Delta\Delta w-8\alpha h\Delta w-8\alpha h(\varphi_{2,1}-\varphi_{1,2})=p(x_1,x_2),\\ A\Delta\varphi_\rho+B\varphi_{\beta,\beta\rho}+8\alpha h(\varepsilon_{\rho\delta}w_\delta-\varphi_\rho)=-q_\rho(x_1,x_2),\ \rho=1,2.\\ \Delta=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2},\ \varphi_{2,1}=\frac{\partial\varphi_2}{\partial x_1},\ \ldots,\\ D=\frac{2h^3E}{3(1-\nu^2)},\quad D'=\frac{2h\bar{E}}{1-\bar{\nu}^2},\ A=\frac{D'(1-\nu)}{2}+\varepsilon_1,\ B=\frac{D'(1+\nu)}{2}-\varepsilon_1,\end{gathered}$$

where w is the deflection function, φ_ρ are the components of the local rotation vector (microrotation) ; E, ν, α, $\bar{E}$, $\bar{\nu}$, ε_1 — the material coefficients, p and q_ρ — the external normal force and tangential couples respectively acting on the lateral plate surfaces.

When the plate rests on an elatic foundation the superficial loads can be decomposed into two parts, [5, 6] :

$$(\mathrm{I}.1.2)\quad \begin{gathered} p(x_1,x_2)=P(x_1,x_2)+p^*(x_1,x_2),\\ q_\rho(x_1,x_2)=Q(x_1,x_2)+q_\rho^*(x_1,x_2),\end{gathered}$$

in which, P and Q_ρ are given loads at the surface, $x_3=+h$, p^* and q_ρ being the normal force and tangential couples appearing as reactions due to the intime contact between the plate and foundation.

Considering the material of foundation be a classical continuum and the response of foundation be of Vinkler type, we can disregard the reaction couples q_ρ and assume the reaction force of the form :

$$p^*(x_1, x_2) = -k_1 w(x_1, x_2), \tag{I.1.3}$$

where k_1 denotes the foundation modulus.

Moreover we can suppose also that

$$Q_\rho = 0, \quad \rho = 1, 2, \tag{I.1.4}$$

that is, no external couples act on the lateral plate surfaces.

In view of assumptions (I.1.2)—(I.1.4), with the notations

$$\sigma = \frac{8\alpha h}{D}, \quad k = \frac{k_1}{D}, \quad \varkappa = \frac{B}{A}, \quad \delta = \frac{8ah}{A}, \qquad q(x_1, x_2) = \frac{P(x_1, x_2)}{D}, \tag{I.1.5}$$

the equilibrium equations can be written under the form :

$$\begin{aligned} &\Delta\Delta w - \sigma(\Delta w + \varphi_{2,1} - \varphi_{1,2}) + kw = q, \\ &\Delta\varphi_\rho + \varkappa\varphi_{\beta,\beta\rho} + \delta(\varepsilon_{\rho\lambda}\, w_{,\lambda} - \varphi_\rho) = 0, \quad \rho = 1, 2. \end{aligned} \tag{I.1.6}$$

For a given problem, the solution of the system (I.1.6) must meet certain boundary conditions along the plate edge.

It was shown, in [2, 4], that no more as four edge conditions can be associated to the system (I.1.6). They are selected in accordance with the physical conditions imposed upon the edges, among the following two sets :

$$w = \hat{w}, \quad w_{,n} = \hat{u}, \quad M_n = \hat{M}, \quad V_n = \hat{V} \tag{I.1.7}$$

or

$$\varphi_1 = \hat{\varphi}_1, \quad \varphi_2 = \hat{\varphi}_2, R_{1n} = \hat{R}_1, \quad R_{2n} = \hat{R}_2, \tag{I.1.8}$$

where $w_{,n}$ denotes the normal derivative along the plate contour M_n — the bending moment, V_n — the transverse shearing force, and $R_{\rho n}$ ($\rho = 1, 2$) — the components of couple-stress resultant ; $(\hat{...})$ — are given functions.

When the plate extends to infinity, we require also some supplimentary conditions assigning the boundless of the stress and couple-stress state for large values of $r = (x_1^2 + x_2^2)^{1/2}$.

Note that, when the material constants verify some positiveness conditions [4], the solution of the boundary value problem consisting from (I.1.6) and appropiate four edge conditions among (I.1.7) and (I.1.8) is uniquely determinated.

I. 2. The general solution of Eqs. (I.1.6). The general solution of equilibrium equations is assumme to be the form :

$$w = w^0 + w^*, \quad \varphi_\rho = \varphi_\rho^0 + \varphi_\rho^*, \quad \rho = 1, 2. \tag{I.2.1}$$

where $(w^*, \varphi_1^*, \varphi_2^*)$ is a particular solution, $(w^0, \varphi_1^0, \varphi_2^0)$ being the general solution of (I.1.6) with $q = 0$. Let us denote this system by $(\text{I.1.6})_0$. Assuming w_ρ^* and φ_ν^* be known, it remains to find functions w_ρ^0 and φ_ρ^0. In this order, we seek the solution of $(\text{I.1.6})_0$ in the form :

$$w^0 = W, \quad \varphi_1^0 = \Phi_{,1} - \Psi_{,2}, \quad \varphi_2^0 = \Phi_{,2} + \Psi_{,1} \tag{I.2.2}$$

where W, Φ and Ψ are new unknown functions sufficiently smooth.

Eqs. $(\text{I.1.6})_0$ are satisfied when the functions (I.2.2) verify :

$$(1+\varkappa)\Delta\Phi - \delta\Phi = 0, \quad \Delta\Psi - \sigma(W + \Psi) = 0, \quad \Delta\Delta W - \sigma(W + \Psi) + kW = 0. \tag{I.2.3}$$

The first equation gives, independentely on the last two, the function Φ.

From the second equation, we can express function Ψ in terms of W when this is known :

$$\Delta\Psi - \delta\Psi = \delta W. \tag{I.2.4}$$

Eliminating Ψ among $(\text{I.2.3})_2$ and $(\text{I.2.3})_3$ we deduce that W must be solution of the equation :

$$\Delta\Delta\Delta W - (\delta + \sigma)\Delta\Delta W + k\Delta W - k\delta W = 0, \tag{I.2.5}$$

which can be written in an alternative form as

$$(\Delta - \lambda_1)(\Delta - \lambda_2)(\Delta - \lambda_3) W = 0, \tag{I.2.6}$$

if $\lambda_1, \lambda_2, \lambda_3$ are the roots of the equation :

$$\lambda^3 - (\delta + \sigma)\lambda^2 + k\lambda - k\delta = 0. \tag{I.2.7}$$

To approximate the roots of the algebraic Eq. (I.2.7), we assume the ratio

$$\eta_1 = \frac{\sigma}{\delta},$$

be small compared to unity so that the terms containing η_1^2 and η_1^3 can be neglected.

This assumption is not so restrictive since in virtue of (I.1.5) and (I.1.1), one has :

$$\eta_1 = \frac{\sigma}{\delta} = \frac{8\alpha h}{D} \cdot \frac{2h(\gamma_1 + \varepsilon_1)}{8\alpha h} = \frac{3(\gamma_1 + \varepsilon_1)(1 - \nu^2)}{h^2 E} = \left(\frac{d}{h}\right)^2,$$

where the constant d^2, is given by

$$d^2 = \frac{2h(\gamma_1 + \varepsilon_1)}{3D},$$

and defines the micropolar characteristic length for bending, which is associated with the microstructure scale [4].

Solving Eq. (I.2.7) in the linear aproximation for η_1 we find :

$$\lambda_1 = \delta\left(1 + \frac{\delta\eta_1}{k+\delta^2}\right),$$

(I.2.8)

$$\lambda_2 = \frac{k\eta}{2(k+\delta^2)} - i\sqrt{k}\left[1 - \frac{\delta\eta_1}{2(k+\delta^2)}\right], \quad \lambda_3 = \bar{\lambda}_2,$$

the bar denoting the complex conjugate number.

It is not dificult to show that

(I.2.9) $$\mathrm{Im}\{\lambda_2\} < 0, \text{ for } 0 < \eta < 4\ k,$$

where λ_1 and $\mathrm{Re}\{\lambda_2\}$ are always possitive.

Putting :

(I.2.10) $$\varkappa_1 = \sqrt{\lambda_1},\ \varkappa_2 = \sqrt{-\lambda_2},\ (\bar{\varkappa}_2 = \sqrt{-\bar{\lambda}_2}),$$

and assuming the function W of the form

(I.2.11) $$W = W_1 + W_2 + W_3,$$

the equation (I.2.6) is satisfied when $W_i =$, $i = 1, 2, 3$, verify equations :

(I.2.12) $$(\Delta - \varkappa_1^2)W_1 = 0,\ (\Delta - \varkappa_2^2)W_2 = 0,\ (\Delta - \bar{\varkappa}_2^2)W_3 = 0.$$

We can also rewrite Eq. (I.2.3), which determine the function Φ, under the form :

(I.2.13) $$(\Delta - \varkappa_0^2)\Phi = 0, \qquad \varkappa_0^2 = \frac{\delta}{1+\varkappa} > 0.$$

Thus, the form of the functions expressing the solution of homogeneous equilibrium equations is entirely determined by the solutions of Eqs. (I.2.12) and (I.2.13).

Particularly, when functions W, Φ and Ψ depend only of radial coordinate r, the general solution of equations (I.2.12) and (I.2.13) can be expressed by means of Bessel functions thus, [6, 7] :

$$\Phi(\xi) = A_0 I_0(\varkappa_0\xi) + B_0 K_0(\varkappa_0\xi),$$

$$W_1(\xi) = A_1 I_0(\varkappa_1\xi) + B_1 K_0(\varkappa_1\xi),$$

(I.2.14) $$W_2(\xi) = C_2 J_0(\varkappa_2\xi) + D_2 H_0^{(1)}(\varkappa_2\xi),$$

$$W_3(\xi) = C_3 J_0(\bar{\varkappa}_2\xi) + D_3 H_0^{(2)}(\bar{\varkappa}_2\xi), \qquad \xi = \frac{x}{R}$$

where R is a standard length, $J_n(\ldots)$ — the Bessel function of the first kind : $I_n(\ldots)$, $K_n(\ldots)$ — the modified Bessel functions, $H_n^{(1,2)}(\ldots)$ the Bessel functions of third kind (Hankel functions), of order n and, $A_0 \ldots B_3$ — some arbitrary constants.

Note that the complex valued functions.

$$J_0(\varkappa_2\xi),\ H_0^{(1)}(\varkappa_2\xi),\ J_0(\bar{\varkappa}_2\xi),\ H_0^{(2)}(\bar{\varkappa}_2\xi),$$

are defined, as results form (I.2.8) and (I.2.10), in the domains :

$$\frac{\pi}{4} \leqslant \arg \varkappa_2 < \frac{\pi}{2}, \qquad -\frac{\pi}{2} < \arg \bar{\varkappa}_2 \leqslant -\frac{\pi}{4}.$$

At this point, we put down some notations which will be used in the following :

$$\varphi = \arg \varkappa_2, \quad (\arg \varkappa_2 = \frac{1}{2}(\pi - \arg \lambda_2)),$$

(I.2.15) $$\varkappa_2 = \eta + i\tau = \sqrt{s}(\cos\varphi + i\sin\varphi),$$

$$\varkappa_2^2 = \beta + i\gamma = s(\cos 2\varphi + i\sin 2\varphi),\ s = |\lambda_2|.$$

In order that W be a real function, we choose the constants C_2, D_2, C_3, D_3 appearing in the expressions of W_2 and W_3 under the form :

(I.2.16) $$C_2 + C_3 = -A_2, \qquad C_2 - C_3 = iA_3,$$

$$D_2 + D_3 = -B_2, \qquad D_2 - D_3 = iB_3,\ i = \sqrt{-1},$$

where A_2, A_3, B_2, B_3 are new arbitrary real constants.

When (I.2.16) are substituted in (I.2.14) and the result in (I.2.11), we find the final form for the deflection function W, namely :

(I.2.17) $$W(\xi) = A_1 I_0(\varkappa_1\xi) + B_1 K_0(\varkappa_1\xi) + A_2 U_0(\xi) + B_2 F_0(\xi) + A_3 V_0(\xi) + B_3 G_0(\xi),$$

in which the real functions U_0, V_0, F_0, G_0 are defined as :

$$2U_0(\xi) = J_0(\varkappa_2\xi) + J_0(\bar{\varkappa}_2\xi),$$

(I.2.18) $$2iV_0(\xi) = J_0(\varkappa_2\xi) - J_0(\bar{\varkappa}_2\xi)$$

$$2F_0(\xi) = H_0^{(1)}(\varkappa_2\xi) + H_0^{(2)}(\bar{\varkappa}_2\xi),$$

$$2iG_0(\xi) = H_0^{(1)}(\varkappa_2\xi) - H_0^{(2)}(\bar{\varkappa}_2\xi).$$

Note that, functions $U_0 \ldots G_0$, have, among others, the following properties :

(I.2.18)' $$\Delta U_0 = -(\beta U_0 - \gamma V_0), \qquad \Delta F_0 = -(\beta F_0 - \gamma G_0),$$

$$\Delta V_0 = -(\gamma U_0 + \beta V_0), \qquad \Delta G_0 = (\gamma F_0 + \beta G_0).$$

Without giving the explicit calculations we note only, that the equalities (I.2.18)' are obtained on the basis of definitions (I.2.18), using the fact that J_0, ..., $H_0^{(2)}$ are solutions of Eqs. (I.2.12).

Setting (I.2.17) into (I.2.4) and seeking the solution of this last equation under the form (I.2.18) with some new coefficients A_1', ..., B_3', i.e.

(I.2.19) $$\Psi(\xi) = A_1' I_0(\varkappa_1\xi) + B_1' K_0(\varkappa_1\xi) + A_2' U_0(\xi) + B_2' F_0(\xi) + A_3' V_0(\xi) + B_3' G_0(\xi),$$

after some calculations we obtain the following relations between the unknowns :

$$A_1' = \frac{\delta A_1}{\varkappa_1^2-\delta}, \qquad B_1' = \frac{\delta B_1}{\varkappa_1^2-\delta},$$

(I.2.20) $$A_2' = -\delta[(\beta+\delta)A_2-\gamma A_3]\Delta_1^{-1}, \qquad A_3' = -\delta[\gamma A_2+(\beta+\delta)\Delta_3]\Delta_1^{-1},$$

$$B_2' = -\delta[(\beta+\delta)B_2-\gamma B_3]\Delta_1^{-1}, \qquad B_3' = -\delta[\gamma B_2+(\beta+\delta)B_3]\Delta_1^{-1},$$

$$\Delta_1 = (\beta+\delta)^2+\gamma^2, \qquad \beta = s\cos 2\varphi, \qquad \gamma = s\sin 2\varphi.$$

Now, having both functions $\Psi(\xi)$ and $\Phi(\xi)$, the microrotation vector components are obtained from (I.2.2).

When the studied problem has an axisymmetrical character, functions Φ, Ψ (and W) due not depend on the polar co-ordinate θ and, (I.2.2) becomes :

(I.2.21) $$\varphi_r = \frac{\partial\Phi}{\partial r} = \frac{1}{R}\frac{\partial\Phi}{\partial\xi}, \qquad \varphi_\theta = \frac{\partial\Psi}{\partial r} = \frac{1}{R}\frac{\partial\Psi}{\partial\xi},$$

where φ_r and φ_θ are the components of microrotation vector about polar reference system.

II. Infinite plate acted by a concentrated force. We suposse a concentrated normal force of intensity Q acting upon an infinite plate in the origin of the co-ordinate axes. The problem being axisymmetrical we assume the displacement microrotation field, be of the form

(II.1.1.) $$W = W(\xi), \quad \varphi_r \equiv 0, \quad \varphi_\theta = \varphi_\theta(\xi).$$

In view of (I.2.21), we have in this case

(II.1.2) $$\frac{\partial\Phi}{\partial\xi} \equiv 0, \qquad \varphi_\theta = \frac{1}{R}\frac{\partial\psi}{\partial\xi},$$

where Φ and Ψ are given by (I.2.17) and (I.2.19), respectively.

Since $\varphi_r = 0$ is possible only if Φ is a constant, the first conditions is satisfied when :

(II.1.3) $$A_0 = B_0 = 0.$$

In addition, the physical nature of the problem requires that $W(\xi)$ and $\Psi(\xi)$ be choosen as :

— $W(\xi)$, be bounded in the origin, that is :

(II.1.4) $$W(\xi) \to W_0, \qquad \text{when } \xi \to 0.$$

— W, $\dfrac{dW}{\partial\xi}$ and φ_θ vanishe for large values of ξ, that means :

(II.1.5) $$W(\xi) \to 0, \quad \frac{dW}{d\xi} \to 0, \quad \varphi_\theta(\xi) \to 0 \text{ when } \xi \to \infty,$$

Since the general solution (I.2.17) and (I.2.19) contains both functions which become unbounded for small and large values of their arguments, as follows [7] :

— for $\xi \to 0$: $[I_0(\varkappa_1\xi), U_0(\xi), V_0(\xi)] = O(1)$, $F_0(\xi) = o(\xi^2 \ln \xi)$,

$[k_0(\varkappa_1\xi), G_0(\xi)] = O(\ln \xi)$,

— for $\xi \to \infty$ $I_0(\varkappa_1\xi) = O(\xi^{-\frac{1}{2}}e^{\xi})$, $\quad [U_0(\xi), V_0(\xi)] = O(1)$,

$K_0(\varkappa_1\xi) = o(\xi^{-\frac{1}{2}})$ $\quad [F_0(\xi), G_0(\xi)] = O(\xi^{-\frac{1}{2}})$,

we can satisfy conditions (II. 1. 4) and (II. 1. 5) only if

(II.1.4)′ $$B_1 = B_3 = 0.$$

(II.1.5)′ $$A_1 = A_2 = 0.$$

Consequently, the solution of our problem resulting from $(I.2.14)_1$, (I.2.17) and (I.2.19) in view of (II.1.3.), (II.1.4)′ (II.1.5)′ are :

(II.1.6) $$\Phi(\xi) = 0, \quad W(\xi) = B_2F_0(\xi),$$
$$\Psi(\xi) = -\frac{\delta(\beta+\delta)}{\Delta_1} B_2F_0(\xi) - \frac{\delta\gamma}{\Delta_1} B_2G_0(\xi).$$

By substituting $\Psi(\xi)$ into (I.2.2) and using the derivation formulae of Hankel functions :

(II.1.6)′ $$\frac{d}{du}[H_0^{(1,2)}(u)] = -H_1^{(1,2)}(u),$$

after some operations, one finds :

(II.1.7.) $$\varphi_\theta(\xi) - = \frac{B_2}{R\Delta_1}[aF_1(\xi)+bG_1(\xi)],$$

where

$$a = \delta(\gamma\tau-(\beta+\delta)\eta], \quad b = \delta[\gamma\eta+(\beta+\delta)\tau].$$

Here, $F_1(\xi)$ and $G_1(\xi)$ are expressed in terms of $J_1(\varkappa_2\xi)$, $J_1(\bar{\varkappa}_2\xi)$, $H_1^{(1)}(\varkappa_2\xi)$, $H_1^{(2)}(\bar{\varkappa}_2\xi)$ by formulae analogous with (I.2.18).

It remains to determine the constant B_2. To this end, we consider a circular plate element of radius $\varepsilon > 0$ $(\varepsilon \to 0)$ having the centre in the origin and express the equilibrium condition between the force Q and the resultant of generalized transverse shearing force V_n along the circumferince C_ε. This means that :

(II.1.8) $$\int_{C_\varepsilon} V_n d + Q = 0,$$

where, according to [2, 4], V_n is given by :

$$V_n = -D\Delta w_{,n} + 8\alpha h(w_{,n} + \varphi_2 n_1 - \varphi_1 n_2) + D(1-\nu)[w_{,12}(n_2^2 - n_1^2) + (w_{,11} - w_{,22})n_1 n_2]_{,s}. \tag{II.1.9}$$

Here $(\ldots)_{,n}$ and $(\ldots)_{,s}$ denote the normal and tangential derivatives along a regular are of normal $n(n_1, n_2)$.

In our case, the arc is the circumference C_ε itself, where the direction of the outward normal coincides with the direction of the radius vector. Moreover, when the expression inside the brackets is an uniform functions, the integral from (II.1.9) extends to the first three terms, only, becoming:

$$\int_{C_\varepsilon} V_n ds = \int_{C_\varepsilon} [-D\Delta w_{,n} + 8\alpha h(w_{,n} + \varphi_2 n_1 - \varphi_1 n_2)] ds. \tag{II.1.10}$$

Since at the cercumference C_ε, one has:

$$\Delta w_{,n} = \frac{1}{R^3}\frac{d}{d\xi}\Delta W,\quad w_{,n} = \frac{1}{R}\frac{dW}{d\xi},\quad \varphi_2 n_1 - \varphi_1 n_2 = \varphi_0\left(= -\frac{1}{R}\frac{\partial \Psi}{\partial \xi}\right),$$

we can write the integrand appearing in (II.1.10) as

$$I = -\frac{D}{R^3}\frac{d}{d\xi}[\Delta W - \sigma R^2(W + \Psi)]. \tag{II.1.11}$$

Further, substituting the term $W+\Psi$ by $(1/\delta)\Delta\Psi$ according to $(\mathrm{I}.2.3)_2$, we get:

$$I = -\frac{D}{R^3}\frac{d}{d\xi}\left[\Delta W - \frac{\sigma R^2}{\delta}\Delta\Psi\right]. \tag{II.1.12}$$

Using the expression of $\Psi(\xi)$ given by $(\mathrm{II}.1.6)_3$, formulae (I.2.20), (I.2.18) and the relations

$$\frac{dF_0}{d\xi} = -(\eta F_1 - \tau G_1),\quad \frac{dG_0}{d\xi} = -(\tau F_1 + \eta G_1),$$

$$(\eta = \sqrt{s}\cos\varphi,\quad \tau = \sqrt{s}\sin\varphi).$$

which are obtained, directly, on the basis of (II.1.6), after some manipulations, we find:

$$\frac{d\Delta W}{d\xi} = B_2 s\sqrt{s}\Lambda(\xi),$$
$$\frac{d\Delta\psi}{d\xi} = -\frac{\delta^2}{\Delta_1}s\sqrt{s}B_2\Lambda(\xi) - \frac{\delta}{\Delta_1}s^2\sqrt{s}\chi(\xi)B_2. \tag{II.1.13}$$

Functions $\Lambda(\xi)$ and $\chi(\xi)$ appearing in (II.1.13) are defined as:

$$\Lambda(\xi) = \cos 3\varphi F_1(\xi) - \sin 3\varphi G_1(\xi),$$
$$\chi(\xi) = \cos\varphi F_1(\xi) - \sin\varphi G_1(\xi). \tag{II.2.14}$$

They are power series in ξ appart from some singular terms which in the neighborhoof) of $\xi = 0$, have the form:

$$\Lambda(\xi) = -\frac{2}{\pi\xi\sqrt{s}}\sin 2\varphi + o(\xi\ln\xi),$$
$$\chi(\xi) = o(\xi\ln\xi). \tag{II.1.15}$$

Inserting (II.1.13) into (II.1.12) and performing the integration in (II.1.10) with $ds = R\varepsilon d\theta$, in view of (II.1.15), one gets:

$$B_2\left[\frac{4Ds}{R^2}\left(1 + \frac{\sigma R^2\delta}{\Delta_1}\right)\sin 2\varphi + \varepsilon(\ldots)\right] + Q = 0.$$

Passing to the limit for $\varepsilon \to 0$, we obtain the value of B_2:

$$B_2 = -\frac{QR^2\Delta_1}{4sD(\Delta_1 + \sigma R^2\delta)\sin 2\varphi}.$$

With this B_2, the deflection function W and microrotation φ_θ are determinated under the form:

$$W(\xi) = -\frac{QR^2\Delta_1}{4sD(\Delta_1 + \sigma R^2\delta)\sin 2\varphi}F_0(\xi),$$
$$\varphi_\theta(\xi) = -\frac{QR\Delta_1}{4\sqrt{s}D(\Delta_1 + \sigma R^2\delta)\sin 2\varphi}[s\Lambda(\xi) + \delta\chi(\xi)]. \tag{II.1.16}$$

The appropriate expressions of stress-couples, couple-stress resultants and transverse shearing force are obtained from the stress-strain relations given in [2, 4].

Discussion. In this paper we initiate a preliminary analysis concerning the deformation of an elastic thin micropolar plate in the interaction with an elastic foundation of Vinkler type.

Although the Vinkler foundations is the simplest model of foundation [3, 5], the analysis of the solution obtained in the case of an infinite plate acted upon by a concentrated force, sugests a conclusion. It appears from (II.1.16) that the structure of the solution characterising the bending of a micropolar plate rested on a Vinkler foundation is the same as that for a classical plate (with a certain rigidity) rested on an „improved" type of foundation, known in the litterature as the generalised or Pasternak foundation. Therefore, we can conclude that in the analysis of the response of a foundation, the nature of the structure rested on the foundation must also be taken into account. Obviously, this is only a theoretical conclusion.

REFERENCES

1. Gauthier R.D. — *Experimental Investigations on Micropolar Media*, CISM Courses and Lecture (1979).
2. Ideloviçi A. — *On a Micropolar Theory for Bending of Plates*. An. şt. Univ. Iaşi, XIX, 1, 1973.
3. Kerr A.D. — *Elastic and Viscoelastic Foundation Models*. *J.* Appl. Mech., 1964.

4. Manolachi A. — *On the Bending of Plates in Asymmetric Elasticity*. An. şt. Univ. Iaşi, XIX, 1, 1973.
5. Timoshenko S., Woinowsky-Krieger S. — *Theory of Plates and Shells*. Mc. Grow-Hill. New York, 1959.
6. Vlasov B.Z. and Leontieff N.N. — *Beams, Plates and Shells on Elastic Foundation* (in Russian). Moskow. 1960.
7. Watson G.N. — *Theory of Bessel Functions* (in Russian) Translation from english, Moscow 1949.

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SUR UN PROBLÈME D'ÉLASTICITÉ PLANE EN POSITION DÉFORMÉE

PAR

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Dédié à la mémoire du Professeur Mendel Haimovici

Dans cet ouvrage on considère un calcul qui se propose, d'une côté répondre aux critiques liées aux équations de l'élasticité en position initiale (non déformée) — et de l'autre côté, éviter les difficultés résultées des équations eu positions déformée.

On propose un calcul en position presque déformée — un calcul du second ordre, qui met en évidence une liaison avecl es tensions qui proviennent d'un calcul en position non déformée-calcul du premier ordre.

On considérera une position provisoirement voisine eventuellement celle déformée, qui sera écrite en fonction des coordonnées initiales (x, y).

Soit en ce sens un rectangle défini par les dimensions (dx, dy) et une position quelconque voisine possible (fig. 1).

Les éléments qui caractérisent la position voisine seront notés par un astérisque. En négligeant pourtant quelques termes on a

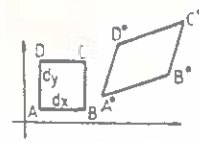

Fig. 1

$$\overline{A^*B^*} = \overline{D^*C^*}\left(\left(1 + \frac{\partial u^*}{\partial x}\right)dx, \quad \frac{\partial v^*}{\partial x}dx\right) \tag{1}$$

$$\overline{A^*D^*} = \overline{B^*C^*}\left(\frac{\partial u^*}{\partial y}dy, \quad \left(1 + \frac{\partial v^*}{\partial y}\right)dy\right).$$

Soit p^*_{xx}, p^*_{xy}, p^*_{yy} les efforts sur des surfaces du parallélogramme $A^*B^*C^*D^*$. On considère aussi une hypothèse sur la nature de la conservation de l'intensité de la force massique autant en intensité qu'en direction. En vertu de cette hypothése on a

$$XdA = X^*dA^*, \quad YdA = Y^*dA^*$$

Une première forme des équations d'équilibre en position voisine est

$$\frac{\partial p^*_{xx}}{dx}|\overline{B^*C^*}| + \frac{\partial p^*_{xy}}{\partial y}dy|\overline{A^*B^*}| + Xdxdy = 0$$

$$\frac{\partial p^*_{yx}}{\partial x}dx|\overline{B^*C^*}| + \frac{\partial p^*_{yy}}{\partial y}dy|\overline{A^*B^*}| + Ydxdy = 0 \tag{2}$$

On tient compte que les formules (1) conduisent à

$$|\overline{A^*B^*}| = \sqrt{\left(1 + \frac{\partial u^*}{\partial x}\right)^2 + \left(\frac{\partial v^*}{\partial x}\right)^2}dx \cong \left(1 + \frac{\partial u^*}{\partial x}\right)dx, \quad |\overline{B^*C^*}| = \left(1 + \frac{\partial v^*}{\partial y}\right)dy.$$