We remark that $l=m$ is necessary for the symmetry of G. For $k=n=1$ and $l=m=\alpha$ one obtains the α-lift of g considered by M. Matsumoto ([5], p. 105). Generally, G with $l=m$ is quasi-Riemannian. The α-lift is Riemannian if $-1<\alpha<1$. By a direct computation it follows:

Theorem 6.1. *a) A CS-prolongation of g with $k=n$ and $l=m=\alpha$ is almost hermitian with respect to F if and only if $\alpha=0$. b) A CS-prolongation of g with $l=m=\alpha$ satisfies $G(PX, PY)=G(X, Y)$ for every $X, Y\in S(TTM)$ if and only if $\alpha=0$. c) A CS-prolongation of g with $l=m$ and $k=m$ satisfies $G(\widetilde{Q}X, \widetilde{Q}Y)=G(X, Y)$ for every $X, Y\in S(TTM)$.*

A pair (G, P) on TM is called almost Hermitian hiperbolic structure if $G(PX, PY)=-G(X, Y)$ holds for every $X, Y\in S(TTM)$. By taking G as CS-prolongation of g, one obtains, after a simple computation

Theorem 6.2. *The pair (G, P), where G is the CS-prolongation of g with $l=m=\alpha$ is an almost Hermitian hiperbolic structure if and only if $k=n=0$ i.e. G is as follows*

$$G(A_1+QB_1, A_2+QB_2)=\alpha(g(A_1, B_2)+g(A_2, B_1)). \tag{6.2}$$

Now we take G given by (6.2) and we obtain

Theorem 6.3. *a) The almost product structures given by (5.5) and (5.6) pairing with G give almost Hermitian hyperbolic structures if and only if $c=0$. b) The almost product structures given by (5.3) and (5.4) pairing with G give almost Hermitian hyperbolic structures.*

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ON SEMI-INVARIANT SUBMANIFOLDS OF AN ALMOST CONTACT METRIC MANIFOLD

BY

AUREL BEJANCU

To the memory of Professor Mendel Haimovici

1. Introduction. Let N be a $(2n+1)$-dimensional almost contact metric manifold with structure tensors (φ, ξ, η, g), where φ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g is the Riemannian metric on N. These tensor fields are related by

$$\varphi^2X=-X+\eta(X)\xi\ ;\ \varphi\xi=0\ ;\quad \eta(\xi)=1\ ;\quad \eta(\varphi X)=0 \text{ and} \tag{1.1}$$

$$g(\varphi X, \varphi Y)=g(X, Y)-\eta(X)\,.\,\eta(Y)\ ;\ \eta(X)=g(X, \xi), \tag{1.2}$$

for any vector fields X, Y tangent to N.

Let M be an m-dimensional submanifold of N. We assume that the structure vector field ξ is tangent to M. Then M is called a *semi-invariant submanifold* of N if there exist on M two differentiable orthogonal distributions D and $D^\perp$ such that the following conditions are satisfied:
(i) $TM=D\oplus D^\perp\oplus\{\xi\}$, where $\{\xi\}$ is the 1-dimensional distribution spanned by ξ on M,
(ii) the distribution D is invariant by φ, that is, $\varphi(D_x)=D_x$ for each $x\in M$ and
(iii) the distribution $D^\perp$ is anti-invariant by φ, that is, $\varphi(D^\perp_x)\subset T_xM^\perp$, for each $x\in M$, where $T_xM^\perp$ is the normal space to M at x, (see [1]).

We denote by $2p$ (resp. q) the real dimension of a fiber of D (resp. $D^\perp$) and by P (resp. Q) the projection morphisms of TM to D (resp. $D^\perp$). Then we see that for $p=0$ (resp. $q=0$) we obtain an anti-invariant submanifold tangent to ξ (see [3]) (resp. an invariant submanifold (see [4])) of N. Moreover, when $q=\dim T_xM^\perp$ we obtain generic submanifolds studied by K. Yano and M. Kon in [2]. It is easy to check that each hypersurface of N which is tangent to ξ inherites a structure of a semi-invariant submanifold of N.

For any vector bundle H on M we denote by $\Gamma(H)$ the module of all differentiable sections of H on a neighborhood coordinates on M.

Now, suppose D' be another invariant distribution on M. Then we have $D'_x\subset D_x$ for any $x\in M$. In fact, if $X\in\Gamma(D')$ then there exists $Y\in\Gamma(D')$ such that $X=\varphi Y$ and we have

$$g(\varphi Y_x, Z_x)=-g(Y_x, \varphi Z_x)=0 \text{ for any } Z\in\Gamma(D^\perp) \text{ and}$$

$$g(\varphi Y_x, \xi_x)=-g(Y_x, \varphi\xi_x)=0.$$

In a similar way follows $D''_x \subset D^\perp_x$ for any anti-invariant distribution D'' on M. Thus we have

Proposition 1.1. [1]. *Let M be a semi-invariant submanifold of an almost contact metric manifold N. Then the distribution D is the maximal invariant distribution on M and $D^\perp$ is the maximal anti-invariant distribution on M.*

In the present paper we obtain in §2 necessary and sufficient conditions for a submanifold of N in order to be a semi-invariant submanifold. Also, in §3, we give integrability conditions for distributions $D\oplus\{\xi\}$ and $D^\perp\oplus\{\xi\}$ by means of some Nijenhuis tensors.

2. Some characterizations of semi-invariant submanifolds. Suppose M be a submanifold of N tangent to the structure vector field ξ. Then we put

$$\varphi X = \psi X + \omega X, \tag{2.1}$$

for any X tangent to M, where ψX is the tangent part of φX and ωX is the normal part of φX. Also, for any V normal to M we put

$$\varphi V = BV + CV \tag{2.2}$$

where BV is the tangent part of φV and CV is the normal part of φV. Now, we can state

Theorem 2.1. *Let M be a submanifold of an almost contact metric manifold N tangent to the structure vector field ξ. Then M is a semi-invariant submanifold if and only if we have*

$$\textit{rank. } \psi = \textit{constant on } M \textit{ and} \tag{2.3}$$

$$\omega \circ \psi = 0. \tag{2.4}$$

Proof. Suppose M is a semi-invariant submanifold of N. Then for any $X\in\Gamma(TM)$ we have

$$X = PX + QX + \eta(X)\xi. \tag{2.5}$$

Thus applying φ to (2.5) we obtain

$$\psi X = \varphi PX \tag{2.6}$$ and $$\omega X = \varphi QX. \tag{2.7}$$

From (2.6) follows $\psi(TM) = \varphi(D) = D$. Hence rank. $\psi = 2p$ on M. Also, by using (2.6) and (2.7) we obtain (2.4).

Conversely, suppose (2.3) and (2.4) be satisfied. We define the distribution D by $D_x = \mathrm{Im}.\ \psi_x$ for each $x\in M$. D is a differentiable distribution since it is the image of a differentiable morphism of vector bundles, which is of constant rank. Moreover, D is an invariant distribution since for each $X = \psi Y \in \Gamma(D)$ we have

$$\varphi X = \varphi\psi Y = \psi^2 Y + (\omega\circ\psi)Y = \psi^2 Y \in \Gamma(\mathrm{Im}.\ \psi) = \Gamma(D).$$

Also, D is orthogonal to $\{\xi\}$. In fact, we have

$$g(\psi Y, \xi) = g(\varphi Y, \xi) = -g(Y, \varphi\xi) = 0$$

Denote by $D^\perp$ the orthogonal complementary distribution to $D\oplus\{\xi\}$ in TM. Then $D^\perp$ is an anti-invariant distribution. In fact, for any $X\in\Gamma(D^\perp)$ and $Y = U + W + \eta(Y)\xi$ where $U\in\Gamma(D)$ and $W\in\Gamma(D^\perp)$ we have

$$g(\varphi X, Y) = -g(X, \varphi U + \varphi W) = -g(X, \psi W) = 0,$$

since $\psi W\in\Gamma(D)$. Thus M is a semi-invariant submanifold of N.

Theorem 2.2. *Let M be a submanifold of an almost contact metric manifold N tangent to the structure vector field ξ. Then M is a semi-invariant submanifold of N if and only if we have*

$$\textit{rank. } B = \textit{constant on } M \textit{ and} \tag{2.8}$$

$$\psi\circ B = 0. \tag{2.9}$$

Proof. Suppose N is a semi-invariant submanifold. First, we see that $\mathrm{Im}.\ B_x \subset D^\perp_x$ *for each* $x\in M$. In fact, by means of (1.1) and (1.2) we have

$$g(BV, Y) = g(\varphi V, Y) = -g(V, \varphi Y) = 0 \text{ and}$$

$$g(BV, \xi) = g(\varphi V, \xi) = -g(V, \varphi\xi) = 0$$

for any $V\in\Gamma(TM^\perp)$ and $Y\in\Gamma(D)$.

On the other hand, we have $D^\perp\subset \mathrm{Im}.\ B_x$ for each $x\in M$. Indeed, if we take $U\in\Gamma(D^\perp)$ then $\varphi U\in\Gamma(TM^\perp)$ and by using (1.1) and (2.2) we obtain

$$-U = B\varphi U + C\varphi U.$$

Hence $U = -B\varphi U \in \Gamma(\mathrm{Im}.\ B)$. Thus we have $\mathrm{Im}.\ B = D^\perp$ which implies rank. B = constant on M.

Next, for each $V\in\Gamma(TM^\perp)$ we have $\psi BV = (\varphi\circ B)V + (\omega\circ B)V$ Hence, $(\psi\circ B)V = 0$ since $(\varphi\circ B)V$ and $(\omega\circ B)V$ are normal to M.

Conversely, suppose conditions (2.8) and (2.9) be satisfied. Then we define the distribution $D^\perp$ by $D^\perp_x = \mathrm{Im}.\ B_x$ for each $x\in M$. We note that $D^\perp$ is a differentiable distribution which is anti-invariant by φ. In fact, $D^\perp$ is a differentiable distribution since it is an image of a differentiable morphism of vector bundles. For each $X\in\Gamma(D^\perp)$ and $Y\in(TM)$ by using (2.9) we have

$$g(\varphi X, Y) = g(\varphi BV, Y) = g(\psi BV, Y) = 0.$$

Thus $D^\perp$ is anti-invariant by φ. Also, we have

$$g(X, \xi) = g(BV, \xi) = g(\varphi V, \xi) = -g(V, \varphi\xi) = 0$$

for each $X\in\Gamma(D^\perp)$. Hence $D^\perp$ is orthogonal to $\{\xi\}$.

We denote by D the complementary orthogonal distribution to $D^\perp\oplus\{\xi\}$ in TM. Then we see that D is an invariant distribution on M. Indeed, for any $X\in\Gamma(D)$, $Y\in\Gamma(D^\perp)$, $V\in\Gamma(TM^\perp)$ we have

$$g(\varphi X, Y) = -g(X, \varphi Y) = 0,$$

$$g(\varphi X, \xi) = -g(X, \varphi\xi) = 0 \text{ and}$$
$$g(\varphi X, V) = -g(X, \varphi V) = -g(X, BV) = 0.$$

Thus M is a semi-invariant submanifold, and the proof is complete.

3. Integrability of distributions $D \oplus \{\xi\}$ and $D^{\perp} \oplus \{\xi\}$. Let M be a semi-invariant submanifold of an almost contact metric manifold N. Then, by (2.6) we have a tensor field ψ of type (1,1) on M. The purpose of the present paragraph is to obtain integrability conditions for $D \oplus \{\xi\}$ and $D^{\perp} \oplus \{\xi\}$ by means of Nijenhuis tensors of φ and ψ.

First, we recall that the Nijenhuis tensor $[\varphi, \varphi]$ of φ is given by

(3.1.) $$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi\{[X, \varphi Y] + [\varphi X, Y]\}$$

for any X, Y tangent to M. Also, we note that ψ satisfies

(3.2) $$\psi^2 = -P$$

Now, we take $X, Y \in \Gamma(D)$ and by a direct computation, using (3.1) and (3.2) obtain

(3.3) $$[\varphi, \varphi](X, Y) = [\psi, \psi](X, Y) - Q([X, Y]) - \omega\{[X, \psi Y] + [\psi X, Y]\}$$

and

(3.4) $$[\varphi, \varphi](X, \xi) = [\psi, \psi](X, \xi) - Q([X, \xi]) - \omega([\psi X, \xi]).$$

For any vector field U on N we denote by $U^{\top}$ (resp. $U^{\perp}$) its tangent part (resp. normal part.) Then, from (3.3) and (3.4) we have

Theorem 3.1. *Let M be a semi-invariant submanifold of an almost contact metric manifold N. Then the distribution $D \oplus \{\xi\}$ is involutive if and only if we have*

(3.5) $$[\varphi, \varphi](X, Y)^{\top} = [\psi, \psi](X, Y),$$

for any $X, Y \in \Gamma(D \oplus \{\xi\})$.

Taking the normal parts in (3.3) and (3.4) we obtain

(3.6) $$[\varphi, \varphi](X, Y)^{\perp} = -\omega([X, \psi Y] + [\Psi X, Y]) \text{ and}$$

(3.7) $$[\varphi, \varphi](X, \xi)^{\perp} = -\omega([\psi X, \xi]).$$

Theorem 3.2. *Let M be a semi-invariant submanifold of an almost contact metric manifold N. Then the distribution $D \oplus \{\xi\}$ is involutive if and only if we have*

(3.8) $[\varphi, \varphi](X, Y)^{\perp} = 0$ *and* (3.9) $Q([\psi, \psi](X, Y)) = 0,$

for any $X, Y \in \Gamma(D \oplus \{\xi\})$.

Proof. Suppose $D \oplus \{\xi\}$ is involutive. Then (3.8) follows from (3.6) and (3.7). Also, taking account of (3.2) we have

$$[\psi, \psi](X, Y) = -P([X, Y]) + [\psi X, \psi Y] - \psi\{[X, \psi Y] + [\psi X, Y]\}$$

for each $X, Y \in \Gamma(D \oplus \{\xi\})$. Hence $[\psi, \psi](X, Y) \in \Gamma(D \oplus \{\xi\})$ and the condition (3.9) is accomplished.

Conversely, suppose (3.8) and (3.9) be satisfied. Then from (3.6), (3.7) and (3.8), follows

(3.10) $$[X, \xi] \in \Gamma(D \oplus \{\xi\}) \text{ for any } X \in \Gamma(D) \text{ and}$$

(3.11) $$Q([X, \varphi Y] + [\varphi X, Y]) = 0 \text{ for any } X, Y \in \Gamma(D).$$

From (3.11) we obtain $Q([\varphi X, \varphi Y] - [X, Y]) = 0$. Thus by using (3.1) we have $Q([\varphi, \varphi](X, Y)^{\top}) = 0$.

On the other hand, from (3.3) we have

$$Q([\varphi, \varphi](X, Y)^{\top}) = Q([\psi, \psi](X, Y)) - Q([X, Y]).$$

Hence, by using (3.9) we obtain

(3.12) $$Q([X, Y]) = 0 \text{ for any } X, Y \in \Gamma(D).$$

Finally, from (3.10) and (3.11) follows the integrability of $D \oplus \{\xi\}$.

Now, we take $U, W \in \Gamma(D^{\perp} \oplus \{\xi\})$ and obtain

$$[\psi, \psi](U, W) = -P([U, W])$$

Thus we have

Theorem 3.3. *Let M be a semi-invariant submanifold of an almost contact metric manifold N. Then the distribution $D^{\perp} \oplus \{\xi\}$ is involutive if and only if the Nijenhuis tensor of ψ vanishes identically on $D^{\perp} \oplus \{\xi\}$.*

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