Les conditions à la limite sont

$$(6_1)\qquad u=0,\quad \sigma_{xy}=0 \text{ sur } x=0 \qquad (6_2)\qquad \sigma_{xx}=-p,\quad \sigma_{xy}=0 \text{ sur } x=h$$

$$(6_3)\qquad \sigma_{xy}=\sigma_{yy}=0 \text{ sur } y=\pm b$$

Des relations (6) on déduit les conditions à la limite :

$$u=0,$$

$$\frac{E}{1-\mu^2}\left(\frac{\partial v}{\partial y}+\mu\frac{\partial u}{\partial x}\right)=0,\quad \text{sur } x=0,\quad n(-1,0),\quad \frac{E}{1-\mu^2}\left[\frac{\partial u}{\partial x}+\mu\frac{\partial v}{\partial y}-p\frac{\partial v}{\partial y}\right]=-p,$$

$$\frac{E}{1-\mu^2}\left(\frac{\partial u}{\partial y}+\mu\frac{\partial u}{\partial x}\right)=0 \text{ sur } x=0,\ n(1,0).\qquad \frac{E}{2(1+\mu)}\left(\frac{\partial u}{\partial y}+\mu\frac{\partial v}{\partial x}\right)=0,$$

$$\frac{E}{2(1+\mu)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+p\frac{\partial v}{\partial x}=0 \text{ sur } y=b,\ n(0,1).\qquad \frac{E}{2(1+\mu)}\left(\frac{\partial u}{\partial y}+\mu\frac{\partial v}{\partial x}\right)=0,$$

$$\frac{E}{2(1+\mu)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+p\frac{\partial v}{\partial x}=0 \text{ sur } y=-b,\ n(0,-1)$$

Evidemment on doit avoir $u(x,-y)=u(x,y)$, $v(x,-y)=-v(x,y)$. En cherchant des solutions de la forme

$$u=a_{11}x^2+a_{22}y^2+2a_{13}x,\quad v=2b_{12}xy+2b_{23}y,$$

on déduit

$$u=-\frac{2p}{E}x,\quad v=\frac{2\mu p}{E}y$$

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MOMENTUM CONNECTIONS, APPLICATIONS IN MECHANICS AND FIELD THEORY

BY

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To the memory of Professor Mendel Haimovici

The below exposed attempt to develop a theory of momentum connections is performed in view of an advanced consistency between the structure of material objects or of physical fields and the geometrical description via the momentum representation of the fundamental characteristics. The respective elements are inspirated by a previously formulated higher order momentum mechanics as well as from a procedure of generating „structural" connections. The results allow subsequent developments concerning the path dynamics, the continuum and structural mechanics as well as the field theory, the gravity, relativity, cosmology, etc. The rather expositive here presented elements are substantiated by a more comprehensive treatment, actually in print.

I. N-dimensional spaces with „primitive" fiber and representative M-spaces. We list below the basic elements which occur in the subsequent definitions of momentum components of different kinds leading to similar covariant transformations laws.

1) The set X_N of elements in a one-to-one bi-continual correspondence with the value of N variables or coordinates $x=(x_1, x_2, \ldots, x_N)$ which stand for the coordinates of the points $P\in X_N$ in direct and converse correspondence with the values x. We assume also the on-to-one bi-continual correspondences $x^I=F^I(y)$ between two arbitrary systems of variables x, y assigned to the spaces X_N, Y_N. If the variables are regarded as cartezian coordinates of an euclidian space E_N it follows that between the corresponding points $P(x)\in X_N$, $Q(y)\in E_N$ holds true an on-to-one bi-continual correspondence, X_N being the topological representation of E_N.

2) The euclidian space E_P (differential fiber) with coordinates $(dx^1, \ldots, dx^N)$ at $P\in X_N$.

3) The differential affinity (differomorphism) at P: $dx^I=A^I_i dx^i$ induced by the coordinate transformation $x^I=x^I(x^i)$, $(A^I_i=\partial x^I/\partial x^i)_P)$.

4) The set Ξ_N of elements assigned to each point $P(x)\in X_N$ in a one-to-one bi-continual correspondence with the values of N variables $\xi=(\xi^1, \ldots, \xi^N)$ with origin at P so that to any point $\Pi\in\Xi_N$ for $P\in X_N$ should correspond a set of values ξ and conversely. We say that Ξ_N is the „primitive" fiber at P. As in 1) we consider the transformation $\xi^I=\Phi^I_P(\eta)$ between the values of coordinates of the point $\Pi(\xi)\in\Xi_N$ at $P\in X_N$ and $\varkappa(\eta)\in\mathcal{Y}_N$ at $Q\in Y_N$ which introduces the N-dimensional space $\mathcal{Y}_N$ with

the variables η, Φ^I_P being a one-to-one bi-continual function. If $\Phi^I_P = \Phi^I(x_p, \xi) = F^I(x_p + \xi)$ we say that the transformation Φ^I_P of the primitive fibers is totally constrained.

5) The euclidian space $\mathcal{E}_N$ (primitive euclidian fiber) of points $\varkappa$ with coordinates η. Ξ_N stands for the topological representation of $\mathcal{E}_N$ at $P \in X_N$, (between $\Pi \in \Xi_N$ for $P \in X_N$ and $\varkappa \in \mathcal{Y}_N$ for $Q \in Y_N$).

6) The euclidian space $\mathcal{E}_\Pi$ („primitive" differential fiber) with coordinates $(\delta\xi^1, \ldots, \delta\xi^N)$ at $\Pi \in \Xi_N$ for $P \in X_N$.

7) The differential primitive affinity (difforomorphism) at Π : $\delta\xi^I = \mathcal{A}^I_i \delta\xi^i$ induced by the transformation $\xi^I = \xi^I(\xi^i)$ for $\mathcal{A}^I_i = (\partial\xi^I/\partial\xi^i)_\Pi$.

The representative apparatus corresponding to the connections assigned to the considered space should be too encumbrant if we resort to the space X_N together with the entire reference frame of Ξ_N assigned to any point P of X_N. Therefore we try to resume ourselves to the space E_N as well as to a restrained set of additional values. As pointed below there is not a single or milky way to characterize by means of the considered values the distributions of points from Ξ_N. Consequently we refer below to some rather illustrative and non-restrictive as pretentions schemes of reduction of the initial pointwise assigned frames of reference. More rigorous correspondences concerning for instance a „smoothening" or homogeneization of the involved functions may be developed subsequently using a more sophisticated tool of analysis. Anyway the resulting correspondences can also be termed as apriorical and eventually linked with physical observations. The corresponding elements are enounced below.

8) The elementar linear and angular momentum components $\delta\xi^i$, $\frac{1}{2}\delta\xi^{ij} = \xi^j\delta\xi^i$ with the associated „primitive" linear and angular distortions $\mathcal{A}^I_i$, $\mathcal{A}^{IJ}_i = 2\xi^J\mathcal{A}^I_i$ assigned to the primitive diffeomorphism as well as to the analogous transformation $\delta\xi^{IJ} = \mathcal{A}^{IJ}_i\,\delta\xi^i$ (angular diffeomorphism) and the primitive linear and angular curvatures at a point $\Pi \in \Xi_N$: $\mathcal{A}^I_{ij} = (\partial^2\xi^I/\partial\xi^j\partial\xi^i)_\Pi = (\partial A^I_i/\partial\xi^j)_\Pi$, $\mathcal{A}^{IJ}_{ij} = (\partial\mathcal{A}^{IJ}_i/\partial\xi^j)_\Pi$.

8.1) For enough small regions $\Lambda : /\xi^i - \xi^i(\Pi)/ \leqslant l^i$, $(\Pi \in \Xi_N)$ and for a continuous variation of distortions and curvatures there holds true the following approximations

$$\mathcal{A}^I_i = A^I_i + A^I_{ij}\xi^j, \quad \mathcal{A}^{IJ}_i = A^{IJ}_i + 2A^{IJ}_{ij}\xi^j$$

for the primitive linear and angular distortions and curvatures of modified kind :

$$A^I_i = \mathcal{A}^I_i - \mathcal{A}^I_{ij}\xi^j(\Pi), \quad A^I_{ij} = \mathcal{A}^I_{ij}, \quad A^{IJ}_i = \mathcal{A}^{IJ}_i - \mathcal{A}^{IJ}_{ij}\xi^j(\Pi), \quad A^{IJ}_{ij} = \frac{1}{2}\mathcal{A}^{IJ}_{ij}.$$

8.2) Meanwhile the elementar and angular momentum satisfy the transformation rules

$$\delta\xi^I = A^I_i\delta\xi^i + \frac{1}{2}A^I_{ij}\delta\xi^{ij}, \quad \delta\xi^{IJ} = A^{IJ}_i\delta\xi^i + A^{IJ}_{ij}\delta\xi^{ij}$$

8.3) In the same regions Λ, for a continous variation of the coordinate transformation at $\Pi \in \Xi_N$ in which we retain also the quadratic terms : $\xi^I = \xi^I(\Pi) + a^I_i(\xi^i - \xi^i(\Pi)) + \frac{1}{2}a^I_{ij}(\xi^i - \xi^i(\Pi))(\xi^j - \xi^j(\Pi))$ the linear and angular distortions or curvatures from 8.1 satisfy the following correlations

$$A^I_i = a^I_i - a^I_{(ij)}\xi^j(\Pi), \quad A^I_{ij} = a^I_{(ij)}, \quad A^{IJ}_i = 2A^I_i\alpha^J,$$

$$A^{IJ}_{ij} = A^I_{ij}\alpha^J + A^I_iA^J_j, \text{ for } \alpha^I = \left(\xi^I - a^I_i\xi^I + \frac{1}{2}a^I_{ij}\xi^i\xi^j\right)_\Pi$$

8.4) If the origin of reference frames in Ξ_N is preserved by the transformations then $\alpha^I = 0$.

8.5) In the case of a symmetric angular momentum (which in subsequent applications is linked with strain-like measures) the elementar angular momentum becomes an effective total differential $\delta\xi^{(IJ)} = \delta(\xi^I\xi^J)$ and the angular distortions become the derivatives $\mathcal{A}^{IJ}_i = \partial(\xi^I\xi^J)/\partial\xi^i$.

8.6) In the same case, for commutative derivation the angular curvatures are also symmetric $(\mathcal{A}^{IJ}_{ij} = \mathcal{A}^{IJ}_{ji} = \mathcal{A}^{JI}_{ij})$ and the elementar angular momentum satisfies the transformation rule from 8.2.

9) The differential linear and angular momentum of an element $\Lambda \in L$ belonging to a partition L defined over a curve in Ξ_N (eventually included in a congruence refered as primitive congruent fiber bundle of Ξ_N)

$$\delta\lambda^i = \int\limits_\Lambda \rho(\xi)\delta\xi^i, \quad \delta\mu^{ij} = \int\limits_\Lambda \rho(\xi)\delta\xi^{ij};$$

$\rho(\xi)$ stands for a weight function eventually of distributional type defined over the support L. These integrals act as „informational" filters between the reference frame of $\mathcal{E}_N$ and the space $\mathcal{E}_{N(N+1)}$ defined by the values $(\delta\lambda^i, \delta\mu^{ij})$

9.1) In the frame of the assumptions from 8.1, 8.2 following transformations hold true

$$\delta\lambda^I = A^I_i\delta\lambda^i + \frac{1}{2}A^I_{ij}\delta\mu^{ij}, \quad \delta\mu^{IJ} = A^{IJ}_i\delta\lambda^i + A^{IJ}_{ij}\delta\mu^{ij}$$

Λ being a small interval $l(\Pi)$ of L where the elementar linear and angular gradients are expandable into linear polynomials of ξ.

9.2) If to any point $\Pi(\Lambda)$ of the partition element $\Lambda \in L$ of a (congruent) curve is assigned a parameter τ and to the element Λ a characteristic value $\tau(\lambda)$, then the differential linear and angular momentum are expressed by means of derivated integrands

$$\delta\lambda^i = \int\limits_\Lambda \frac{d\xi^i}{d\tau}\,p(\tau)d\tau, \quad \delta\mu^{ij} = \int\limits_\Lambda \frac{d\xi^{ij}}{d\tau}\,\rho(\tau)d\tau$$

for

$$\Lambda : \tau(\Lambda) - 0 < \tau < \tau(\Lambda) + 0, \quad \xi^i(\Lambda) = \xi^i(\tau(\Lambda)), \quad /\xi^i(\tau(\Lambda) \pm 0) - \xi^i(\Lambda)/ \leqslant l.$$

9.3) The distortions and curvatures depend upon the variables x, $\xi(\Lambda)$ but such dependences are reducible to dependences upon the variables (x, λ, μ), fact which is shown, using parametric polynomials in the extensive exposition under press.

9.4) If the dependences from 9.3 can be replaced by the dependences upon $(\bar{x} = x + \lambda, \mu)$ we say that the M-space is *linearly constrained*.

9.5) The set of variables from 9.3 or 9.4 is refered below as the *momentum space* (M).

10) The static and inertial momentum of the partition element $\Lambda \in L$

$$\delta\sigma^i = \int_\Lambda \xi^i \rho d\tau, \quad d\eta^{ij} = \int_\Lambda \xi^i \xi^j \rho \theta\tau$$

τ being the (congruence) curve parameters.

10.1) Different properties and the transformation of these components, analogous to 9.1 are also exposed separately. A more complete system of representation is based on canonical expansion and furnishes an operative tool of analysis. The below exposed elements are also abstracted shortly from the same mentioned material.

11) Parametric primitive (congruent) fiber bundle $\mathcal{L}(t)$ with supports $L: \tau_0 \leqslant \tau \leqslant \tau_1$ of the (congruent) curves $\mathcal{L}(t)$ for $t =$ the parameter of $\mathcal{L}(t)$, $\tau =$ the parameter of L and the canonical decomposition of orthonormal type of the elementar linear momentum assigned to the primitive fiber-bundle

$$\xi^i(\tau, t) = \varphi^i_j(\tau)\chi^j(t) + \varphi^i_{jk}(\tau)\chi^{jk}(t)$$

for

$$<\varphi^i_j \varphi^j_k> = \delta^i_k, \quad <\varphi^i_j \varphi^j_{kl}> = 0, \quad <\varphi^{ij}_k \varphi^k_{lm}> = \delta^i_l \delta^k_m, \quad <\varphi^{ij}_k \varphi^k_l> = 0.$$

$$\Pi(\xi) \in \Xi_N$$

Here occur the notation $<f> = \int_L f(\tau)\rho(\tau)\,d\tau$; φ^{ij}_k stands for the inverse of φ^k_{lm}.

11.1) The canonical momentum components of first and second degree

$$\chi^i = <\varphi^i_j \xi^j>, \quad <\Upsilon^{ij}> = <\varphi^{ij}_k \xi^k>$$

11.2) The differentiation, transformation laws and correlations with momentum components of other kinds are also established.

To this analysis are also added the following momentum components.

12) The linear and angular momentum defined by integration over the entire partition $L = \bigcup \Lambda$ of the (congruence) curves

$$\lambda^i = \int_L \delta\lambda^i, \quad \mu^{ij} = \int_L \delta\mu^{ij}.$$

13) The differential of the parametric linear and angular momentum defined over the support L of the (congruence) curves $\mathcal{L}(t)$

$$dl^i = <d\xi^i>, \quad dm^{ij} = <d\xi^{ij}>.$$

The above listed representation tools of the primitive fiber will be refered as momentum (M)-spaces (with $N(p + qN)$ variables (x^i, χ^i, χ^{ij})); in the general case $p = 2$, $q = 1$; for a constrained space $p = 1$; for non-constrained with antisymmetric angular momentum spaces $p = 3/2$, $q = 1/2$; idem with symmetric a.m.s $p = 5/2$, $q = 1/2$; for constrained spaces with symmetric a.m.s. $p = 3/2, q = 1/2$; idem with antisymmetric a.m.s. $p = 1/2 = q$.

II. Correspondences in M-spaces. We consider a constrained M-space with the coordinates $(x^i + \chi^i, \chi^{ij})$ refered in generic sense as linear and angular momentum and assigned to points P of this space. We denote the differentials $X^I = d(x^i + \chi^i)(P)$, $X^{IJ} = d\chi^{ij}(P)$ and consider them as coordinates of the euclidian space E_p.1. The residual linear and angular correspondence functions between the coordinates (X^I, X^{IJ}), (Y^I, Y^{IJ}) of the points located in the euclidian spaces E_P, E_Q for P, Q belonging to a constrained M-space are defined by the linear expressions

$$F^I = Y^I - (A^I + A^I_J X^J + A^I_{JK} X^{JK}), \quad F^{IJ} = Y^{IJ} - (A^{IJ} + A^{IJ}_K X^K + A^{IJ}_{KL} X^{KL}).$$

According to the derived transformation laws of the momentum differentials from I we introduce the following assumptions .

I. The *l.* and *a.* distortions and curvatures are independent upon the variables (X^I, X^{IJ}).

II. The *l.* and *a.* rezidual functionals depend upon the variables of higher order $(X^{IJK}, \ldots)$.

III. Due to the assumed complexity of the primitive fiber Ξ_N at P, the whole set of momentum components characterizing in a determined sense the distribution of points L from Ξ_N has to be considered „a priori" as composed by strictly independent (holonomic) variables.

IV. The values of coordinates (X^I, X^{IJ}) of the point $\Pi \in E_P$ vanish identically at $P: {}^I_P X^I_P = 0$, $X^{IJ}_P = 0$ and analogously the variables (Y^I, Y^{IJ}) of the point $\varkappa \in E_Q$ vanish identically at Q: $Y^I_Q = 0$, $Y^{IJ}_Q = 0$.

V. In the same frame we assume the restriction, as extension of the assumption IV that $F^I = F^I(\varkappa, Q)$, $F^{IJ} = F^{IJ}(\varkappa, Q)$ vanish identically for $Q \equiv P$, $\varkappa \equiv \Pi$: $F^I(\pi, P) = 0$, $F^{IJ}(\varkappa, P) = 0$, the rezidual functionals being dependent upon higher order momentum differentials, vanishing identically at P, according to a similar construction.

VI. The momentum correspondences include the identical correspondence so that for $Q \to P$ we have $\varkappa \to \pi$ or $Y \to X$ and $(X^I, X^{IJ})_Q \to (X^I, X^{IJ})_P$, $F^I(\varkappa, Q) \to F^I(\pi, P)$ $(=0)$, $F^{IJ}(\varkappa, Q) \to F^{IJ}(\varkappa, P)$ $(=0)$.

VII. The momentum correspondences include the inverse ones (corresponding to the replacements $X \leftrightarrows Y$, $P \leftrightarrows Q$, $\varkappa \leftrightarrows \pi$.

These assumptions lead to the following final form of the rezidual functionals

$$F^I - F^I_Q = Y^I - Y^I_Q - \left[X^I - X^I_Q - K^I_J(X^J - X^J_Q) - \frac{1}{2} K^I_{JK}(X^{JK} - X^{JK}_Q)\right]$$

$$F^{IJ}-F^{IJ}_Q=Y^{IJ}-Y^{IJ}_Q-[X^{IJ}-X^{IJ}_Q-K^{IJ}_K(X^K-X^K_Q)-K^{IJ}_{KL}(X^{KL}-X^{KL}_Q)]$$

for

$$K^I_J=M^I_{JK}(X^K_Q-X^K_P)+N^I_{JKL}(X^{KL}_Q-X^{KL}_P),\ K^I_{JK}=M^I_{JKL}(X^L_Q-X^L_P)+$$
$$+2N^I_{JKLM}(X^{LM}_Q-X^{LM}_P),\ K^{IJ}_K=M^{IJ}_{KL}(X^L_Q-X^L_P)+N^{IJ}_{KLM}(X^{LM}_Q-X^{LM}_P)$$
$$K^{IJ}_{KL}=M^{IJ}_{KLM}(X^M_Q-X^M_P)+N^{IJ}_{KLMN}(X^{MN}_Q-X^{MN}_P),\ F^I_Q=F^I(Q,\ Q)\ \text{etc.}$$

Taking into account the inverse relations in which we neglect higher order terms we obtain the compatibility conditions

$$Y^I_P+X^I_Q=-(Y^I_Q+X^I_P)\ (=0),\ Y^{IJ}_P+X^{IJ}_Q=-(Y^{IJ}_Q+X^{IJ}_P)\ (=0).$$

III. The transformation rules of the momentum connections with constrained distortions and curvatures. The quite general correspondences obtained above must be corelated with the variance rule from I concerning transformation of the corrdinated differentials. We restrain ourselves in the below exposed analysis to the case of a *constrained connection* characterized by the distortions and curvatures

$$A^I_i=\varkappa^I_{,i}(=\partial x^I/\partial x^i),\ A^I_{ij}=\varkappa^I_{,ij},\ A^{IJ}_i=0,\ A^{IJ}_{ij}=\varkappa^I_{,i}\varkappa^J_{,j}$$

where we denote by $\varkappa^I$ instead $x^I+\varkappa^I$ the total linear momentum and assume transformations of primitive fibers which preserve the origin. This connection is closer to the affine (linear) connections and present a special interest since it describes a variation in the primitive fiber identical with the variation in the space X_N in distortions and curvatures of a determined point in Ξ_N.

Hence we consider the transformations

$$X^I=(\varkappa^I_{,i})_P\,X^i+\frac{1}{2}(\varkappa^I_{,ij})_P\,X^{ij},\ X^{IJ}=(\varkappa^I_{,j}\,\varkappa^J_{,i})_P\,X^{ij}$$

$$Y^I=(\varkappa^I_{,i})_Q\,Y^i+\frac{1}{2}(\varkappa^I_{,ij})_Q\,Y^{ij},\ Y^{IJ}=(\varkappa^I_{,i}\varkappa^J_{,j})_Q\,Y^{ij}$$

After substitution in the last two relations the expressions for Y^i,... furnished by the momentum correspondences and after approximation of the distortions and curvatures by linear polynomials in $(X^i,\ X^{ij})_Q$ around the point P, we obtain following transformation rules of the rezidual functionals

$$F^I=(\varkappa^I_{,i})_Q\,F^i+\frac{1}{2}(\varkappa^I_{,ij})_Q\,F^{ij}+\left(F^I_{i:j}X^j_Q+\frac{1}{2}F^I_{i:jk}X^{jk}_Q\right)(X^i-X^i_Q)+$$
$$+(F^I_{ij:k}X^k_Q+F^I_{ij:kl}(X_Q-X_P)^{kl})(X^{ij}-X^{ij}_Q)+\Delta F^I$$
$$F^{IJ}=(X^I_{,i}X^J_{,j})_Q\,F^{ij}+(F^{IJ}_{i:j}X^j_Q+F^{IJ}_{i:jk}X^j_Q)(X^i-X^i_Q)+$$
$$+(F^{IJ}_{ij:k}X^k_Q+F^{IJ}_{ij:kl}X^{kl}_Q)(X^{ij}-X^{ij}_Q)+\Delta F^{ij}$$

where

$$F^I_{i:j}=(\varkappa^I_{,ij})_P-(\varkappa^I_{,k})_P\,M^k_{ij}+(\varkappa^J_{,i}\varkappa^K_{,j})_P\,M^I_{JK}-\frac{1}{2}(\varkappa^I_{,kl})_P\,M^{kl}_{ij}$$

$$F^I_{ij:k}=(\varkappa^I_{,ijk})_P-(\varkappa^I_{,m})_P\,M^m_{ijk}-(\varkappa^I_{,mn})_P\,M^{mn}_{ijk}+$$
$$+(\varkappa^J_{,ij}\varkappa^K_{,k})_P\,M^I_{JK}+(\varkappa^J_{,i}\varkappa^K_{,j}\varkappa^L_{,k})_P\,M^I_{JKL}$$

$$\frac{1}{2}F^I_{i:jk}=\frac{1}{2}(\varkappa^J_{,i}\varkappa^K_{,jk})_P\,M^I_{JK}+(\varkappa^J_{,i}\varkappa^K_{,j}\varkappa^L_{,k})_P\,N^I_{JKL}-$$
$$-(\varkappa^I_{,m})_P\,N^m_{ijk}-\frac{1}{2}(\varkappa^I_{,lm})_P\,N^{lm}_{ijk}+(\partial\varkappa^I_{,i}/\partial\varkappa^{JK})_P$$

$$F^I_{ij:kl}=\frac{1}{4}(\varkappa^J_{,ij}\varkappa^K_{,kl})_P\,M^I_{JK}+\frac{1}{4}(\varkappa^J_{,i}\varkappa^K_{,j}\varkappa^L_{,kl})_P\,M^I_{JKL}+$$
$$+\frac{1}{2}(\varkappa^J_{,ij}\varkappa^K_{,k}\varkappa^L_{,l})_P\,N^I_{JKL}+(\varkappa^J_{,i}\varkappa^K_{,j}\varkappa^L_{,k}\varkappa^M_{,l})_P\,N^I_{JKLM}-$$
$$-(\varkappa^I_{,m})_P\,N^m_{ijkl}-\frac{1}{2}(\varkappa^I_{,mn})_P\,N^{mn}_{ijkl}+\frac{1}{2}\left(\frac{\partial\varkappa^I_{,ij}}{\partial\varkappa^{kl}}\right)_P$$

$$F^{IJ}_{i:j}=(\varkappa^K_{,i}\varkappa^L_{,j})_P\,M^{IJ}_{KL}-(\varkappa^I_{,k}\varkappa^J_{,l})_P\,M^{kl}_{ij}$$

$$F^{IJ}_{ij:k}=(\varkappa^I_{,ik}\varkappa^J_{,j}+\varkappa^I_{,i}\varkappa^J_{,jk})_P-(\varkappa^I_{,l}\varkappa^J_{,m})_P\,M^{lm}_{ijk}+$$
$$+\frac{1}{2}(\varkappa^K_{,ij}\varkappa^L_{,k})_P\,M^{IJ}_{KL}+(\varkappa^K_{,i}\varkappa^L_{,j}\varkappa^M_{,k})_P\,M^{IJ}_{KLM}$$

$$F^{IJ}_{i:jk}=\frac{1}{2}(\varkappa^K_{,i}\varkappa^L_{,jk})_P\,M^{IJ}_{KL}+(\varkappa^K_{,i}\varkappa^L_{,j}\varkappa^M_{,k})_P\,N^{IJ}_{KLM}-(\varkappa^I_{,l}\varkappa^J_{,m})_P\,N^{lm}_{ijk}$$

$$F^{IJ}_{ij:kl}=\frac{1}{2}(\varkappa^K_{,i}\varkappa^L_{,j}\varkappa^M_{,kl})_P\,M^{IJ}_{KLM}+\frac{1}{4}(\varkappa^K_{,ij}\varkappa^L_{,kl})_P\,M^{IJ}_{KL}+$$
$$+(\varkappa^K_{,i}\varkappa^L_{,j}\varkappa^M_{,k}\varkappa^N_{,l})\,N^{IJ}_{KLMN}-(\varkappa^I_{,n}\varkappa^J_{,n})_P\,N^{mn}_{ijkl}+\frac{1}{2}(\varkappa^K_{,ij}\varkappa^L_{,k}\varkappa^M_{,l})_P\,N^{IJ}_{KLM}+\left(\frac{\partial\varkappa^I_{,ij}}{\partial\varkappa^{kl}}\right)_P$$

We define now the momentum connection by the correlations between the coefficients M,N refered as the momentum connections symbols, which correspond to the independence conditions of the rezidual functionals $F^I-F^I_Q(=0)$, $F^{IJ}-F^{IJ}_Q(=0)$ upon the relative positions measured in E_P by the coordinates $X^I-X^I_Q$, $X^{IJ}-X^{IJ}_Q$ of an arbitrary point $\pi(X)$ and of the point Q as origin of E_Q. Consequently, if we observe that for $\pi\equiv P$ the independence refers also to the relative positions of the points P, Q refered by the differences $X^I_P-X^I_Q$ etc., and taking into account the above derived expressions of the rezidual functionals, we get the conditions

$$F^I_{i:j}=0,\ F^I_{i:jk}=0,\ F^{IJ}_{ij:k}=0,\ F^{IJ}_{ij:kl}=0.$$

IV. Axiomatic aspects of momentum connections. The already formulated conditions characterizing the momentum correspondence and connections correspond to determined assumptions concerning various dependences in the sense of a generalized non-holonomy between the distortions and curvatures, as exposed in the more comprehensive material under press.

The respective M-spaces are refered as M-spaces with constrained connections. Meanwhile more general invariantive features of the rezidual functional involve an extension of the above presented transformation rules of the connection symbols. Consequently, in order to cover this field with a more systematic treatment which is ressented as necessary we formulate below an axiomatic frame and we resume here only to the basic assumptions occuring in the sections II and III.

The final form of the rezidual functionals in section II furnishing the correspondences of linear and angular differential momentum corresponds to the following assumption :

I. *Absolute objectivity of the rezidual functional*

— *The rezidual functionals are independent upon the absolute positions of the corresponding points in* E_P *respectively* E_Q or

— *The rezidual functionals are invariant for arbitrary absolute and uniform variations of the variables :* $X^I \to X^I + \Delta X^I$ (const.), $X^{IJ} \to X^{IJ} + \Delta X^{IJ}$ (const.) ; $Y^I \to Y^I + \Delta Y^I$ (const.), $Y^{IJ} \to Y^{IJ} + \Delta Y^{IJ}$ (const.).

It is obvious that this invariance holds true also at the point Q: $X^I_Q \to X^I_Q + \Delta X^I_Q$ (const.) ; $Y^I_Q \to Y^I_Q + \Delta Y^I_Q$ (const.) etc.

The conditions for the transformation of the symbols of momentum connections were derived according to the below formulated assumption.

II. *Relative linear and angular objectivity of the rezidual functionals*

— *The rezidual functionals are independent upon the position in of points in* E_P *relative to the origin* Q *of* E_Q

or

— *The rezidual functionals are invariant for arbitrary uniform variations of the variables of points in* E_p *relative to the variables of the origin* Q *of* E_Q: $X^I - X^I_Q \to X^I - X^I_Q + \Delta(X^I - X^I_Q)$ (const.); $X^{IJ} - X^{IJ}_Q \to X^{IJ} - X^{IJ}_Q + \Delta(X^{IJ} - X^{IJ}_Q)$ (const.). Implicitely the invariance holds true for $X^I \to X^I_P$.

On the basis of different assumptions mentioned above result different transformation laws of momentum connections. Accordingly, the symbols of connection are endowed with different specific differential structure. These results are presented separately.

V. *Covariant differentiation and applications*

We define the contravariant *versator* (generalized vector of linear and angular type V^I, V^{IJ}) as the quantities which transform according to the following laws

$$V^I = A^I_i V^i + \frac{1}{2} A^I_{ij} V^{ij}, \quad V^{IJ} = A^{IJ}_i V^i + A^{IJ}_{ij} V^{ij}$$

We restrain ourselves as before to the case of constrained distortion and curvature. If the angular part is antisymmetric ($V^{IJ} = -V^{JI}$) we refer to the respective versator as to a *dynamo*, according to a previously by H. Schaefer adopted terminology in continuum mechanics. More general quantities which transform as products of versators shall be refered as *tractors* (or generalized tensors of linear and angular type).

Further we define the *covariant differentials* of versators

$$DV^I = dV^I + M^I_{JK} V^J d\varkappa^K + N^I_{JKL} V^J d\varkappa^{KL} + \frac{1}{2} M^I_{JKL} V^{JK} d\varkappa^L +$$

$$+ N^I_{JKLM} dV^{JK} d\varkappa^{LM}$$

$$DV^{IJ} = dV^{IJ} + M^{IJ}_{KL} V^K d\varkappa^L + \frac{1}{2} N^{IJ}_{KLM} V^K d\varkappa^{LM} + \frac{1}{2} M^{IJ}_{KLM} V^{KL} d\varkappa^M +$$

$$+ \frac{1}{2} N^{IJ}_{KLMN} V^{KL} d\varkappa^{MN}.$$

The (generalized) covariant differentials of versators in a space with momentum connections are versators, the respective transformation laws being identical with the above formulated laws and result by means of the substitutions $(V^I, V^{IJ}) \to (DV^I, DV^{IJ})$.

We define the parallel transfer of versators by the conditions $DV^I = 0$, $DV^{IJ} = 0$. Let us now consider a path in the M-spaces and the associated parameter t. We identify the versator (V^I, V^{IJ}) with the derivatives of momentum over the path $(d\varkappa^I/dt,\ d\varkappa^{IJ}/dt)$. Accordingly the equations of the parallel transfer of this path versator (the generalized auto-parallel curves) written under the form

$$D(d\varkappa^I/dt)/dt = 0, \quad D(d\varkappa^{IJ}/dt)/dt = 0$$

become

$$d^2\varkappa^I/dt^2 + M^I_{JK}(d\varkappa^J/dt)(d\varkappa^K/dt) + (N^I_{JKL} + \frac{1}{2} M^I_{KLJ})(d\varkappa^J/dt).$$

$$(d\varkappa^{KL}/dt) + N^I_{JKL}\ (d\varkappa^{JK}/dt)(d\varkappa^{KL}/dt) = 0$$

$$d^2\varkappa^{IJ}/dt^2 + M^{IJ}_{KL}(d\varkappa^K/dt)(d\varkappa^L/dt) + \frac{1}{2}(N^{IJ}_{KLM} + M^{IJ}_{LMK})(d\varkappa^K/dt).$$

$$(d\varkappa^{LM}/dt) + \frac{1}{2} N^{IJ}_{KLMN}\ (d\varkappa^{KL}/dt)\ (d\varkappa^{MN}/dt) = 0.$$

The first two terms in the first equation correspond to the equation of path dynamics and geodesics for affine connections.

Covariant formulations similar to the above ones proceed from the definition of covariant versators which follows from the definition of invariant products of versators $I = V_I V^I + V_{IJ} V^{IJ}$. If we substitute the contravariant versators by the expressions furnished by the transformation laws formulated above and impose the invariance condition $I = V_i V^i + V_{ij} V^{ij}$ we get the covariance conditions

$$V_i = A^I_i V_I, \quad V_{ij} = A^I_i A^J_i V_{IJ} + \frac{1}{2} A^I_{ij} V_I$$

This results can be applied to the analysis of forces (F) and displacements (U) of continua with „primitive" structure, in the representation system of momentum components :

$$I = F_I U^I + F_{IJ} U^{IJ}$$

If we take $I = dW$, $F_I = \sigma_I^J\, da_J$, $F_{IJ} = \sigma_{IJ}^K\, da_K$, da^I being the usual area element in the underspace of linear momentum, we get after integration over the region including the body

$$dW = (\sigma_I^J U^I_{;J} + \sigma_{IJ}^K U^{IJ}_{;K} + \sigma^J_{I;J} U^I + \sigma^K_{IJ;K} U^{IJ})\, dv$$

for $U^I_{;J}$ the generalized covariant derivatives $= U^I_{,J} + M^I_{KJ} U^K + M^I_{KLJ} U^{KL}$.

A more complete extension of the obtained expression of the energy is analysed in other paper.

VI. Metric momentum spaces and applications. Let us consider a riemannian „primitive" metrics in Ξ_N: $d\xi^2 = \gamma_{ij}\, d\xi^i d\xi^j$. We define the canonic momentum metrics assigned to a M-space by the relation

$$ds^2 = g_{ij} d\varkappa^i d\varkappa^j + g_{ijk}\, d\varkappa^i d\varkappa^{jk} + g_{ijkl} d\varkappa^{ij} d\varkappa^{kl}$$

$\varkappa^i$, $\varkappa^{ij}$ stand for canonic variables and $g_{ij} = <\varphi_i^l \varphi_j^m \gamma_{lm}>$,

$$g_{ijk} = <(\varphi_j^l \varphi_{jk}^m + \varphi_{jk}^l \varphi_i^m)\, \gamma_{lm}>, \quad g_{ijkl} = <\varphi_{ij}^l \varphi_{kl}^m \gamma_{mn}>.$$

This metrics leads to an extension of the riemannian geometry and of the euclidian geometry. In this last case we consider the vector $\vec{\gamma}_i = \vec{e}_I \xi^I_{,i}$ refered to a cartezian unit base so that $\gamma_{ij} = \vec{\gamma}_i \vec{\gamma}_j$. Consequently, starting from the „primitive" element of area $d\vec{\alpha} = \vec{\gamma}_i \times \vec{\gamma}_j d\xi^i d\xi^j$ and from the vector γ in Ξ_N we formulate the invariant $Y = \gamma d\vec{\alpha}$.

Taking into account a canonic representation of the primitive fiber we obtain by integration $I = Y = V^I da_I + V^{IJ} da_{IJ}$. The contravariant factors stand for the coefficients of the canonic representation of γ and $da_I = <\varphi_I^J e_J d\vec{\alpha}>$, $da_{IJ} = <\varphi^K_{IJ} \vec{e}_K\, d\vec{\alpha}>$. After application of the canonic expression for the variables $d\xi^I$ we obtain for the above quantities expressions in terms of $(d\varkappa^I, d\varkappa^{IJ})$ nor exposed here. As a matter of fact these aspects are linked with the formula of the canonic area

$$d\vec{a} = <d\vec{\alpha}> = \vec{g}_{ij} d\varkappa^i d\varkappa^j + \vec{g}_{ijk} d\varkappa^i d\varkappa^{jk} + \vec{g}_{ijkl}\, d\varkappa^{ij} d\varkappa^{kl}$$

analogous to the above relation for the metrics, also derived by integration of the „primitive" area element.

As application we consider the energy from section V for $F_I = \sigma_I^J da_J + \sigma_I^{JK} da_{JK}$, $F_{IJ} = \sigma_{IJ}^K da_K + \sigma_{IJ}^{KL} da_{KL}$. The stress field is completed by additional terms and the respective integration can be performed on the generalized integration theorems which shall be published later.

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THEOREMS ON EXISTENCE AND UNIQUENESS OF SOLUTION TO BOUNDARY-VALUE PROBLEMS OF PLACE AND MICROROTATION FOR COSSERAT BODIES

BY

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To the memory of Professor Mendel Haimovici

1. Introduction. Theorems on existence and uniqueness of solution to boundary value problems of place for hyperelastic bodies are given by Beju [1]. Beju's analysis of boundary-value problems of place in the nonlinear theory is based on four basic theorems of Langenbach [2] in a theory of minimization of functionals.

Using Langenbach's theorems, Chiriţă [3] has established theorems of existence in the theory of continuum with deformable directors.

In this paper we shall study the existence and uniqueness of solution to boundary-value problem of place and microrotation for hyperelastic Cosserat bodies.

2. Langenbach's theorems. Let Ω be a bounded domain in the Euclidean space R^n and let $H(\Omega)$ be a Hilbert space on Ω. The boundary surface $\partial\Omega$ is assumed sufficiently smooth to assure the validity of the common laws of transformation of surface integrals.

Langenbach [2] has considered the nonlinear equation

(2.1) $$P(u) = f,$$

with linear and homogeneous boundary conditions

(2.2) $$L_i(u) = 0, \quad i = 1, 2, \ldots, p,$$

where f is a given element of $H(\Omega)$.

The domain $D(P)$ and the range $R(P)$ of the nonlinear map are contained in $H(\Omega)$.

We put

$$D_0(P) = \{u \in D(P),\ L_i(u) = 0,\ i = 1, 2, \ldots, p\}.$$

Having introduced the necessary basic concepts, we can now state the theorems of Langenbach.

Theorem 1. *If the following conditions are satisfied:*

1) *$D_0(P)$ and $D(P)$ are linear manifolds dense in $H(\Omega)$.*

2) *P has a linear Gateaux differential $P'(u)h$ such that for every $h \in D(P)$ the field $P'(u)h$ is continuous on each 2-dimensional hyperplane passing through u in $D(P)$.*