If we take $I=dW$, $F_I=\sigma_I^J\, da_J$, $F_{IJ}=\sigma_{IJ}^K\, da_K$, da^I being the usual area element in the underspace of linear momentum, we get after integration over the region including the body

$$dW=(\sigma_I^J U^I_{;J}+\sigma^K_{IJ}U^{IJ}_{;K}+\sigma^J_{I;J}U^I+\sigma^K_{IJ;K}U^{IJ})\, dv$$

for $U^I_{;J}$ the generalized covariant derivatives $=U^I_{,J}+M^I_{KJ}U^K+M^I_{KLJ}U^{KL}$.

A more complete extension of the obtained expression of the energy is analysed in other paper.

VI. Metric momentum spaces and applications. Let us consider a riemannian „primitive" metrics in Ξ_N: $d\xi^2=\gamma_{ij}\, d\xi^i d\xi^j$. We define the canonic momentum metrics assigned to a M-space by the relation

$$ds^2=g_{ij}dx^i dx^j+g_{ijk}\, dx^i dx^{jk}+g_{ijkl}dx^{ij}dx^{kl}$$

x^i, x^{ij} stand for canonic variables and $g_{ij}=<\varphi_i^l\varphi_j^m\gamma_{lm}>$,

$$g_{ijk}=<(\varphi_j^l\varphi^m_{jk}+\varphi^l_{jk}\varphi^m_i)\,\gamma_{lm}>,\quad g_{ijkl}=<\varphi^l_{ij}\varphi^m_{kl}\gamma_{mn}>.$$

This metrics leads to an extension of the riemannian geometry and of the euclidian geometry. In this last case we consider the vector $\vec{\gamma}_i=\vec{e}_l\xi^l_{,i}$ refered to a cartezian unit base so that $\gamma_{ij}=\vec{\gamma}_i\,\vec{\gamma}_j$. Consequently, starting from the „primitive" element of area $d\vec{\alpha}=\vec{\gamma}_i\times\vec{\gamma}_j d\xi^i d\xi^j$ and from the vector γ in Ξ_N we formulate the invariant $Y=\gamma d\vec{\alpha}$.

Taking into account a canonic representation of the primitive fiber we obtain by integration $I=Y=V^I da_I+V^{IJ}da_{IJ}$. The contravariant factors stand for the coefficients of the canonic representation of γ and $da_I=$ $=<\varphi^J_I e_J d\vec{\alpha}>$, $da_{IJ}=<\varphi^K_{IJ}\vec{e}_K\, d\vec{\alpha}>$. After application of the canonic expression for the variables $d\xi^i$ we obtain for the above quantities expressions in terms of $(dx^I,\, dx^{IJ})$ nor exposed here. As a matter of fact these aspects are linked with the formula of the canonic area

$$d\vec{a}=<d\vec{\alpha}>=\vec{g}_{ij}dx^i dx^j+\vec{g}_{ijk}dx^i dx^{jk}+\vec{g}_{ijkl}\, dx^{ij}dx^{kl}$$

analogous to the above relation for the metrics, also derived by integration of the „primitive" area element.

As application we consider the energy from section V for $F_I=\sigma_I^J da_J+$ $+\sigma_I^{JK}da_{JK}$, $F_{IJ}=\sigma^K_{IJ}da_K+\sigma^{KL}_{IJ}da_{KL}$. The stress field is completed by additional terms and the respective integration can be performed on the generalized integration theorems which shall be published later.

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Supliment la tomul XXVII, s. I a, 1981

THEOREMS ON EXISTENCE AND UNIQUENESS OF SOLUTION TO BOUNDARY-VALUE PROBLEMS OF PLACE AND MICROROTATION FOR COSSERAT BODIES

BY

I. NISTOR

To the memory of Professor Mendel Haimovici

1. Introduction. Theorems on existence and uniqueness of solution to boundary value problems of place for hyperelastic bodies are given by Beju [1]. Beju's analysis of boundary-value problems of place in the nonlinear theory is based on four basic theorems of Langenbach [2] in a theory of minimization of functionals.

Using Langenbach's theorems, Chiriţă [3] has established theorems of existence in the theory of continuum with deformable directors.

In this paper we shall study the existence and uniqueness of solution to boundary-value problem of place and microrotation for hyperelastic Cosserat bodies.

2. Langenbach's theorems. Let Ω be a bounded domain in the Euclidean space R^n and let $H(\Omega)$ be a Hilbert space on Ω. The boundary surface $\partial\Omega$ is assumed sufficiently smooth to assure the validity of the common laws of transformation of surface integrals.

Langenbach [2] has considered the nonlinear equation

$$P(u)=f, \tag{2.1}$$

with linear and homogeneous boundary conditions

$$L_i(u)=0,\quad i=1,\ 2,\ \dots,\ p, \tag{2.2}$$

where f is a given element of $H(\Omega)$.

The domain $D(P)$ and the range $R(P)$ of the nonlinear map are contained in $H(\Omega)$.

We put

$$D_0(P)=\{u\in D(P),\ L_i(u)=0,\ i=1,\ 2,\ \dots,\ p\}.$$

Having introduced the necessary basic concepts, we can now state the theorems of Langenbach.

Theorem 1. *If the following conditions are satisfied:*

1) *$D_0(P)$ and $D(P)$ are linear manifolds dense in $H(\Omega)$.*

2) *P has a linear Gâteaux differential $P'(u)h$ such that for every $h\in$ $\in D(P)$ the field $P'(u)h$ is continuous on each 2-dimensional hyperplane passing through u in $D(P)$.*

3) $P_0(0)=0$.

4) *$P'(u)h$ is symmetric with respect to $D_0(P)$,*

(2.3) $$(P'(u)h^1, h^2)=(P'(u)h^2, h^1), \; u\in D(P), \; h^1, h^2\in D_0(P).$$

5) *$P'(u)h$ is positive-definite with respect to $D_0(P)$,*

(2.4) $$(P'(u)h, h)>0,$$

for all non-zero $h\in D_0(P)$, then;

a) *If a solution u_0 to the problem (2.1), (2.2) exists, then it is unique; further, it establishes on $D_0(P)$ an absolute minimum of the functional*

(2.5) $$F(u) = \int_0^1 (P(tu), u)dt-(f, u).$$

b) *Conversely, if on $D_0(P)$, $F(u)$ attains an absolute minimum at u_0, then u_0 is a solution of (2.1) and (2.2).*

We note that the preceding is a theorem of uniqueness, not a theorem of existence. In general, the functional $F(u)$ need not attain an absolute minimum on $D_0(P)$.

Theorem 2. *If condition 5) of Theorem 1 is replaced by 5') $P'(u)h$ is strictly positive with respect to $D_0(P)$,* i.e., *there exists a positive constant c such that*

(2.6) $$(P'(u)h, h)\geq c\|h\|^2 \qquad \forall h\in D_0(P),$$

then the following additional assertions are true:

a) *The functional $F(u)$ is bounded from below on $D_0(P)$.*

b) *The functional $F(u)$ is strictly convex on $D_0(P)$, i.e., for each unequal $u, v\in D_0(P)$ and each $t\in(0, 1)$*

$$F(tu+(1-t)v)<tF(u)+(1-t)F(v).$$

c) *Any minimizing sequence for $F(u)$ on $D_0(P)$ is convergent in $H(\Omega)$.*

By assertion c) above, a minimizing sequence for $F(u)$ has a limit in $H(\Omega)$. If that limit belongs to $D_0(P)$, by Theorem 1 it is the solution of the problem (2.1) and (2.2). On the other hand if that limit does not belong to $D_0(P)$, then the problem (2.1) and (2.2) has no solution, but we call the limit of a minimizing sequence a generalized solution.

Theorem 3. *Under the assumptions of Theorem 2, the generalized solution of (2.1) and (2.2) is unique.*

Theorem 4. *If condition 5') of Theorem 2 is replaced by*

5'') *There exist an element $\tilde{u}_0\in D_0(P)$ and positive numbers c_1, c_2 such that*

(2.7) $$(P'(u)h, h)\geq c_1(P'(\tilde{u})h, h)\geq c_2\|h\|^2, \quad \forall h\in D_0(P),$$

then:

a) *The generalized solution of (2.1) and (2.2) belongs to the completion of $D_0(P)$ with respect to the inner product $(u, v)=(P'(\tilde{u})u, v)$ induced by $P'(\tilde{u})$.*

Proofs of these theorems are given by Langenbach [2].

3. The first boundary-value problem for Cosserat bodies. Let $\mathcal{B}$ be an Cosserat body. We assume that the reference configuration of $\mathcal{B}$ is a bounded region $\Omega\subset R^3$, with boundary surface $\partial\Omega$. The deformation of a Cosserat body is given by [4]

(3.1) $$y_i=y_i(x_j), \quad R_{ij}=R_{ij}(x_k),$$

where x_i and y_j are the coordinates of the place occupied by the body point in the reference configuration Ω and the deformed configuration, respectively; R_{ij} are the components of a proper orthogonal tensor.

The components of a proper orthogonal tensor (relative to an orthonormal basis) may be written [5] in the form

(3.2) $$R_{ij}=\frac{1}{1+\frac{1}{4}\varphi_q\varphi_q}\left[\left(1-\frac{1}{4}\varphi_k\varphi_k\right)\delta_{ij}+\frac{1}{2}\varphi_i\varphi_j+e_{jip}\varphi_p\right],$$

where φ_i are the components of microrotation vector, e_{ijk} is alternating symbol.

We consider as strain tensors

(3.3) $$e_{ij}=y_{k,i}R_{kj}, \quad \gamma_{ij}=\frac{1}{2}e_{jmn}R_{pn}R_{pm,i},$$

where the symbol „i", denotes partial derivation with respect to the variable x_i.

Using (3.2) the deformation tensor γ_{ij} may be represented in the form

(3.4) $$\gamma_{ij}=H_{jk}\varphi_{k,i}$$

where

(3.5) $$H_{jk}=\frac{1}{2\left(1+\frac{1}{4}\varphi_p\varphi_p\right)}(2\delta_{jk}+e_{jkq}\varphi_q).$$

(3.6) $$\det(H_{jk})\neq 0.$$

It is easy to verify that

(3.7) $$H_{ki}=R_{ij}H_{jk}=H_{ij}R_{jk},$$

(3.8) $$\frac{\partial R_{ij}}{\partial\varphi_k}=e_{inm}H_{kn}R_{mj}.$$

From (3.5), by means of the relations

(3.9) $$e_{ijk}e_{pjk}=2\delta_{ip}, \quad e_{ijk}e_{pqk}=\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp},$$

we obtain

(3.10) $$\frac{\partial H_{jq}}{\partial\varphi_t}e_{kqt}=\frac{1}{2}e_{kqt}e_{jps}H_{pq}H_{st}.$$

Multiplying both sides of relation (3.9) by e_{knm} and making use of the identity $(3.9)_2$ one obtains

(3.11) $$\frac{\partial H_{jn}}{\partial\varphi_m}=\frac{\partial H_{jm}}{\partial\varphi_n}+e_{jps}H_{pn}H_{sm}.$$

By use of the relation (3.7) and the identity

$$e_{kpq}R_{qj}=e_{jih}R_{ki}R_{ph},$$

we may express (3.11) in the form

(3.12) $$\frac{\partial H_{jn}}{\partial\varphi_n}=\frac{H_{mk}}{\partial\varphi_n}R_{kj}.$$

The equations of equilibrium of Cosserat media are [6]

$$-(R_{ij}t_{kj})_{,k}=\rho F_i,\quad (R_{ij}m_{kj})_{,k}+e_{ijk}y_{j,p}R_{kq}t_{pq}+\rho e_{ijk}R_{jp}B_{pk}=0, \tag{3.13}$$

where

$$t_{ij}=\frac{\partial W}{\partial c_{ij}},\quad m_{kj}=\frac{\partial W}{\partial \gamma_{kj}},\quad W=W(c_{ij},\ \gamma_{pq}). \tag{3.14}$$

In these equations we have used the following notations : t_{ij} and m_{ij}-components of stress tensor and couple-stress tensor, respectively, measured per unit area in reference configuration ; F_i-components of the body force vector ; B_{jk}-components of the body generalized forces vectors ; ρ-density of the body in reference configuration ; W-strain energy function per unit volume in reference configuration.

In case when displacements $u_i^0(x_j)$ and microrotation $\varphi_i^0(x_j)$ are prescribed on the surface $\partial\Omega$ the boundary condition takes the form

$$u_i(x_k)=u_i^0(x_k),\quad \varphi_i(x_j)=\varphi_i^0(x_j). \tag{3.15}$$

Taking into account the condition (3.6) we find that the equation $(3.13)_2$ is equivalent to

$$(R_{ip}m_{jp})_{,j}H_{ki}+e_{ijh}y_{j,p}R_{hq}t_{pq}H_{ki}+\rho e_{ijh}R_{jp}B_{ph}H_{ki}=0. \tag{3.16}$$

Using (3.8) the relation (3.16) may be written in the form

$$-(R_{ip}m_{jp})_{,j}H_{ki}+y_{j,p}t_{pq}\frac{\partial R_{jq}}{\partial \varphi_k}-\rho B_{ji}\frac{\partial R_{ij}}{\partial \varphi_k}=0. \tag{3.17}$$

Let us denote by u the vector $(u_i,\ \varphi_j)$ and by u^0 the element $(u_i^0,\ \varphi_j^0)$. The equations $(3.13)_1$, (3.15) and (3.17) may be written in the form

$$A(u)=g,\quad x_i\in\Omega,\quad u=u^0,\quad x_j\in\partial\Omega, \tag{3.18}$$

where A is the operator defined by the left-hande side of the relations (3.13) and (3.17), and g is a prescribed element defined by $g=(-\rho F_i,0,0,\ 0)$, $(i=1,2,3)$.

We choose a function v which attains the boundary-value u^0 on $\partial\Omega$. We put $w=u-v$. The boundary condition in terms of w becomes the linear homogeneous condition

$$w=0,\quad x_i\in\partial\Omega. \tag{3.19}$$

Introducing the operator E defined by

$$E(w)=A(u)-A(v)=A(w+v)-A(v), \tag{3.20}$$

the boundary-value problem of place and microrotation is given by the system

$$E(w)=f,\ x_i\in\Omega,\quad w(x_j)=0,\quad x_j\in\partial\Omega, \tag{3.21}$$

where

$$f=g-A(v). \tag{3.22}$$

Let $L_2(\Omega)$ be the Hilbert space of square-integrable vector fields on Ω. The norm of this space is generated by the scalar product

$$(u,\ v)=\int(u_jv_j+\varphi_i\psi_i)d\tau,\ (i,\ j=1,\ 2,\ 3),$$

where $u=(u_j,\ \varphi_i)$, $v=(v_j,\ \psi_i)$. We define $D(E)$ as the linear manifolds of $L_2(\Omega)$ consisting in all vector fields of class C^2 on Ω, and we define $D_0(E)$ as the submanifold of $D(E)$ subject to the linear homogeneous boundary condition (3.19). Then $D(E)$ and $D_0(E)$ are dense in $L_2(\Omega)$. From (3.20), it is clear that the mapping E satisfies the hypothesis 3) of theorem 1.

We obtain (3.13) and (3.17) the coordinate form for the linear Gâteaux differential of E at any point $w\in D(E)$

$$\begin{aligned} E'_q(w)h &= -\left[\left(\frac{\partial W}{\partial c_{kj}}\frac{\partial R_{qj}}{\partial \varphi_p}+\frac{\partial^2 W}{\partial c_{kj}\partial \varphi_p}R_{qj}\right)h_{p+3}+R_{qj}\frac{\partial^2 W}{\partial c_{kj}\partial u_{\alpha,p}}h_{\alpha,p}\right]_{,k} \\ E'_{q+3}(w)h &= -\left[\left(\frac{\partial W}{\partial \gamma_{kj}}\frac{\partial R_{ij}}{\partial \varphi_p}+\frac{\partial^2 W}{\partial \gamma_{kj}\partial \varphi_p}R_{ij}\right)h_{p+3}+\right. \\ &\quad \left.+R_{ij}\frac{\partial^2 W}{\partial \gamma_{kj}\partial u_{\alpha,p}}h_{\alpha,p}\right]_{,k}H_{qi}+\left(R_{ij}\frac{\partial W}{\partial \gamma_{kj}}\right)_{,k}\frac{\partial H_{qi}}{\partial \varphi_p}h_{p+3}+ \\ &\quad +\frac{\partial W}{\partial c_{pk}}\frac{\partial R_{jk}}{\partial \varphi_q}h_{j,p}+\frac{\partial W}{\partial c_{jk}}\frac{\partial^2 R_{ik}}{\partial \varphi_p\partial \varphi_q}y_{i,j}h_{p+3}-\rho\frac{\partial^2 R_{ij}}{\partial \varphi_q\partial \varphi_p}B_{ji}h_{p+3}+ \\ &\quad +\left(\frac{\partial^2 W}{\partial c_{jk}\partial u_{\alpha,p}}h_{\alpha,p}+\frac{\partial^2 W}{\partial c_{jk}\partial \varphi_q}h_{p+3}\right)\frac{\partial R_{ik}}{\partial \varphi_q}y_{i,j},\ (\alpha=1,\ 2,\ \ldots,\ 6). \end{aligned} \tag{3.23}$$

From (3.23) it is clear that for each h the differential is continuous on each 2-dimensional hyperplane passing through w. Using the relations (3.7), (3.12) and the divergence theorem, we obtain

$$\begin{aligned} (E'(w)h^1,\ h^2) &= \int_\Omega\left[\frac{\partial^2 W}{\partial c_{kj}\partial c_{pq}}\delta^1c_{kj}\delta^2c_{pq}+\frac{\partial^2 W}{\partial c_{kj}\partial \gamma_{pq}}(\delta^1c_{kj}\delta^2\gamma_{pq}+\right. \\ &\quad +\delta^2c_{kj}\delta^2\gamma_{pq})+\frac{\partial^2 W}{\partial \gamma_{kj}\partial \gamma_{pq}}\delta^1\gamma_{kj}\delta^2\gamma_{pq}+\left(\frac{\partial R_{ij}}{\partial \varphi_q}h^1_{i,k}h^2_{q+3}+\frac{\partial R_{ij}}{\partial \varphi_q}h^2_{i,k}h^1_{q+3}+\right. \\ &\quad \left.+\frac{\partial^2 R_{ij}}{\partial \varphi_p\partial \varphi_q}y_{i,k}h^1_{p+3}h^2_{q+3}\right)\frac{\partial W}{\partial c_{kj}}+\left(\frac{\partial H_{jp}}{\partial \varphi_q}h^2_{q+3}h^1_{p+3,k}+\frac{\partial H_{jp}}{\partial \varphi_q}h^1_{q+3}h^2_{q+3,k}+\right. \\ &\quad \left.\left.+\frac{\partial^2 H_{ji}}{\partial \varphi_q\partial \varphi_p}\varphi_{i,k}h^1_{q+3}h^2_{p+3}\right)\frac{\partial W}{\partial \gamma_{kj}}-\rho\frac{\partial^2 R_{ij}}{\partial \varphi_p\partial \varphi_q}B_{ji}h^1_{p+3}h^2_{q+3}\right]d\tau=(E'(w)h^2,\ h^1). \end{aligned} \tag{3.24}$$

where

$$\delta^\alpha c_{kj}=R_{pj}h^\alpha_{p,k}+\frac{\partial R_{pj}}{\partial \varphi_q}y_{p,k}h^\alpha_{q+3},\ \delta^\alpha\gamma_{kj}=H_{jp}h^\alpha_{p+3,k}+\frac{\partial H_{jp}}{\partial \varphi_q}h^\alpha_{q+3}\varphi_{p,k},\ (\alpha=1,\ 2).$$

From relation (3.24) it follows that

$$(E'(w)h\ h)=\int_\Omega\left[\frac{\partial^2 W}{\partial c^\alpha_{kj}\partial c^\beta_{pq}}\delta c^\alpha_{kj}\delta c^\beta_{pq}+\left(2\frac{\partial R_{ij}}{\partial \varphi_q}h_{i,k}h_{q+3}+\right.\right.$$

$$
(3.25)\quad \begin{aligned} &+\frac{\partial^2 R_{ij}}{\partial\varphi_p\partial\varphi_q}\gamma_{i,k}h_{q+3}h_{p+3}\Big)\frac{\partial W}{\partial c_{kj}}+\Big(2\,\frac{\partial H_{jp}}{\partial\varphi_q}h_{q+3,k}+\frac{\partial^2 H_{ji}}{\partial\varphi_p\partial\varphi_q}\varphi_{ik}h_{p+3}h_{q+3}\Big)\\ &\cdot\frac{\partial W}{\partial\gamma_{kj}}-\rho\frac{\partial^2 R_{ij}}{\partial\varphi_p\partial\varphi_q}B_{ji}h_{p+3}h_{q+3}\Big]d\tau,\ (\alpha,\beta=1,2)\end{aligned}
$$

where $c^1_{kj}=c_{kj}$, $c^2_{pq}=\gamma_{pq}$.

The condition (2.4) is satisfied if and only if

$$
(3.26)\qquad (E'(w)h,\,h)>0,
$$

for all non-zero $h\in D_0(L)$.

Similarly, condition (2.6) is satisfied if and only if

$$
(3.27)\qquad (E'(w)h,\,h)\geqslant c\int_\Omega(v_iv_i+\psi_j\psi_j)d\tau,
$$

and condition (2.7) is satisfied if and only if

$$
(3.28)\qquad (E'(w)h,\,h)\geqslant c_1(L'(w)h,\,h)\geqslant c_2\int_\Omega(v_iv_i+\psi_j\psi_j)d\tau,
$$

where $h=(v_i,\psi_j)$, $(i\,j=1,2,3)$, w is a particular element of $D_0(L)$ and $(E'(w)h,h)$ is given by (3.25).

Having set up assumptions sufficient to satisfy the premisses of Langenbach's theorems for the system (3.21), we have the following results:

Theorem 5. *Let $\mathcal{B}$ be a homogeneous hyperelastic Cosserat body. If the strain energy function and the generalized body force together satisfy the inequality* (3.26) *then conditions a) and b) of theorem 1 hold for the system* (3.21).

Theorem 6. *If the inequality* (3.26) *in theorem 5 is replaced by the inequality* (3.27), *then also conditions a)–c) of theorem 2 and theorem 3 hold for the system* (3.21).

Theorem 7. *If the inequality* (3.27) *in theorem 6 is replaced by the inequality* (3.28), *then condition a) of theorem 4 holds for the systzm* (3.21).

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Supliment la tomul XXVII, s. I a, 1981

L'APPROXIMATION PAR PÉNALISATION-ÉLÉMENTS FINIS D'UNE CLASSE D'INÉQUATIONS VARIATIONNELLES DE PREMIÈRE ESPECE

BY

A. RADOSLOVESCU

Dédié à la mémoire du Professeur Mendel Haimovici

Nous chercherons de donner une approximation au problème suivant sous contraintes unilatérales.

Soit Ω un ouvert borné de R^n, de frontière $\partial\Omega$ régulière.

On désigne par $H^1(\Omega)$ l'espace de Sobolev des functions v telles que:

$$
(1)\qquad v\in L_2(\Omega),\quad \frac{\partial v}{\partial x_i}\in L_2(\Omega),\quad i=1,\ \ldots,\ n,
$$

où $L_2(\Omega)$ est l'espace des (classes de) fonctions de carré sommable sur Ω et où les dérivés $\partial v/\partial x_i$ sont prises au sens des distributions sur Ω. Muni du produit scalaire:

$$
(2)\qquad (u,\,v)_1=\int_\Omega uv\,dx+\sum_{i=1}^n\int_\Omega\frac{\partial u}{\partial x_i}\cdot\frac{\partial v}{\partial x_i},
$$

$H^1(\Omega)$ est un espace de Hilbert.

On désigne par $H^1_0(\Omega)$ la clôture de $C^1_0(\Omega)$ dans $H^1(\Omega)$.

Nous pouvons alors définir le sous-ensemble suivant K, des contraintes de $H^1_0(\Omega)$:

$$
(3)\qquad K=\{v\in H^1_0(\Omega)/v-g\leqslant 0\ \text{p.p. dans}\ \Omega\},
$$

où g est une fonction donnée à valeurs réelles définie dans $\bar\Omega$, avec $g\geqslant 0$ sur $\partial\Omega$. On vérifie que K est un ensemble convexe, fermé de $H^1_0(\Omega)$.

Introduisons maintenant la forme bilinéaire:

$$
(4)\qquad a(u,\,v)=\int_\Omega\sum_{i=1}^n\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx.
$$

Alors pour $f\in L_2(\Omega)$ donné, considérons l'inéquation variationnelle suivante:

$$
(5)\qquad \begin{cases}\text{on cherche } u\in K \text{ tel que}\\ a(u,\,v-u)\geqslant(f,\,v-u)\quad\forall v\in K,\end{cases}
$$

où $(\ldots)$ est le produit scalaire usuel dans $L_2(\Omega)$. Ce problème correspond [11] avec la formulation faible d'un problème de l'obstacle.