$$\begin{bmatrix} P_{\varepsilon_1 h_1} & P_{\varepsilon_1 h_2} & \dots & P_{\varepsilon_1 h_n} & \dots \\ P_{\varepsilon_2 h_1} & P_{\varepsilon_2 h_2} & \dots & P_{\varepsilon_2 h_n} & \dots \\ \vdots & \vdots & & \vdots & \\ P_{\varepsilon_m h_1} & P_{\varepsilon_m h_2} & & P_{\varepsilon_m h_n} & \dots \\ \vdots & \vdots & & \vdots & \end{bmatrix}$$

peut être réduite à la résolution de quelques problèmes de la première colonne (ce qui correspond à quelques pas dans la méthode de pénalisation chaque pas revenant à la résolution d'un problème discret sur le même espace V_h des éléments finis). La limite de $u_{\varepsilon h}$, lorsque $\varepsilon \to 0$, est alors une bonne approximation de u.

D'autres résolutions de la matrice ci-dessus peuvent se baser, par exemple, sur un choix de type $\varepsilon = h$ ou $\varepsilon = h^2$ et alors nous avons, évidemment, $u_{\varepsilon h} \to u$ avec $\varepsilon \to 0$. Mais la stabilisation du choix de la relation optime entre ε et h dans le sens de la rapidité de la convergence $u_{\varepsilon h} \to u$ avec $\varepsilon \to 0$ et $h \to \to 0$ est un problème qui demande une analyse plus fine, différent d'un type à l'autre de données du problème.

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THE PROPAGATION OF THERMAL STRESSES IN THIN METALLIC RODS

BY

ELISABETA RUSU

To the memory of Professor Mendel Haimovici

The problem of propagation of thermal stresses in thin rods was considered by L. Sneldon [1], O. W. Dillon [2], in the linear theory of coupled thermoelesticity.

The generalized theory of thermoelasticity has been derived by A. E. Green and K. A. Lindsay [3].

In this paper we consider the problem of determining the displacement and temperature in a semiinfinite bar $x \geqslant 0$ for which the end $x = 0$ of a bar is subject to *a*) constant strain ; *b*) constant velocity (impact), in the generalized theory of linear thermoelasticity [3]. The Laplace technique transform is used to develop the solutions, for small times.

The basic equations of the linear theory for an isotropic solid with a centre of symmetry at each point (one-dimensional case) are [4] :

$$(1)\qquad \begin{cases} (\lambda + 2\mu)\dfrac{\partial^2 u}{\partial x^2} - m\dfrac{\partial}{\partial x}(\theta + \alpha\dot{\theta}) = \rho\dfrac{\partial^2 u}{\partial t^2} \\[2ex] k\dfrac{\partial^2 \theta}{\partial x^2} - d\dot{\theta} - h\ddot{\theta} - \theta_0 m\dfrac{\partial^2 u}{\partial x \partial t} = 0 \end{cases}$$

where u denote the displacement in the axial direction x. The initial conditions are

$$(2)\qquad u(x, 0) = 0 \ ; \ \dot{u}(x, 0) = 0 \ ; \ \dot{\theta}(x, 0) = 0 \ ; \ \theta(x, 0) = 0 \ ; \ x \geqslant 0.$$

It is convenient to write the basic equations (1) in dimensionless form. We consider the following variables :

$$(3)\qquad X = ax \ ; \ U = au \ ; \ a = \frac{v_1 d}{k} \ ; \ \tau = a v_1 t \ ; \ v_1^2 = \frac{\lambda + 2\mu}{\rho}$$

$$(4)\qquad A = \frac{h v_1^2}{k} \ ; \ B = \alpha a v_1 \ ; \ \varepsilon = \frac{\theta_0 m^2}{d(\lambda + 2\mu)}$$

The equations (1) respectively assume the forms

$$(5)\qquad \begin{aligned} &\frac{\partial^2 U}{\partial X^2} - \frac{\partial^2 U}{\partial \tau^2} = \frac{m}{\lambda + 2\mu}\frac{\partial}{\partial X}\left(\theta + B\frac{\partial \theta}{\partial \tau}\right) \\ &\frac{\partial^2 \theta}{\partial X^2} - \frac{\partial \theta}{\partial \tau} - A\frac{\partial^2 \theta}{\partial \tau^2} = \frac{m\theta_0}{d}\frac{\partial^2 U}{\partial X \partial \tau} \end{aligned}$$

We put ($H(\tau)$ is the Heaviside step functions):

$$U^* = H(\tau)\, U(X, \tau)\ ;\ \theta^* = H(\tau)\, \theta(X, \tau). \tag{6}$$

Let „-" denote the Laplace transform of distribution (6).

In all the problems considered herein, the initial conditions are the quiescent ones (2) and only solutions which remain finite as x approach infinity are wanted. The equations (5) becomes

$$\begin{aligned} &\left(\frac{d^2}{dX^2} - p^2\right)\overline{U}^* = \frac{m}{\lambda + 2\mu}(1 + Bp)\frac{d\overline{\theta}^*}{dX} \\ &\left(\frac{d^2}{dX^2} - p - Ap^2\right)\overline{\theta}^* = \frac{m\theta_0}{d}\, p\, \frac{d\overline{U}^*}{dX}. \end{aligned} \tag{7}$$

Then $\overline{U}^*$ and $\overline{\theta}^*$ obey the equation

$$\left\{\frac{d^4}{dX^4} - [p^2 + p + Ap^2 + \varepsilon p(1 + Bp)]\frac{d^2}{dX^2} + p^3(1 + Ap)\right\}\{\overline{U}^*, \overline{\theta}^*\} = 0 \tag{8}$$

The general solution of equation (8) is of the form

$$\overline{U}^* = C_1 e^{-\lambda_1 X} + C_2 e^{-\lambda_2 X} \qquad \overline{\theta}^* = C_3 e^{-\lambda_1 X} + C_4 e^{-\lambda_2 X} \tag{9}$$

where λ_1 and λ_2 are the roots of the equation

$$\lambda^4 - \lambda^2 [p^2 + p + Ap^2 + \varepsilon p(1 + Bp)] + p^3(1 + Ap) = 0 \tag{10}$$

By (9) and (7) we find the general solution of equations (7)

$$\begin{aligned} &\overline{U}^* = C_1 e^{-\lambda_1 X} + C_2 e^{-\lambda_2 X} \\ &\overline{\theta}^* = -\frac{m\theta_0}{d}\left(\frac{p\lambda_1}{\lambda_1^2 - p - Ap^2} C_1 e^{-\lambda_1 X} + \frac{p\lambda_2}{\lambda_2^2 - p - Ap^2} C_2 e^{-\lambda_2 X}\right). \end{aligned} \tag{11}$$

Boundary value problems. a) Step in strain. The problem in which the end of the bar, initially at rest, is subjected to a sudden charge in strain of magnitude ε_0, while the temperature on $x = 0$ is helt constant is developed by B. A. B o l e y and I. S. T o l i n s, for the half-space, and by O. W. D i l l o n [2], for the semiinfinite bar, in the linear theory of coupled (classical) thermoelasticity. The boundary conditions are:

$$u'(0, t) = H(t)\varepsilon_0\ ;\ \theta(0, t) = 0\ ;\ t > 0\ ;\ x = 0\ ; \tag{12}$$

We note that we can give instead of u' in (12), the stress $t_{11} = H(t)\sigma_0$; then $\varepsilon_0 = \dfrac{\sigma_0}{\lambda + 2\mu}$.

The transformed solution (11), subjected to the conditions

$$\frac{d\overline{U}^*}{dX} = \frac{\varepsilon_0}{p}\ ;\ \overline{\theta}^*(0, p) = 0 \quad \text{on} \ = 0 \tag{13}$$

is

$$\begin{aligned} &\overline{U}^* = \varepsilon_0\left\{\frac{\lambda_1^2 - p - Ap^2}{p\lambda_1(\lambda_2^2 - \lambda_1^2)} e^{-\lambda_1 X} - \frac{\lambda_2^2 - p - Ap^2}{p\lambda_2(\lambda_2^2 - \lambda_1^2)} e^{-\lambda_2 X}\right\} \\ &\overline{\theta}^* = \frac{\theta_0 m\varepsilon_0}{d}\left\{\frac{1}{\lambda_1^2 - \lambda_2^2} e^{-\lambda_1 X} - \frac{1}{\lambda_1^2 - \lambda_2^2} e^{-\lambda_2 X}\right\}. \end{aligned} \tag{14}$$

Let us obtain the inverse Laplace transform, for smal values of time. From Abel's theorem it follows that high values of p correspond to small values of τ. Expanding the expression of roots λ_1, λ_2 in MacLaurin series, relatining the first four terms and disregarding ε^2 in comparison wit ε, we have [5]

$$\begin{aligned} &\lambda_1 = p\left[1 + \frac{B\varepsilon}{2(1-A)} + \frac{(1-A+B)\varepsilon}{2p(1-A)^2} + \frac{(1-A+B)\varepsilon}{2p^2(1-A)^3} + \frac{(1-A+B)\varepsilon}{2p^3(1-A)^4}\right] \\ &\lambda_2 = p\sqrt{A}\left\{1 - \frac{B\varepsilon}{2(1-A)} + \frac{1}{p}\left[\frac{1}{2A} + \frac{(2A^2 - AB - 2A - B)\varepsilon}{4A(1-A)^2}\right] + \right. \\ &\quad + \frac{1}{p^2}\left[-\frac{1}{8A^2} + \frac{(4A^3 - 3A^2B - 6AB - 4A - B)\varepsilon}{16A^2(1-A)^3}\right] + \\ &\quad + \frac{1}{p^3}\left[\frac{(6A^4 - 5A^3B + 5A^2B + 6A^3 + 5AB - 14A^2 + 2A - B)\varepsilon}{32A^2(1-A)^4} + \right. \\ &\qquad\qquad \left.\left. + \frac{1}{16A^3}\right]\right\}. \end{aligned} \tag{15}$$

We take

$$\begin{aligned} &k_1 = X\left(1 + \frac{B\varepsilon}{2(1-A)}\right);\ k_2 = \sqrt{A}\, X\left(1 - \frac{B\varepsilon}{2(1-A)}\right);\ \omega_1 = \frac{X(1-A+B)}{2(1-A)^3}; \\ &k_3 = \frac{X}{2\sqrt{A}}\left[1 + \frac{(2A^2 - AB - 2A - B)\varepsilon}{2(1-A)^2}\right];\ C = \\ &= \frac{X(4A^3 - 3A^2B - 6AB - 4A - B)\varepsilon}{16A\sqrt{A}(1-A)^3} \\ &D = \frac{X(6A^4 - 5A^3B + 5A^2B + 6A^3 - 14A^2 + 5AB + 2A - B)\varepsilon}{32A^2\sqrt{A}(1-A)^4}; \end{aligned} \tag{16}$$

$$\omega_2 = \frac{X}{8A\sqrt{A}}\ ;\ \omega_3 = \frac{X}{16A^2\sqrt{A}}\ ;\ e^{-\lambda_1 X} \simeq e^{-k_1 p}\, e^{-(1-A)\omega_1\varepsilon}\left(1 - \frac{\omega_1\varepsilon}{p} - \frac{\omega_1\varepsilon}{p^2(1-A)}\right)$$

$$e^{-\lambda_2 X} \simeq e^{-k_2 p}\, e^{-k_3}\left\{1 + \frac{1}{p}(\omega_2 - C\varepsilon) + \frac{1}{p^2}\left[\left(\frac{\omega_2^2}{2} - \omega_3\right) - \varepsilon(C\omega_2 + D)\right]\right\}$$

From (4) we have

$$B - A = \frac{v_1^2}{k}(\alpha d - h)$$

In [3] on proves that $\alpha d - h \geqslant 0$. It follows that $1 - A + B \geqslant 0$ and $(1-A)\omega_1\varepsilon = \dfrac{(1-A+B)X\varepsilon}{2(1-A)^2} \geqslant 0$ for $X \geqslant 0$, and because h and ε are small $k_2 \geqslant 0$ and $k_3 \geqslant 0$.

From (14), (15), and (16) we obtain

$$
\begin{aligned}
\overline{U}^* &= \varepsilon_0 \left\{ \frac{\varepsilon}{\sqrt{A}(1-A)^2 p^2} \left[R_0 + \frac{R_1}{p} + \frac{R_2}{p^2} + \frac{R_3}{p^3} \right] e^{-k_1 p} e^{-k_2} - \right. \\
&\qquad \left. - \frac{1}{p_2} \left[L_0 + \frac{L_1}{p} + \frac{L_2}{p^2} + \frac{L_3}{p^3} \right] e^{-k_1 p} e^{-(1-A)\omega_1 \varepsilon} \right\} \\
\overline{\theta}^* &= \frac{\varepsilon_0 \theta_0 m}{d(1-A)p^2} \left\{ \left[M_0 + \frac{M_1}{p} + \frac{M_2}{p^2} + \frac{M_3}{p^3} \right] e^{-k_1 p} e^{-(1-A)\omega_1 \varepsilon} - \right. \\
&\qquad \left. - \left[N_0 + \frac{N_1}{p} + \frac{N_2}{p^2} + \frac{N_3}{p^3} \right] e^{-k_1 p} e^{-k_2} \right\}
\end{aligned}
\tag{17}
$$

where

$$M_0 = 1 - \frac{B\varepsilon(1+A)}{(1-A)^2}$$

$$M_1 = \frac{1}{1-A} - \frac{\varepsilon[2B(1+A) - (A^2 - 2B - 1)]}{(1-A)^3} - \varepsilon\omega_1$$

$$M_2 = \frac{1}{(1-A)^2} - \frac{\varepsilon[3B(1+A) - 2(A^2 - 2B - 1) + 2(1 - A + B)]}{(1-A)^4} - \frac{2\omega_1\varepsilon}{1-A}$$

$$M_3 = \frac{1}{(1-A)^3} - \frac{\varepsilon(4B(1+A) - 3(A^2 - 2B - 1) + 6(1 - A + B)]}{(1-A)^5} - \frac{2\omega_1\varepsilon}{(1-A)^2} - \frac{\omega_1\varepsilon}{1-A}$$

$$N_0 = M_0$$

$$N_1 = \frac{1}{1-A} - \frac{\varepsilon(2B(1+A) - (A^2 - 2B - 1)]}{(1-A)^3} + \omega_2 - \varepsilon\left[C + \omega_2 \frac{B(1+A)}{(1-A)^2}\right]$$

$$
\begin{aligned}
N_2 = \frac{1}{(1-A)^2} &- \frac{[3B(1+A) - 2(A^2 - 2B - 1) + 2(1 - A + B)]\varepsilon}{(1-A)^4} + \\
&+ \frac{\omega_2 - C\varepsilon}{(1-A)} - \varepsilon\omega_2 \frac{2B(1+A) - (A^2 - 2B - 1)}{(1-A)^3} + \frac{\omega_2^2}{2} - \\
&- \omega_3 - \varepsilon(C\omega_2 + D) - \varepsilon\left(\frac{\omega_2^2}{2} - \omega_3\right)\frac{B(1+A)}{(1-A)^2}
\end{aligned}
$$

$$
\begin{aligned}
N_3 = \frac{1}{(1-A)^3} &- \frac{\varepsilon[4B(1+A) - 3(A^2 - 2B - 1) + 6(1 - A + B)]}{p^3(1-A)^5} + \\
&+ \frac{\omega_3 - C\varepsilon}{(1-A)^2} - \varepsilon\omega_2 \frac{3B(1+A) - 2(A^2 - 2B - 1) + 2(1 - A + B)}{(1-A)^4} + \\
&+ \frac{1}{1-A}\left[\frac{\omega_2^2}{2} - \omega_3 - \varepsilon(C\omega_2 + D)\right] - \varepsilon\left(\frac{\omega_2^2}{2} - \omega_3\right)\frac{2B(1+A) - (A^2 - 2B - 1)}{(1-A)^3}
\end{aligned}
$$

$$
\begin{aligned}
R_0 &= -M_0 AB, \\
R_1 &= M_0\left(\frac{A^2 - A - B}{1-A} + \frac{B}{2}\right) - N_1 AB, \\
R_2 &= -M_0\left(\frac{1 - A + B}{(1-A)^2} + \frac{A^2 - A - B}{2A(1-A)} + \frac{3B}{8A}\right) + \\
&\quad + N_1 \frac{2(A^2 - A - B) + B(1-A)}{2(1-A)} - N_2 AB, \\
R_3 &= -ABN_3 + N_2 \frac{2(A^2 - A - B) + B(1-A)}{2(1-A)} - N_1\left(\frac{1 - A + B}{(1-A)^2} + \right. \\
&\quad \left. + \frac{A^2 - A - B}{2A(1-A)} + \frac{3B}{8A}\right) + M_0\left(-\frac{1 - A + B}{(1-A)^3} + \frac{1 - A + B}{2A(1-A)^2} + \right. \\
&\quad \left. + \frac{3(A^2 - A - B)}{8A^2(1-A)} + \frac{5B}{16A^2}\right).
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
L_0 &= M_0\left(1 + \frac{B\varepsilon(1+A)}{2(1-A)^2}\right), \\
L_1 &= M_1\left(1 + \frac{B\varepsilon(1+A)}{2(1-A)^2}\right) - M_0\left(\frac{1}{1-A} + \frac{(A^2 - 2B - 1)\varepsilon}{2(1-A)^3}\right). \\
L_2 &= M_2\left(1 + \frac{B\varepsilon(1+A)}{2(1-A)^2}\right) - M_1\left(\frac{1}{1-A} + \frac{(A^2 - 2B - 1)\varepsilon}{2(1-A)^3}\right) + M_0 \frac{(1 - A + B)\varepsilon}{(1-A)} \\
L_3 &= M_3\left(1 + \frac{B\varepsilon(1+A)}{2(1-A)^2}\right) - M_2\left(\frac{1}{1-A} + \frac{(A^2 - 2B - 1)\varepsilon}{2(1-A)^3}\right) + \\
&\quad + M_1 \frac{(1 - A + B)\varepsilon}{(1-A)^3} + M_0 \frac{(1 - A + B)\varepsilon}{(1-A)^5},
\end{aligned}
$$

Using the relations

$$\mathcal{L}^{-1}\left(\frac{1}{p^n} e^{-\alpha p}\right) = \frac{1}{(n-1)!}(\tau - \alpha)_+^{n-1} = \frac{1}{(n-1)!}(\tau - \alpha)^{n-1} H(\tau - \alpha).$$

and performing the inverse Laplace transform in (17) we find the solution of Boundary-value problem (7) (12)

$$U^*(\chi, \tau) = \varepsilon_0 \left\{ \frac{\varepsilon}{\sqrt{A}(1-A)^2} H(\tau - k_2) \left[[(\tau - k_2) R_0 + \frac{1}{2}(\tau - k_2)^2 R_1 + \right.\right.$$

$$\left. + \frac{1}{3!}(\tau - k_2)^3 R_2 + \frac{1}{4!}(\tau - k_2)^4 R_3 \right] e^{-k_2} - H(\tau - k_1) \left[(\tau - k_1) L_0 + \right.$$

$$\left.\left. + \frac{1}{2}(\tau - k_1)^2 L_1 + \frac{1}{3!}(\tau - k_1)^3 L_2 + \frac{1}{4!}(\tau - k_1)^4 L_3 \right] e^{-(1-A)\omega_1 \varepsilon} \right\}$$

$$\theta^*(X, \tau) = \frac{\varepsilon_0 \theta_0 m}{d(1-A)} \left\{ H(\tau - k_1) \left[(\tau - k_1) M_0 + \frac{1}{2!}(\tau - k_1)^2 M_1 + \right.\right.$$

$$\left. + \frac{1}{3!}(\tau - k_1)^3 M_2 + \frac{1}{4!}(\tau - k_1)^4 M_3 \right] e^{-(1-A)\omega_1 \varepsilon} - H(\tau - k_2) \left[(\tau - k_2) N_0 + \right.$$

$$\left.\left. + \frac{1}{2}(\tau - k_2)^2 N_1 + \frac{1}{3!}(\tau - k_2)^3 N_2 + \frac{1}{4!}(\tau - k_2)^3 N_3 \right] e^{-k_2} \right\}.$$

Note that in the generalized theory of linear thermoelasticity U and θ are continuous and there are not displacement and temperature at any x prior to arrival of the wave; there are two fronts of wave which arrive in any x.

b) Constant Velocity Impact Problem. The problem in which constant velocity is imposed on the face $x=0$, when no heat flows across this face, was considered in [3].

Thus the boundary conditions are

$$\begin{cases} u(0, t) = v_0 H(t) t \\ \theta'(0, t) = 0\,; \quad t > 0\,; \quad x = 0\,; \end{cases} \tag{20}$$

In this case, the transformed solution (11) for the problem (7) (20) is

$$\begin{aligned} \overline{U}^* &= v_0 \left\{ \frac{\lambda_2^2(\lambda_1^2 - p - A p^2)}{p^2(p + A p^2)(\lambda_1^2 - \lambda_2^2)} e^{-\lambda_1 X} - \frac{\lambda_1^2(\lambda_2^2 - p - A p^2)}{p^2(p + A p^2)(\lambda_2^2 - \lambda_1^2)} e^{-\lambda_2 X} \right\}, \\ \overline{\theta}^* &= -\frac{m\theta_0}{d} V_0 \left\{ \frac{\lambda_1 \lambda_2^2}{p(p + A p^2)(\lambda_1^2 - \lambda_2^2)} e^{-\lambda_1 X} - \frac{\lambda_1^2 \lambda_2}{p(p + A p^2)(\lambda_1^2 - \lambda_2^2)} e^{-\lambda_2 X} \right\}. \end{aligned} \tag{21}$$

Now we find

$$\begin{aligned} \overline{U}^* &= v_0 \left\{ \frac{1}{p^2} \left[S_0 + \frac{S_1}{p} + \frac{S_2}{p^2} + \frac{S_3}{p^3} \right] e^{-k_1 p} e^{-(1-A)\omega_1 \varepsilon} + \right. \\ &\quad \left. + \frac{1}{p^2(1-A)} \left[T_0 + \frac{T_1}{p} + \frac{T_2}{p^2} + \frac{T_3}{p^3} \right] e^{-k_2 p} e^{-k_2} \right\}, \\ \overline{\theta}^* &= -\frac{m\theta_0 v_0}{d} \left\{ \frac{1}{p^2(1-A)} \left[V_0 + \frac{V_1}{p} + \frac{V_2}{p^2} + \frac{V_3}{p^3} \right] e^{-k_1 p} e^{-(1-A)\omega_1 \varepsilon} - \right. \\ &\quad \left. - \frac{1}{p^2 \sqrt{A^1}(1-A)} \left[z_0 + \frac{z_1}{p} + \frac{z_2}{p^2} + \frac{z_3}{p^3} \right] e^{-k_2 p} e^{-k_2} \right\}. \end{aligned} \tag{22}$$

where

$$S_0 = M_0 \left(1 + \frac{AB\varepsilon}{(1-A)^2} \right),$$

$$S_1 = M_1 \left(1 + \frac{AB\varepsilon}{(1-A)^2} \right) - M_0 \left(\frac{1}{1-A} + \frac{(A^2 - A - B)\varepsilon}{(1-A)^3} \right),$$

$$S_2 = M_2 \left(1 + \frac{AB\varepsilon}{(1-A)^2} \right) - M_1 \left(\frac{1}{1-A} + \frac{(A^2 - A - B)\varepsilon}{(1-A)^3} \right) + M_0 \frac{(1 - A + B)\varepsilon}{(1-A)^4},$$

$$S_3 = M_3 \left(1 + \frac{AB\varepsilon}{(1-A)^2} \right) - M_2 \left(\frac{1}{1-A} + \frac{(A^2 - A - B)\varepsilon}{(1-A)^3} \right) +$$

$$+ M_1 \frac{(1 - A + B)\varepsilon}{(1-A)^3} + M_0 \left(\frac{(1 - A + B)\varepsilon}{(1-A)^5} \right.$$

$$T_0 = \frac{B\varepsilon}{1-A} N_0$$

$$T_1 = \frac{B\varepsilon}{1-A} N_1 + \frac{(1 - A + B)\varepsilon}{(1-A)^2} N_0,$$

$$T_2 = \frac{B\varepsilon}{(1-A)} N_2 + \frac{(1 - A + B)\varepsilon}{(1-A)^2} N_1 + \frac{(1 - A + B)\varepsilon}{(1-A)^3} N_0 \tag{23}$$

$$T_3 = \frac{B\varepsilon}{1-A} N_3 + \frac{(1 - A + B)\varepsilon}{(1-A)^2} N_2 + \frac{(1 - A + B)\varepsilon}{(1-A)^3} N_1 + \frac{(1 - A + B)\varepsilon}{(1-A)} N_0$$

$$V_0 = M_0 \left(1 - \frac{B\varepsilon}{2(1-A)} \right),$$

$$V_1 = M_1 \left(1 - \frac{B\varepsilon}{2(1-A)} \right) + M_0 \frac{(A - B - 1)\varepsilon}{2(1-A)^2},$$

$$V_2 = M^2 \left(1 - \frac{B\varepsilon}{2(1-A)} \right) + M_1 \frac{(A - B - 1)\varepsilon}{2(1-A)^2} + M_0 \frac{(A - B - 1)\varepsilon}{2(1-A)^3},$$

$$V_3 = M_3 \left(1 - \frac{B\varepsilon}{2(1-A)} \right) + M_2 \frac{(A - B - 1)\varepsilon}{2(1-A)^2} + M_1 \frac{(A - B - 1)\varepsilon}{2(1-A)^3} +$$

$$+ M_0 \frac{(A - B - 1)\varepsilon}{2(1-A)^4},$$

$$Z_0 = N_0 \left(1 + \frac{B\varepsilon}{2(1-A)} \right),$$

$$Z_1 = N_1 \left(1 + \frac{B\varepsilon}{2(1-A)} \right) + N_0 \left(-\frac{1}{2A} + \frac{(1 - A - B)\varepsilon}{2(1-A)^2} - \frac{B\varepsilon}{4A(1-A)} \right),$$

$$Z_2=N_2\left(1+\frac{B\varepsilon}{2(1-A)}\right)+N_1\left(-\frac{1}{2A}+\frac{(1-A+B)\varepsilon}{2(1-A)^2}-\frac{B\varepsilon}{4A(1-A)}\right)+$$
$$+N_0\left(\frac{\varepsilon}{8A^2}+\frac{(1-A+B)\varepsilon}{2(1-A)^3}-\frac{(1-A+B)\varepsilon}{4A(1-A)^2}+\frac{3B\varepsilon}{16A^2(1-A)}\right),$$
$$Z_3=N_3\left(1+\frac{B\varepsilon}{2(1-A)}\right)+N_2\left(-\frac{1}{2A}+\frac{(1-A+B)\varepsilon}{2(1-A)^2}-\frac{B\varepsilon}{4A(1-A)}\right)+$$
$$+N_1\left(\frac{3}{8A^2}+\frac{(1-A+B)\varepsilon}{2(1-A)^3}-\frac{(4-A+B)\varepsilon}{1A(1-A)^2}+\frac{3B\varepsilon}{16A^2(1-A)}\right)+$$
$$+N_0\left(-\frac{5}{16A^3}+\frac{(1-A+B)\varepsilon}{2(1-A)^4}-\frac{(1-A+B)\varepsilon}{4A(1-A)^3}+\right.$$
$$\left.+\frac{3(1-A+B)\varepsilon}{16A^2(1-A)^2}-\frac{5B\varepsilon}{32A^3(1-A)}\right).$$

By performing the inverse Laplace transform in (22) we find the solution of boundary value problem (7) (20)

$$U^*(X,\tau)=V_0\left\{H(\tau-k_1)\left[(\tau-k_1)S_0+\frac{1}{2!}(\tau-k_1)^2S_1+\frac{1}{3!}(\tau-k_1)^3S_2+\right.\right.$$
$$\left.+\frac{1}{4!}(\tau-k_1)^4S_3\right]e^{-(1-A)\omega_1\varepsilon}+\frac{1}{1-A}H(\tau-k_2)\left[(\tau-k_2)T_0+\frac{(\tau-k_2)^2}{3}T_1+\right.$$
$$\left.\left.+\frac{(\tau-k_2)^3}{3!}T_2+\frac{(\tau-k_2)^4}{4!}T_3\right]e^{-k_2}\right\}$$

$$\theta^*(X_1\tau)=-\frac{m\theta_0V_0}{d}\left\{\frac{H(\tau-k_1)}{1-A}\left[(\tau-k_1)V_0+\frac{(\tau-k_1)^2}{2!}V_1+\frac{(\tau-k_1)^3}{3!}V_2+\right.\right.$$
$$\left.+\frac{(\tau-k_1)^4}{4!}V_3\right]e^{-(1-A)\omega_1\varepsilon}-\frac{H(\tau-k_2)}{\sqrt{A}(1-A)}\left[(\tau-k_2)Z_0+\frac{(\tau-k_2)^2}{2!}Z_1+\right.$$
$$\left.\left.+\frac{(\tau-k_2)^3}{3!}Z_2+\frac{(\tau-k_2)}{4!}Z_3\right]e^{-k_2}\right\}.$$

As in the Boley-Tollins problem, U and θ are continuous and there are two fronts of wave which arrive in any x. For $x\to\infty$, $U\to 0$ and $\theta\to 0$.

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AN ENERGY EQUATION IN THE THEORY OF MICROPOLAR FLUIDS

BY

VALERIU AL. SAVA

To the memory of Professor Mendel Haimovici

It is shown that a weak solution of the equations governing the behaviour of an incompressible micropolar fluid lying in a space $\mathbf{L}^q((0,T):\mathcal{L}^p\times\mathbf{L}^p)$ where $2/p+2/q\leq 1$ and $p\geq 4$, satisfies an energy equality rather than the usual energy inequality.

1. Preliminary results. It is known [1] that a weak solution $\{v,\nu\}$ of the equàtions governing the behaviour of an incompressible micropolar fluid, exists satisfying the energy inequality

$$\|v(t)\|_2^2+j\|\nu(t)\|_2^2+\int_0^t\{2\mu\|\nabla v(\tau)\|_2^2+2(\alpha+\beta)\|\nabla\cdot\nu(\tau)\|_2^2+$$
$$+2\gamma\|\nabla\nu(\tau)\|_2^2+2\varkappa\|b(\tau)\|_2^2\}\,d\tau\leq\|v(0)\|_2^2+j\|\nu(0)\|_2^2, \tag{ε}$$

whenever $\{v(0),\nu(0)\}\in\mathcal{L}^2\times\mathbf{L}^2$. Here $b(\tau)$ is à màtrix of components $b_{ij}=$ $=(\partial/\partial x_j)v_i-\varepsilon_{mij}\nu_m$.

In this pàper we shall show that the weak solution lying in a space $\mathbf{L}^q((0,T);\mathcal{L}^p\times\mathbf{L}^p)$, where $2/p+2/q\leq 1$, and $p\geq 4$, satisfies an energy equa lity rather than the energy inequality (1).

The equations governing the behaviour of an incompressible micropolar fluid, in the absence of body force and body couple are

$$\dot v(t)+(v(t)\cdot\nabla)\,v(t)=(\mu+\varkappa)\nabla^2v(t)+\varkappa\nabla\times\nu(t)-1/\rho\,\nabla p(t), \tag{2}$$

$$j\dot\nu(t)+j(v(t)\cdot\nabla)\,\nu(t)=(\alpha+\beta)\,\nabla\nabla\cdot\nu(t)+\gamma\nabla^2\nu(t)+\varkappa\nabla\times v(t)-2\varkappa\nu(t), \tag{3}$$

$$\nabla\cdot v(t)=0, \tag{4}$$

where $v:t\to v(t)$ and $\nu:t\to\nu(t)$ are vector-functions with values in one of the function spàces defined below, representing velocity and microrotation respectively, and $p:t\to p(t)$ is a thermodynamic pressure ; μ, $\varkappa$, α, β and γ are the material constants. These equations must be supplemented by boundary and initial conditions. The usual boundary conditions can be written in the form

$$\{v(t),\ \nu(t)\}\ \in\mathscr{H}_0^{1,2}\times\mathbf{H}_0^{1,2},\quad t>0, \tag{5}$$

which implies thàt $v(t)=0$ and $\nu(t)=0$ on the boundary. In addition, we suppose :