$$Z_2 = N_2\left(1 + \frac{B\varepsilon}{2(1-A)}\right) + N_1\left(-\frac{1}{2A} + \frac{(1-A+B)\varepsilon}{2(1-A)^2} - \frac{B\varepsilon}{4A(1-A)}\right) +$$
$$+ N_0\left(\frac{\varepsilon}{8A^2} + \frac{(1-A+B)\varepsilon}{2(1-A)^3} - \frac{(1-A+B)\varepsilon}{4A(1-A)^2} + \frac{3B\varepsilon}{16A^2(1-A)}\right),$$
$$Z_3 = N_3\left(1 + \frac{B\varepsilon}{2(1-A)}\right) + N_2\left(-\frac{1}{2A} + \frac{(1-A+B)\varepsilon}{2(1-A)^2} - \frac{B\varepsilon}{4A(1-A)}\right) +$$
$$+ N_1\left(\frac{3}{8A^2} + \frac{(1-A+B)\varepsilon}{2(1-A)^3} - \frac{(4-A+B)\varepsilon}{1A(1-A)^2} + \frac{3B\varepsilon}{16A^2(1-A)}\right) +$$
$$+ N_0\left(-\frac{5}{16A^3} + \frac{(1-A+B)\varepsilon}{2(1-A)^4} - \frac{(1-A+B)\varepsilon}{4A(1-A)^3} + \right.$$
$$\left. + \frac{3(1-A+B)\varepsilon}{16A^2(1-A)^2} - \frac{5B\varepsilon}{32A^3(1-A)}\right).$$

By performing the inverse Laplace transform in (22) we find the solution of boundary value problem (7) (20)

$$U^*(X, \tau) = V_0\left\{H(\tau - k_1)\left[(\tau - k_1)S_0 + \frac{1}{2!}(\tau - k_1)^2 S_1 + \frac{1}{3!}(\tau - k_1)^3 S_2 + \right.\right.$$
$$\left. + \frac{1}{4!}(\tau - k_1)^4 S_3\right] e^{-(1-A)\omega_1\varepsilon} + \frac{1}{1-A} H(\tau - k_2)\left[(\tau - k_2)T_0 + \frac{(\tau - k_2)^2}{3} T_1 + \right.$$
$$\left.\left. + \frac{(\tau - k_2)^3}{3!} T_2 + \frac{(\tau - k_2)^4}{4!} T_3\right] e^{-k_2}\right\}$$

$$\theta^*(X_1 \tau) = -\frac{m\theta_0 V_0}{d}\left\{\frac{H(\tau - k_1)}{1-A}\left[(\tau - k_1)V_0 + \frac{(\tau - k_1)^2}{2!} V_1 + \frac{(\tau - k_1)^3}{3!} V_2 + \right.\right.$$
$$\left. + \frac{(\tau - k_1)^4}{4!} V_3\right] e^{-(1-A)\omega_1\varepsilon} - \frac{H(\tau - k_2)}{\sqrt{A}(1-A)}\left[(\tau - k_2)Z_0 + \frac{(\tau - k_2)^2}{2!} Z_1 + \right.$$
$$\left.\left. + \frac{(\tau - k_2)^3}{3!} Z_2 + \frac{(\tau - k_2)}{4!} Z_3\right] e^{-k_2}\right\}.$$

As in the Boley-Tollins problem, U and θ are continuous and there are two fronts of wave which arrive in any x. For $x \to \infty$, $U \to 0$ and $\theta \to 0$.

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AN ENERGY EQUATION IN THE THEORY OF MICROPOLAR FLUIDS

BY

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To the memory of Professor Mendel Haimovici

It is shown that a weak solution of the equations governing the behaviour of an incompressible micropolar fluid lying in a space $\mathbf{L}^q((0, T): \mathscr{L}^p \times \mathbf{L}^p)$ where $2/p + 2/q \leq 1$ and $p \geq 4$, sàtisfies an energy equality rather than the usual energy inequality.

1. Preliminary results. It is known [1] that a weak solution $\{v, \nu\}$ of the equàtions governing the behaviour of an incompressible micropolar fluid, exists satisfying the energy inequality

$$\| v(t) \|_2^2 + j \| \nu(t) \|_2^2 + \int_0^t \{2\mu \| \nabla v(\tau) \|_2^2 + 2(\alpha + \beta) \| \nabla . \nu(\tau) \|_2^2 + \tag{1}$$
$$+ 2\gamma \| \nabla \nu(\tau) \|_2^2 + 2\varkappa \| b(\tau) \|_2^2\} \, d\tau \leq \| v(0) \|_2^2 + j \| \nu(0) \|_2^2 ,$$

whenever $\{v(0), \nu(0)\} \in \mathscr{L}^2 \times \mathbf{L}^2$. Here $b(\tau)$ is à màtrix of components $b_{ij} = (\partial/\partial x_j)v_i - \varepsilon_{mij}\nu_m$.

In this pàper we shall show that the weak solution lying in a space $\mathbf{L}^q((0, T); \mathscr{L}^p \times \mathbf{L}^p)$, where $2/p + 2/q \leqq 1$, and $p \geqq 4$, satisfies an energy equality rather than the energy inequaiity (1).

The equations governing the behaviour of an incompressible micropolar fluid, in the absence of body force and body couple are

$$\dot{v}(t) + (v(t) . \nabla) v(t) = (\mu + \varkappa)\nabla^2 v(t) + \varkappa \nabla \times \nu(t) - 1/\rho \, \nabla p(t), \tag{2}$$

$$j\dot{\nu}(t) + j(v(t) . \nabla) \nu(t) = (\alpha + \beta) \nabla\nabla . \nu(t) + \gamma \nabla^2 \nu(t) + \varkappa \nabla \times v(t) - 2\varkappa \nu(t), \tag{3}$$

$$\nabla . v(t) = 0, \tag{4}$$

where $v : t \to v(t)$ and $\nu : t \to \nu(t)$ are vector-functions with values in one of the function spàces defined below, representing velocity and microrotation respectively, and $p : t \to p(t)$ is a thermodynamic pressure; μ, $\varkappa$, α, β and γ are the material constants. These equations must be supplemented by boundary and initial conditions. The usual boundary conditions can be written in the form

$$\{v(t), \nu(t)\} \in \mathscr{H}_0^{1,2} \times \mathbf{H}_0^{1,2}, \quad t > 0, \tag{5}$$

which implies thàt $v(t) = 0$ and $\nu(t) = 0$ on the boundary. In addition, we suppose:

$$v(0)=v^0, \qquad \mathrm{v}(0)=\mathrm{v}^0, \tag{6}$$

where v^0 and v^0 are given.

Definition. *A pair of functions* $\{v, \mathrm{v}\}: t \to \{v(t), \mathrm{v}(t)\}$ *is called a weak solution of the initial-boundary-value problem* (2)–(6) *if it belongs to* $\mathbf{L}^2((0,\infty)\,;\, \mathcal{H}_0^{1,2}\times \mathbf{H}_0^{1,2})\cap \mathbf{L}^{\infty}((0,\infty)\,;\,\mathcal{L}^2\times\mathbf{L}^2)$ *and satisfies*

$$\int_0^\infty \{(v(t)),\dot{u}(t)) + (v(t)\,.\,\nabla u(t),\, v(t)) + (\mu+k)\,(v(t),\,\nabla^2 u(t)) + $$
$$+\varkappa(\mathrm{v}(t),\,\nabla\times u(t))\}\,dt = -(v^0,\ u(0)),$$

$$\int_0^\infty \{j(\mathrm{v}(t),\ \dot{\psi}(t) + j(v(t)\,.\,\nabla\psi(t),\ \mathrm{v}(t)) + (\alpha+\beta)(\mathrm{v}(t),\,\nabla\nabla\cdot\psi(t)) +$$
$$+\gamma(\mathrm{v}(t),\,\nabla^2\psi(t)) + \varkappa(\mathrm{v}(t),\,\nabla\times\psi(t)) - \tag{7}$$
$$-2\varkappa(\mathrm{v}(t),\,\psi(t))\}dt = -j(\mathrm{v}^0,\,\psi(0)),$$

for all $u\in \mathbf{C}_0^1([0,\infty)\,;\,\mathcal{C}_0^\infty)$ *and* $\psi\in C_0^1([0,\infty)\,;\,\mathbf{C}_0^\infty)$.

In [1] it was shown that problem (2)–(6) has a weak solution whenever $\{v^0, \mathrm{v}^0\}\in\mathcal{L}^2\times\mathbf{L}^2$, and satisfying the energy inequality (1). Here C_0^∞ is the set of vector-functions defined in Ω (a domain in R^3), having continuous derivatives of all orders and being zero in a neighbourhood of $\partial\Omega$ (the smooth boundary of Ω), $\mathcal{C}_0^\infty$ is the set of vector-functions in $\mathbf{C}_0^\infty$ that are divergence free. $\mathbf{H}_0^{1,2}$ and $\mathcal{H}_0^{1,2}$ are the completions of $\mathbf{C}_0^\infty$ and $\mathcal{C}_0^\infty$, respectively, in the norm

$$\|u\|_H = \|\nabla u\|_2 \equiv \left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2}.$$

The space $\mathbf{C}_0^1([0,\infty)\,;\,\mathcal{C}_0^\infty)$ is the set of al functions $\varphi: t\to\varphi(t)$ defined and with continuous derivative for all t from $[0,\infty)$, with φ taking values in $\mathcal{C}_0^\infty$.

The weak solution has certain properties which were also given in [1]. Two of these, that we need in following, are:

Theorem 1. *The weak solution* $\{v, \mathrm{v}\}$ *of* (2)–(6) *is continuous as a function of* t *in the weak topology of* $\mathcal{L}^2\times\mathbf{L}^2$.

Theorem 2. *Let* $\{v^0, \mathrm{v}^0\}\in\mathcal{L}^2\times\mathbf{L}^2$ *and let* $\{v, \mathrm{v}\}$ *be a weak solution of* (2)–(6). *Then, after suitable redefinition of* $\{v, \mathrm{v}\}$ *on a set of values of* t *of one-dimensional measure zero, we have,*

$$(v(t),\ u(t)) + j(\mathrm{v}(t),\ \psi(t)) = (v^0,\ u(0)) + (\mathrm{v}^0,\ \psi(0)) +$$
$$+\int_0^t \{(v(\tau),\ \dot{u}(\tau)) - (v(\tau)\cdot\nabla v(\tau),\ u(\tau)) - (\mu+\varkappa)(\nabla v(\tau),\ \nabla u(\tau)) +$$
$$+\varkappa(\mathrm{v}(\tau),\ \nabla\times u(\tau)\}d\tau + \tag{8}$$
$$+\int_0^t \{j(\mathrm{v}(\tau),\ \dot{\psi}(\tau)) - j(v(\tau)\cdot\nabla\mathrm{v}(\tau),\ \psi(\tau)) - (\alpha+\beta)\,(\nabla\cdot\mathrm{v}(\tau)\ \nabla\cdot\psi(\tau)) -$$

$$-\gamma(\nabla\mathrm{v}(\tau),\ \nabla\psi(\tau)) - 2\varkappa(\mathrm{v}(\tau),\ \psi(\tau)) + \varkappa(\nabla\times v(\tau),\ \psi(\tau)\}d\tau,$$

for all $\{u,\psi\}\in\mathbf{C}_0^\infty([0,\infty)\,;\,\mathcal{C}_0^\infty\times\mathbf{C}_0^\infty)$, *and all* $t\geq 0$.

2. The energy equality. Our main result is

Theorem 3. *Let* $\{v^0, \mathrm{v}^0\}\in\mathcal{L}^2\times\mathbf{L}^2$, *and let* $\{v, \mathrm{v}\}$ *be a weak solution of* (2)–(6). *If* $\{v, \mathrm{v}\}\in\mathbf{L}^q((0,T)\,;\,\mathcal{L}^p\times\mathbf{L}^p)$ *where* $p\geq 4$ *and*

$$2/p + 2/q \leq 1 \tag{9}$$

then $\{v, \mathrm{v}\}$ *satisfies the energy equality*

$$\|v(t)\|_2^2 + j\,\|\mathrm{v}(t)\|_2^2 +$$
$$+\int_0^t \{2\mu\,\|\nabla v(\tau)\|_2^2 + 2(\alpha+\beta)\,\|\nabla\cdot\mathrm{v}(\tau)\|_2^2 + 2\gamma\,\|\nabla\mathrm{v}(\tau)\|_2^2 + \tag{10}$$
$$+2\varkappa\,\|b(\tau)\|_2^2\}d\tau = \|v(0)\|_2^2 + j\,\|\mathrm{v}(0)\|_2^2, \quad 0\leq t\leq T.$$

Proof. First, notice that we may assume $T<\infty$. When $t=0$, (10) is true because of (8). Let t_0 be fixed, $0<t_0\leq T$. Let

$$(k_\varepsilon\ast\varphi)\,(t) = \int_0^{t_0} k_\varepsilon(t-\tau)\varphi(\tau)d\tau,\quad (l_\varepsilon\ast\psi)(t) = \int_0^{t_0} l_\varepsilon(t-\tau)\psi(\tau)d\tau,$$

be the mollifiers, so that k_ε, l_ε are $\mathbf{C}^\infty$, real-valued, nonnegative functions supported in $[-\varepsilon,\varepsilon]$, and integrating to unity. Let $(\{v^i, \mathrm{v}^i\})\subset\mathbf{C}_0^\infty([0,\infty)\,;\,\mathcal{C}_0^\infty\times\mathbf{C}_0^\infty)$ be a sequence converging to $\{v, \mathrm{v}\}$ in $\mathbf{L}^2((0,T)\,;\,\mathcal{L}^2\times\mathbf{L}^2)\cap\mathbf{L}^2((0,T)\,;\,\mathcal{H}_0^{1,2}\times\mathbf{H}_0^{1,2})\cap\mathbf{L}^q((0,T)\,;\,\mathbf{L}^p\times\mathbf{L}^p)$. Such à sequence exists because $\{v, \mathrm{v}\}$. is a weak solution and the assumption $T<\infty$. Set $t=t_0$ and $u=k_\varepsilon\ast v^i$, $\psi=l_\varepsilon\ast\mathrm{v}^i$ in (8). One obtains

$$\int_0^{t_0} k_\varepsilon(t_0-t)(v(t_0),\ v^i(t))dt + j\int_0^{t_0} l_\varepsilon(t_0-t)(\mathrm{v}(t_0),\ \mathrm{v}^i(t))dt =$$
$$=\int_0^{t_0} k_\varepsilon(-t)(v^0,\ v^i(t))dt + \int_0^{t_0} l_\varepsilon(-t)(\mathrm{v}^0,\ \mathrm{v}^i(t))dt +$$
$$+\int_0^{t_0}\int_0^{t_0} \dot{k}_\varepsilon(t-\tau)(v(t),\ v^i(\tau))d\tau\,dt + j\int_0^{t_0}\int_0^{t_0} \dot{l}_\varepsilon(t-\tau)(\mathrm{v}(t),\ \mathrm{v}^i(\tau))d\tau\,dt -$$
$$-\int_0^{t_0}\int_0^{t_0} k_\varepsilon(t-\tau)\,\{(v(t)\cdot\nabla v(t),\ v^i(\tau)) + (\mu+\varkappa)(\nabla v(t),\ \nabla v^i(\tau)) -$$
$$-\varkappa(\mathrm{v}(t),\ \nabla\times v^i(\tau))\}\,d\tau\,dt - \int_0^{t_0}\int_0^{t_0} l_\varepsilon(t-\tau)\{j(v(t)\cdot\nabla v(t),\ \mathrm{v}^i(\tau)) +$$

$$+(\alpha+\beta)(\nabla\cdot v(t),\ \nabla\cdot v^{i}(\tau))+\gamma(\nabla v(t),\ \nabla v^{i}(\tau))-\varkappa(\nabla\times v(t),\ v^{i}(\tau))+$$
$$+2\varkappa(v(t),\ v^{i}(\tau))\}\,d\tau\,dt.$$

In this formula, i must tend to infinity. The only term that gives some trouble is the nonlinear one, but adopting for our use the calculation in [3], one obtains

$$\left|\int_0^{t_0}\int_0^{t_0} k_\varepsilon(t-\tau)(v(t)\cdot\nabla v(t),\ v^{i}(\tau)-v(\tau))\,d\tau\,dt\right|\leq$$
$$\leq a\int_0^{t_0}\|v(t)\|_p\,\|\nabla v(t)\|_2\,dt\int_0^{t_0}\|v^{i}(\tau)-v(\tau)\|_2^{2-q/2}\,\|v^{i}(\tau)-v(\tau)\|_p^{q/2-1}d\tau,$$
$$\left|\int_0^{t_0}\int_0^{t_0} l_\varepsilon(t-\tau)\ (v(t)\cdot\nabla v(t),\ v^{i}(\tau)-v(\tau))d\tau\,dt\right|\leq$$
$$\leq b\int_0^{t_0}\|v(t)\|_p\,\|\nabla v(t)\|_2\,dt\int_0^{t_0}\|v^{i}(\tau)-v(\tau)\|_2^{2-q/2}\,\|v^{i}(\tau)-v(\tau)\|_p^{q/2-1}d\tau,$$

where a, b are positive constants. Here, we may reduce q to obtain a pair (p, q) satisfying $2/p+2/q=1$. The first integrals on the right hand side of these formulas are bounded by $\|v\|_{p,2}\,\|v\|_{2,2}$ and $\|v\|_{p,2}\,\|v\|_{2,2}$ respectively, which are finite since $q\geq 2$ and $T<\infty$. The second integrals do not exceed a constant time $\|v^{i}-v\|_{2,2}^{2-q/2}\cdot\|v^{i}-v\|_{p,q}^{q/2-1}$ and $\|v^{i}-v\|_{2,2}^{2-q/2}\,\|v^{i}-v\|_{p,q}^{q/2-1}$ respectively, and these go to zero. Thus, we obtain

$$\int_0^{t_0} k_\varepsilon(t_0-t)(v(t_0),\ v(t))dt+j\int_0^{t_0} l_\varepsilon(t_0-t)(v(t_0),\ v(t))dt=$$
$$=\int_0^{t_0} k_\varepsilon(-t)(v^0,\ v(t))dt+\int_0^{t_0} l_\varepsilon(-t)(v^0,\ v(t))dt-$$
$$-\int_0^{t_0}\int_0^{t_0} k_\varepsilon(t-\tau)[(v(t)\cdot\nabla v(t),\ v(\tau))+(\mu+\varkappa)(\nabla v(t),\ \nabla v(\tau))]d\tau\,dt+$$
$$+\int_0^{t_0}\int_0^{t_0} k'_\varepsilon(t-\tau)(v(t),\ v(\tau))d\tau\,dt+j\int_0^{t_0}\int_0^{t_0} l'_\varepsilon(t-\tau)(v(t),\ v(\tau))d\tau\,dt-$$
$$-\int_0^{t_0}\int_0^{t_0} l_\varepsilon(t-\tau)\{j(v(t)\cdot\nabla v(t),\ v(\tau))+(\alpha+\beta)(\nabla\cdot v(t),\ \nabla\cdot v(\tau))+ \tag{11}$$

$$+\gamma(\nabla v(t),\ \nabla v(\tau))+2\varkappa(v(t),\ v(\tau))\}d\tau\,dt+$$
$$+\varkappa\int_0^{t_0}\int_0^{t_0} k_\varepsilon(t-\tau)(v(t),\ \nabla\times v(\tau))d\tau\,dt+$$
$$+\varkappa\int_0^{t_0}\int_0^{t_0} l_\varepsilon(t-\tau)\ (\nabla\times v(t),\ v(\tau))\ d\tau\,dt.$$

The terms here involving the derivatives of k_ε and l_ε vanish if k_ε and l_ε are chosen to be even. By letting ε go to zero in the remaining terms in (11), and using the similar arguments as in [2] we get

$$\|v(t_0)\|_2^2+j\,\|v(t_0)\|_2^2+2\mu\int_0^{t_0}\|\nabla v(t)\|_2^2dt+$$
$$+2(\alpha+\beta)\int_0^{t_0}\|\nabla\cdot v(t)\|_2^2\,dt+2\gamma\int_0^{t_0}\|\nabla v(t)\|_2^2dt+$$
$$+2\varkappa\int_0^{t_0}\|b(t)\|_2^2\,dt=\|v^0\|_2^2+j\,\|v^0\|_2^2,$$

which is (10) for $t=t_0$. Since t_0 is arbitrary, the proof of theorem 3 is complete.

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