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NEW CONSIDERATIONS CONCERNING THE CAUCHY TYPE PROBLEM IN THE NONLINEAR BENDING OF A STRAIGHT BAR

BY

P. P. TEODORESCU and I. TOMA

To the memory of Professor Mendel Haimovici

The Bernoulli-Euler equation represents a model for the nonlinear deformations of the elastic bar i.e.

$$\frac{d^2y}{dx^2} = f(x)\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}, \quad x \in [0, l]. \tag{1}$$

Here l represents the bar length and f is given by

$$f(x) = -\frac{M(x)}{E(x)I(x)}, \tag{2}$$

where M is the bending moment, and EI is the flexional rigidity (E — the elasticity modulus, I — the moment of inertia of the cross section).

Solving (1) under initial homogeneous conditions

$$y(0) = y'(0) = 0 \tag{3}$$

results in the obtaining of the deflection in the case of the cantilever bar, while the bilocal conditions correspond to the simply supported bar.

To build up a general process of calculus of the deflection under both types of conditions, a finite analytic expression of the solution of (1) is needed. As far as it is known, this is possible only for constant f.

Numerical methods do not lead to a satisfactory result in this direction, as they essentially depend on the conditions and consequently do not allow qualitative interpretations of the results.

Therefore, in order to obtain a good — even if approximating — representation formula for the solutions of (1), we firstly introduced a new variable and made a function change that led to an invariant over the Bernoulli-Euler class equation. The solution of this equation allowed a rapidly convergent expansion with respect to the newly introduced variable. However, getting back to the solution of (1) involved restrictions on f, besides smoothness — which is obviously ensured by physical reasons. Namely, we succeeded in establishing [1] a good approximating representation for functions f belonging to the class

$$\mathcal{G} = \{f \in C^\infty([0, l]) \mid \exists h = \mathcal{D}f, \ \mathcal{D}h^k, \ \forall k \in \mathbf{N}\}, \tag{4}$$

$\mathcal{D}f$ representing the primitive of f. Besides the existence, these primitives must allow finite analytic expressions, in order to ensure easy computation.

In what follows, we shall enlarge the class of Bernoulli-Euler equations whose solutions allow a general representation formula as mentioned above, and apply it in the case of the cantilever bar.

This will be performed by using the linear equivalence method for polynomial operators, introduced and developed in [2], [3] for instance. In order to apply this method, let us make in (1) a change of function

$$z = \frac{dy}{dx}, \quad u = \sqrt{1+z^2}, \tag{5}$$

that leads to the system

$$\begin{cases} \dfrac{du}{dx} = fzu^2, \\ \dfrac{dz}{dx} = fu^3, \end{cases} \tag{6}$$

whose solutions obviously satisfy

$$u(0) = 1\ , \ z(0) = 0. \tag{7}$$

The nonlinear system (6) is of *polynomial type* (for the general definition of a polynomial operator see [4]).

The representation of the solution of (6) is the consequence of a more general result, valid for *polynomial systems* [3].

Indeed, let us consider *the polynomial initial problem*

$$\begin{cases} \mathbf{w}'(x) = \mathbf{P}(x, \mathbf{w}), \ \mathbf{w} \in (C^1([a, b]))^n, \\ \mathbf{P}(x, \mathbf{w}) = [P_j(x, \mathbf{w})]_{1\leq j\leq n}, \ P_j(x, \mathbf{w}) = \sum\limits_{|\nu|\leq p_j} a_{j\nu}(x)\mathbf{w}^\nu, \\ \qquad \nu \in (\mathbf{N} \cup \{0\})^n, \ p_j \in \mathbf{N} \cup \{0\}, \end{cases} \tag{8}$$

$$\mathbf{w}(a) = \mathbf{w}_0. \tag{8'}$$

Consider the mapping $v(x, \xi) = \exp\,(<\xi, \mathbf{w}>)$, $\xi \in \mathbf{R}^n$, that transforms (6) into the linear PDE

$$\frac{\partial v}{\partial x} = <\xi, \mathbf{P}(x, \mathbf{D})> v, \tag{9}$$

and the initial conditions (8′) into

$$v(a, \xi) = \exp\,(<\xi, \mathbf{w}_0>). \tag{9'}$$

It is important to notice that the linear problem (9), (9′) fully represents the polynomial problem (8), (8′), and is *not* its approximation.

More precisely, it was emphasized in [2] that v establishes an one-to-one correspondence between the set of solutions of the nonlinear problem (8), (8′) and the set of analytic with respect to ξ solutions of (9), (9′). In other words, the polynomial problem (8), (8′) is equivalent to the linear problem (9), (9′) modulo analiticity of v. This last condition yields for v the expansion $v(x, \xi) = \sum\limits_{|\nu|\in \mathbf{N}\cup\{0\}} v_\nu(x)\,\xi^\nu/\nu!$, $v_0(x) = 1$, and by thus (9) leads to the linear infinite system

$$\frac{d\mathbf{V}}{dx} = \mathbf{A}\mathbf{V}, \ \mathbf{V} = [v_\nu(x)]_{\nu\in\mathbf{N}}, \tag{10}$$

to be solved under the (initial) conditions

$$\mathbf{V}(a) = [\mathbf{w}_0^\nu]_{\nu\in\mathbf{N}}. \tag{10'}$$

The associate infinite matrix $\mathbf{A}$ depends only on the coefficients $a_{j\nu}(x)$ and is row-and column-finite.

With these preparations we may enounce the following existence result:

Proposition [3]. *The linear problem* (10), (10′) *allows the following expansion of the solution*

$$\mathbf{V}(x) = \sum_{k\geq 0} \mathbf{A}_k(a)\frac{(x-a)^k}{k!}\mathbf{V}(a) \equiv \Pi(x, a)\,\mathbf{V}(a), \tag{11}$$

where $\mathbf{A}_k$ *are obtained by recurrence*

$$\mathbf{A}_k = \frac{d\mathbf{A}_{k-1}}{dx}\bigg|_{x=a} + \mathbf{A}_{k-1}\mathbf{A}\bigg|_{x=a}, \quad \mathbf{A}_0 = \mathbf{I}. \tag{12}$$

The first components of $\mathbf{V}$ *in* (11) *represent the Taylor series development arround* a *of the solution of the polynomial initial problem* (8), (8′). *The expansion* (11) *holds at least on the interval*

$$|\,x-a\,| < 1/cpQ, \tag{13}$$

where $c = \max\limits_{\substack{1\leq j\leq n\\ |\nu|\leq p_j}} \{\sup\,|\,a_{j\nu}(x)|\}$, $p = \max\limits_{1\leq j\leq n}\{p_j\}$, $Q = \sum\limits_{|\nu|\leq p}|\,y_0^\nu|$.

We shall not give the proof here ; this was already done in [3].

Remark. The first n lines of the matrix $\Pi(x, a)$ represent in a way a local inverse for the system (8), independent on the initial conditions (8′). Therefore, it will allow to build up a *transport matrix* for the solution over a choosen interval, provided regularity holds, taking steps such that (13) be satisfied, and considering the first n lines of $\Pi(x, a)$, successively calculated at the points x_k, $a < x_1 < x_2 < \cdots < x_k < \cdots < x_s = b$. This matrix is a characteristic of the equation only, as it depends only on the coefficients a_j and is independent on $\mathbf{w}_0$.

Let us now apply this proposition to the system (6). In this case, the corresponding linear PDE by the transformation $v(x, \sigma, \xi) = \exp\,(\sigma u + \xi w)$ will be

$$\frac{\partial v}{\partial x} = f(x)\left(\sigma\frac{\partial^3 v}{\partial\sigma^2\partial\xi} + \xi\frac{\partial^3 v}{\partial\sigma^3}\right). \tag{14}$$

and, consequently, the matrix **A** is of the form

$$(15)\qquad \mathbf{A} = f\begin{bmatrix} 0 & \mathbf{B}_1 & 0 & 0\dots \\ 0 & 0 & \mathbf{B}_3 & 0\dots \\ \dots & \dots & \dots & \dots \end{bmatrix},$$

$$\mathbf{B}_{2j-1} \in \mathcal{M}(2j, 2j+2),\ \mathbf{B}_{2j-1} = [k\delta^l_{k+1} + (2j-l)\,\delta^{l+1}_k]_{\substack{1 \leqslant l \leqslant 2j \\ 1 \leqslant k \leqslant 2j+2}},$$

δ^l_i being the Kronecker delta.

For direct applications (by means of programming), the most interesting and important types of f were treated, for a cantilever steel bar of 100 cm of length and of rectangular cross section, acted upon by a concentrated moment, respectively a concentrated force at the free end.

TABLE I

Case	$f = -\dfrac{M}{EI}$	Values of γ	Results by linear equivalence method	Results by using the invariant equation
1	$\dfrac{\gamma x + 1}{\alpha x + \beta}$ cross section of linearly varying width $\alpha = -1.3125$ $\beta = 393.75$ cm	1.1 $\gamma = 0$ const. bending moment	$u_l = 1.66$ cm $y(100 - u_l) = 14.21$ cm $\operatorname{tg}\theta = 0.3168$	$u_l = 1.46$ cm $y(100 - u_l) = 14.28$ cm $\operatorname{tg}\theta = 0.3183$
		1.2 $\gamma = -0.01$ linearly varying bending moment	$u_l = 0.57$ cm $y(100 - u_l) = 9.26$ cm $\operatorname{tg}\theta = 0.1453$	$f \notin \mathcal{G}$
2	$\dfrac{\gamma x + 1}{(\alpha x + \beta)^3}$ cross section of linearly varying height $\alpha = -3.201 \times$ $\times 10^{-2}$ cm$^{2/3}$ $\beta = 9.603$ cm$^{1/3}$	2.1 $\gamma = 0$ const. bending moment	$u_l = 0.41$ cm $y(100 - u_l) = 8.39$ cm $\operatorname{tg}\theta = 0.2118$	$u_l = 0.55$ cm $y(100 - u_l) = 8.45$ cm $\operatorname{tg}\theta = 0.2143$
		2.2 $\gamma = -0.01$ linearly varying bending moment	$u_l = 0.18$ cm $y(100 - u_l) = 5.06$ cm $\operatorname{tg}\theta = 0.0862$	$f \notin \mathcal{G}$
3	$c(\gamma x + 1)$ constant cross section $c = 3.809 \times$ $\times 10^{-5}$ cm^{-1}	3.1 $\gamma = 0$ const. bending moment	$u_l = 2.88$ cm $y(100 - u_l) = 18.64$ cm $\operatorname{tg}\theta = 0.3979$	$u_l = 2.78$ cm $y(100 - u_l) = 18.67$ cm $\operatorname{tg}\theta = 0.3981$
		3.2 $\gamma = -0.01$ linearly varying bending moment	$u_l = 1.01$ cm $y(100 - u_l) = 12.60$ cm $\operatorname{tg}\theta = 0.1897$	$u_l = 0.97$ cm $y(100 - u_l) = 12.66$ cm $\operatorname{tg}\theta = 0.1937$

The nonlinear problem naturally involves the calculus of $u_l = \frac{1}{2}\int_0^l y'^2\,dx$, and $100 - u_l$ represents the final point at which the deflection has to be determined. As by the above exposed lineari equiva'en e method, representations for both y and $\operatorname{tg}\theta = -\dfrac{dy}{dx}$ are obtained, the calculus of u_l was performed without difficulties.

In all the enlisted in table 1 cases, corresponding to the numerical cases considered in [1], the construction of an approximate of Π up to the 6th order only was necessary to ensure a relative error of 10^{-3} order, which corresponds to 10^{-2} cm of absolute error. The comparison with the results obtained by using the invariant equation method (see [1]), provided $f \in \mathcal{G}$, — which is true for the point 3 and only for $\gamma = 0$ at the points 1 and 2 in table 1 — emphasized that the convergence in $h = \mathcal{D}f$ is not sensitively better that the convergence in x, and that considering in the linear equivalen e method an 8th order of approximation only, we should obtain the same error. The seresults were obtained by using an IBM 360 computer.

The point 3.1 corresponds to a constant f, therefore comparison with the exact solution can be made. These exact solution values are: $u_l = 2.66^c$ cm, $y\,(100 - u_l) = 18.71$ cm, $\operatorname{tg}\theta = 0.39928$.

Therefore, the representation formula obtained by linear equivalence is sufficiently accurate.

These results put into evidence the possibility of studying cases of charge and flexional rigidity, i.e. forms of f which do not belong to the class $\mathcal{G}$ and conse juently cannot be treated by using the variable $h = \mathcal{D}f$.

REFERENCES

1. Teodorescu, P. P., Toma, I. — *On the Cauchy Type Problem in the Non-Linear Bending of a Straight Bar.* Mechanics Research Communications, 9 (1982) — to appear.
2. Toma, I. — *On Polynomial Differential Equations.* Bull. Soc. Sci. Math. de la R.S.R. 24 (72), nr. 4, 1981.
3. Toma I. — *Extensions of a Linearization method for Polynomial Operators.* Rev. Roum. de Mat. Pures et Appl., 27 (1982) — to appear.
4. Rall, L. B. — *Solutions for Abstract Polynomial Equations by Iterative Methods.* MRC Report # 892, August, 1968.

Faculty of Mathematics
University of Bucharest
and
ICPE, Bucharest