Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Supliment la tomul XXVII, s. I a, 1981

ON DEFORMATIONS IN RIEMANNIAN CATEGORY

BY

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To the memory of Professor Mendel Haimovici

In this autumn we have commemorated the 75^{th} birthday of M e n d e l H a i m o v i c i, one of the great mathematicians of Romania. From the begining I should say that professor M. Haimovici was a brilliant geometer, his important contributions being mainly in modern domains of differential geometry : nonholonomic spaces, spaces with affine connection, integral geometry, Finsler spaces, Lie groups and pseudo-groups of transformations.

Also, his original work in mechanics is well known by the mathematical world. The elasticity school from Iaşi is—in fact — a creation due to M. Haimovici.

On the other hand, Prof. M. Haimovici had a valuable contribution in a crucial mathematical problem of our century. He created an algebraic and analytical theory for the study of involutive and noninvolutive differential systems. It is known that even É. Cartan and M. Kuranishi, two proeminent personalities in the field, have studied fo rmany years especially the noninvolutive case. Just in this case M. Haimovici, in the last years of his life, has created a geometrical theory of exterior differential sytems sketched in three Notes in *Comptes Rendus*. His methods is also suitable for the study of prolongations of differential systems.

M. Haimovici does not make any difference between mathematics and mechanics. For instance, the study of nonholonomic mechanical systems, the geometry of Vrănceanu spaces, and the theory of partial differential equations are only some aspects of the same mathematical problem. For this reason, we think that his notes in *Comptes Rendus* must be finished, both in the direction of geometrical structures of superior order and new mechanical nonholonomic systems. In this way we shall bring an homage to the great scientist.

Finally, we mention that this paper is dedicated to the memory of Professor Mendel H a i m o v i c i.

2. We begin with an historical view on the appearance of the notion of deformation in science and on some topical interest of it in mathematics.

It is known that in antiquity the architects used only empirical tables based on a long practice. Only in time of Renaissance, when there have been constructed monumental buildings, appeared the necessity of some mathematical criteriums on the resistence of materials. They are related to the name of two great scientists : L e o n a r d o d a V i n c i and G.

G a l i l e i. In the history of science it is mentioned that elasticity appeared at the same time with the „basis problem" of G a l i l e i [5]. But he did not succed to solve it since for him both, beams and columns were immutable. Only the English scientist R. H o o k e established in 1660 the first criterium on deformations. He found the existence of a linear relation between the deformation of the length of a thin beam and its tensions. Hook's law has been generalized by A. C a u c h y in 1822. It appears now as a linear relation between tensors of deformations and tension tensor. He also found the two elastical constants for the case of isotropic medium. Later, in 1837 G. G r e e n has generalized this theory to anisotropic medium and conjectured that there are no more than 21 constants. This was proven by M. B o r n in 1914. Finally, I have to mention that nonlinear relations between tensions and graduated deformations have been established in the second part of this century.

Nowadays a deformation appears in any field of mathematics where are investigated families of objects which depend on a parameter, or some structures, or spaces and manifolds, or morphisms. Thus we have deformations of analytical structures and more general deformations of pseudogroup structures. In particular, there are deformations of complex manifolds and of their submanifolds, deformations of algebraic varieties, deformations of associative linear algebras but mostly deformations of classical Lie algebras. Taking into account their mechanical source, we found here: theorems of rigidity, theorems of infinitesimal deformations and others. A typical example of deformation in topology is the homotopy theory. By means of this theory two continuous mappings of a domain in a topological space are called homotopical if one of them is obtained from another by continuous deformations

The theory of deformations in geometry appears even in the classical theory of curves and surfaces which is as long as differential geometry is. In fact, roughly speaking locally, a curve in any space is nothing but a continuous deformation, (with some regularity conditions, of a portion of a line. In a similar way a surface appears locally as a continuous deformation of a domain in R^2 and more general, a manifold as a continuous deformation of a domain of its model.

3. In the sequel we are concerned only with deformations in differential geometry. This theory is right connected with infinitesimal motions in Riemannian spaces and with affine connections. Such kind of researches have appeared in the last decades of the 19th century and they are stil interesting in our days. We give here only few examples: the intrinsic theory of deformations (E. Cesaro), infinitesimal deformations of surfaces (A Voss), deformations of remarkable families on surfaces (J.N. Hatzidakis) deformations of superior order (G. Ţiţeica) or infinitesimal transformations of surfaces in centro-affine and projective spaces (O. Mayer). Many notes and memoirs which have been devoted to deformations of geometrical objects in spaces with fundamental group have been put together in a treatise of G. Darboux and in our century in the celebrated *Lezioni di Geometria differenziale* of L. Bianchi. In the last decade of the 19 th century, S. Lie (1893) and W. Killing (1895) investigated finite and infinitesimal motions in geometry of Riemannian spaces. They studied transformations for which the metric of these spaces is invariant and obtained the well known from that time „Killing equations and vectors". But a great step in the study of deformations has been made by S. Lie whose method is suitable for the study of deformations in any space. Obviously we refered here to the operator called *Lie derivative*.

The appearance of Ricci calculus and of Levi-Civita connection should also be considered of great importance for the theory of geometrical deformations. From now on there are studied proiective and affine transformations which are more general than isometries. Many papers have been concerned with problems of classifications of Riemannian spaces with respect to their fundamental group and spaces with affine connection. In this direction we have to quote papers of L. Bianchi, G. Fubini, E. Cartan, J. Schouten, W. Slebodzinski L. Eisenhart and in our country the papers of G. Vrănceanu, M. Haimovici, A Haimovici, R. Miron, C. Teleman, V. Cruceanu V. Oproiu, M Craioveanu, Gh. Gheorghiev.

Infinitesimal deformations of submanifolds in manifolds endowed with a geometrical structure (Riemannian structure, Kählerian structure, Sasakian structure) have been studied intensively by K. Yano, B.Y. Chen, A. Bejancu, N. Papaghiuc.

4. The geometrical school of Iassy has an important contribution to the study of deformations in Riemannian spaces and G-structures on a manifold. The following fields of interest have been intensively studied: 1) graduated deformations of the fundamental metric tensor: deformations of the Levi-Civita connection and of volum element [2, 3]; 2) the finite and infinitesimal variation of the metric; 3) deformations of geometrical objects along a C^∞ mapping between Riemannian spaces; 4) some generalizations of this theory to some Riemannian fibre bundles and to classical structures on a finite or infinite dimensional differentiable manifold [4].

In the next section we shall make a survey on the theory of deformations in Riemannian geometry. The main purpose is to setle the ideas of theory and to avoid the technical parts of the problems.

5. Let (M, g) be a C^∞ Riemannian manifold. We denote by TM (resp. T^*M) the tangent bundle (resp. the cotagent bundle) to M. The Levi-Civita connection on M is denoted by ∇ and the Riemannian metric on T^*M by g^*.

We start with the deformation of the metric g and of the volume element $(\det g)^{1/2}$ with respect to a vector field $\vec{v} \in \mathscr{X}(M)$. This deformation is given by [1].

$$\mathcal{L}_{\vec{v}}\, g = g_1(\vec{v})\ ;\ \mathcal{L}_{\vec{v}}(\det\ g)^{1/2} = \operatorname{div}\ \vec{v}(\det g)^{1/2} \tag{1}$$

where $\mathcal{L}_{\vec{v}}$ is the Lie derivative with respect to $\vec{v}$, $g_1(\vec{v})$ and $\operatorname{div} \vec{v} = \,<g^*, g_1(\vec{v})>$ are the infinitesimal deformation of the metric and respectively of the volume element [1]. This process can go further, by applying again the Lie derivative with respect to $\vec{v}$. Thus we have the following sequence of symmetrical tensors of type (0, 2).

$$\mathcal{L}_{\vec{v}}\, g_{\alpha-1}(\vec{v}) = g_\alpha(\vec{v})\quad \alpha = 1, 2, 3, \ldots, \text{ where } g_0(\vec{v}) = g. \tag{2}$$

Interesting deformations are obtained when $g_\alpha = 0$ or $g_\alpha = \lambda g_\beta$ where $(\beta < \alpha)$. If for exemple, we have $g_1 = 0$ or $g_1 = \lambda g$, we obtain isometrical infinitesimal deformations or respectively conformal infinitesimal deformations spanned by the vector field $\vec{v}$. If the following inequality holds [2]

$$(3) \qquad \left\| g_1(\vec{v}) - \frac{\operatorname{div} v}{\dim M} g \right\| < \text{constant} \;\forall\; x \in U \subset M,$$

where $\|\,.\,\|$ is the tensorial norm and U is a convex domain homeomorphic to a ball from the model space, then we obtain quasiconformal infinitesimal transformations. In a similar way, we obtain for $\alpha > 1$ isometrical and conformal transformations of order α. Now, by applying the operator ∇ to the sequence $g_\alpha(\vec{v})$ we obtain a sequence of tensor fields of type $[0, 3]$ given by

$$(4) \qquad \nabla g_\alpha(\vec{v}), \quad \alpha = 1, 2, \ldots$$

where the first term of the sequence is just the Lie derivative of the connection ∇ with respect to the vector field $\vec{v}$. Hence (4) measures the graduated deformations of the connection. When the sequence stops, we have also some special deformations. Thus for $\alpha = 1$ we obtain the affine deformations and the projectiv transformations on M spanned by $\vec{v}$. Therefore, when $\alpha > 1$ we obtain affine or projective transformations of order α.

Let us take other two sequences of tensors obtained by suitable contractions given by

$$(5) \qquad \operatorname{div}_\alpha \vec{v} = C_2(g^* \odot g_\alpha(\vec{v})), \quad \operatorname{div} g_\alpha(\vec{v}) = C_2(g^* \odot \nabla g_\alpha(\vec{v})) = \Delta_\alpha \vec{v}.$$

It is easily seen that for $\alpha = 1$ from the first relation, we obtain $\operatorname{div} \vec{v}$ and from the second we have $\operatorname{div} g_1(\vec{v}) = \Delta \vec{v}$ which is the Laplace operator on a Riemannian manifold. Here there are also interesting some particular cases. Thus, since for $\alpha = 1$, $\operatorname{div}_1 \vec{v} = \operatorname{div} \vec{v} = 0$ implies the existence of unimodular infinitesimal deformations, we say that for $\alpha > 1$ we have similar deformations of order α.

When we have

$$(6) \qquad \Delta_\alpha \vec{v} = 0 \text{ or } \Delta_\alpha \vec{v} = -\nabla \varphi_\alpha,$$

we say that we have harmonical infinitesimal transformations or respectively almost harmonical transformations of order $\alpha > 1$ [3].

6. Some applications. If in particular, $\dim M = 3$ and the Riemannian structure defines the structure model space for the mechanics of continuous mediums in sense of Noll [6, 7, 8], then $g_\alpha(\vec{v})$ define just the sequence of Rivlin-Ericksen tensors for the deformations of velocity of a non-stationary motion.

It is known that for materials of differential type, the tension tensor is given by a polynomial in $g_\alpha(\vec{v})$ as follows [7].

$$(7) \qquad T = -pg + \alpha_1 g_1(\vec{v}) + \beta_1 g_2(\vec{v}) + \beta_2 [g_1(\vec{v})]^2 + \gamma_1 g_3(\vec{v}) + \ldots$$

For the case of incompressible mediums the pressure is an arbitrary function and the other coefficients are constant.

Therefore, the tension of a continuum medium depends only on pressure, of the metric of the model space and of its successive deformations with respect to the direction of the motion.

For the case of fluids of differential type, the universal deformations of an arbitrary order are given by means of almost harmonical infinitesimal transformations given by the sequence

$$(6_2) \qquad \Delta_\alpha \vec{v} = -\nabla \varphi_\alpha.$$

More precisely, for the case of an incompressible fluid we get universal solutions of zero degree if we have $\dfrac{d\vec{v}}{dt} = \nabla f_0$ (Euler equation); universal solutions of first degree if besides of this we have $\Delta \vec{v} = \Delta f_1$ (Navier-Stokes equations); universal solutions of second degree if besides of them we have $\operatorname{div} g_2(\vec{v}) = \Delta_2 \vec{v} = \nabla f_2$ (Reiner-Rivlin equations).

In particular, for incompressible fluids of Navier-Stokes type the universal deformations are characterized by the following: the acceleration of the motion is a potential and its velocity is almost harmonical.

Remark. It is an open question the mechanical significance for the terms of the sequence (5_1) and the significance of their vanishing for $\alpha > 1$.

7. The purpose of this section is to study Riemannian morphisms, that is, mappings of class C^∞, $f_0 : (M, g) \to (M', g')$ where M, M' are two Riemannian manifolds of class C^∞. Next, we consider a morphism of vectorial bundles $f : V(M) \to V(M')$ defined by $\pi : V(M) \to M$, $\pi' : V(M') \to M'$ such that $\pi' \circ f = f_0 \circ \pi$. We recall that the Riemannian vector bundle $V(M) \xrightarrow{\pi} (M, g)$ has as fiber a vectorial space E endowed with an inner product γ, which by means of the isomorphism $\theta_x : (E, \gamma) \to V_x = \pi^{-1}(x)$ defines on $V(M)$, a Riemannian metric denoted also by γ. It is known that on $V(M)$ there exists a Riemannian connection D induced by the metric γ that is, $D\gamma = 0$. We need also the induced vector bundle $f_0^*(V(M')) = M \times \operatorname{Im}(f_0)$ and the notion of comorphism of vector bundles expressed by the following commutative diagram

$$(8) \qquad \begin{tikzcd} & f_0^*(V(M')) \arrow[r] \arrow[ld, "j"'] \arrow[d, "f_0^*\pi'"] & V(M') \arrow[d, "\pi'"] \\ V(M) \arrow[r, "\pi"'] & M \arrow[r, "f_0"] & M' \end{tikzcd}$$

Here, the mapping j is the f_0-comorphism of $V(M')$ in $V(M)$ which for each $x \in M$ establishes the continuous linear mapping $j_x : [V(\operatorname{Im}(f_0))]_{x f_0(x)} \to (V(M))_x$ between local fibres.

In particular, when we take the tangent bundles TM and TM', this mapping is called the pull-back by f_0 of the tangent bundle TM' and the sections of $f_0^*\pi'$ a vector field „along f_0". In a similar way there are obtained: tensor fields, Levi-Civita connections and Lie derivatives „along f_0".

Let $\vec{w}$ be a vector field along f_0 which defines an one parametrical family of morphisms $(f_0)_t$ such that for $t=t_0$ we have $(f_0)_{t_0}=f_0$. Every morphism is defined by

$$(f_0)_t(x) = \exp_{\hat{f}_0(x)} t\vec{w}(x) \text{ for any } x \in M.$$

By using (8) and derivations of mappings, we find the pull-back of restrictions $\mathrm{Im}(f_0)$ of g' and ∇' expressed by

$$(9) \qquad j(f_0^* g') = \bar{g} \qquad j(\overset{*}{{}_0}\nabla') = \bar{\nabla}$$
$$j[(f_{0t}^* g')] = g(t) \qquad j[(f_0^*)_t \nabla'] = \nabla(t).$$

All these geometrical objects are defined on $\pi_0(\mathrm{Im}(j))$. If f_0 is a diffeomorphism on Riemannian spaces then the new objects are defined on the whole manifold (M, g).

Remark. Since the Riemannian bundle are endowed with pairs of metrics (g, γ) and corresponding with pairs of operators of derivations (∇, D), from (8) written first for tangent bundles and then for Riemannian bundles, we can get the pull-backs of such pairs of geometrical objects and operators.

We pay more attention to the comorphism from (9) of the tangent bundle to M' and obtain that the deformation of the Riemannian structure and of the Levi-Civita connection along a morphism is just a finite variation or respectively an infinitesimal variation of them on (M, g) or on a domain of it. In other words, the deformations of morphism of Riemannian manifolds (bundles) are reduced to finite variations or infinitesimal variations of the metric (pairs of metrics) and of the connection for which the metric is invariant (of the pairs of derivations operators induced for which the pairs of metrics are invariant). For this reason we can use the derivation operators induced by these metrics. Such a study has been done in [3].

Final remark. It appears as necessity the study of special Riemannian morphisms: immersions, submersions, transversality to a submanifold $N \subset M'$, embeddings, diffeomorphisms. The study sketched before could be useful to some problems on the stability of morphisms. Finally, one can try to bring together by classifications of f_0-morphisms some characteristics of subsets $\mathrm{Im}\, f_0 \subset M'$ and $\pi o j(f_0^* (\mathrm{Im}\, f_0)) \subset M$.

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Supliment la tomul XXVII, s. I a, 1981

GÉOMÉTRIES NON-CLASSIQUES ET PRÉ-RELATIVISTES CAPABLES DE DECRIRE LA PHÉNOMENALITÉ RELATIVISTE II. LE FORMALISME COMPLEX VECTORIEL

PAR

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Dédié à la mémoire du Professeur Mendel Haimovici

§ 1. La géométrie de Plücker-Cayley et le formalisme complex-vectoriel. En 1868—1869 J. Plücker et A. Cayley ont introduit en géométrie la notion de coordonée de ligne, associée aux coordonnées tetrahédrales de deux points sur une ligne droite, dont on attache quatre nombres ; les coordonnées tetrahédrales $P_1(x_0, x_1, x_2, x_3)$; $P_2(y_0, y_1, y_2, y_3)$. Les coordonnées de ligne de la droite D sont définies par

$$(1) \qquad p_{mn} = (x_m y_n - x_n y_m) \; ; \; (n, \; m = 0, \; 1, \; 2, \; 3).$$

Les coordonnées p_{mn} ont une signification purement géometrique, mais par leur construction se recommandent comme des éléments d'une représentation 4-dimensionelle. On considère la transformation suivante $\dot{T}_{pc}$ qui s'applique sur les $\{x_1\}$ ainsi que sur les $\{p_{mn}\}$:

$$(2) \qquad x_0 = x_0' \operatorname{ch} \alpha + x_1' \operatorname{sh} \alpha, \quad x_2 = x_2', \quad x_1 = x_1' \operatorname{ch} \alpha + x_0' \operatorname{sh} \alpha, \quad x_3 = x_3'$$

$$p_{01} = x_0 y_1 - x_1 y_0 = (x_0' \operatorname{ch} \alpha + x_1' \operatorname{sh} \alpha)(y_1' \operatorname{ch} \alpha + y_0' \operatorname{sh} \alpha) -$$
$$-(x_1' \operatorname{ch} \alpha + x_0' \operatorname{sh} \alpha)(y_0' \operatorname{ch} \alpha + y_1' \operatorname{sh} \alpha = x_0' y_1' - x_1' y_0' = p_{01}',$$
$$(3) \qquad p_{02} = p_{02}' \operatorname{ch} \alpha + p_{12}' \operatorname{sh} \alpha \; ; \quad p_{03} = p_{03}' \operatorname{ch} \alpha - p_{31}' \operatorname{sh} \alpha \; ;$$
$$p_{31} = p_{31}' \operatorname{ch} \alpha - p_{03}' \operatorname{sh} \alpha \; ; \quad p_{12} = p_{12}' \operatorname{ch} \alpha + p_{02}' \operatorname{sh} \alpha \; ; \quad p_{23} = p_{23}'.$$

Un ensamble de six grandeurs, qui se transforment par rapport à $\hat{T}_{pc}$ comme les coordonnées de ligne, forme un 6-vecteur (le caractère vectoriel se justifie *à posteriori*). L'invariant introduit par $\hat{T}_{pc}$ est

$$(4) \qquad I_1 = (p_{01}^2 + p_{02}^2 + p_{03}^2) - (p_{23}^2 + p_{31}^2 + p_{12}^2).$$

Les coordonnées de ligne (p_{mn} : réeles) sont associées, d'une manière organique, au grandeurs complexes

$$(5) \qquad Q_1 = p_{01} + ip_{23}, \; Q_2 = p_{02} + ip_{31}, \; Q_3 = p_{03} + ip_{12},$$

$$(6) \qquad \Rightarrow Q_1^2 + Q_2^2 + Q_3^2 = \{(p_{01}^2 + p_{02}^2 + p_{03}^2) - (p_{23}^2 + p_{31}^2 + p_{12}^2\}$$
$$+ 2i\{p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12}\} = I_1 + 2iI_2 \; ; \; (I_1, I_2 \text{ — invariants}).$$

Les Q_K sont les vecteurs complexes (Q-vecteurs) qui se transforment par rapport à $\hat{T}_{pc}$ de la manière suivante