On obtient

$$Z^{\alpha} = \left(\frac{i}{c}\frac{\partial \vec{Q}}{\partial t} - \nabla \times \vec{Q}, \quad \nabla \vec{Q}\right). \tag{44}$$

Le 4-vecteur complexe se décompose dans un ortho-vecteur $\mathcal{T}^{\alpha}$ et un pseudo-vecteur $\mathcal{Q}^{\alpha}$.

$$Z^{\alpha} = -i\mathcal{F}^{\alpha} + \widetilde{\mathcal{Q}}^{\alpha} \tag{45}$$

$$\mathcal{F}^{\alpha} = \left(+\nabla \times \vec{\tilde{U}} - \frac{1}{c}\frac{\partial \vec{U}}{\partial t}, \quad i\nabla \vec{U}\right) \quad \mathcal{Q}^{\alpha} = \left(-\nabla \times \vec{U} - \frac{1}{c}\frac{\partial \vec{\tilde{U}}}{\partial t}, \quad i\nabla \vec{\tilde{U}}\right). \tag{46}$$

Les composantes de la décomposition sont significatives, car on constate leur caractère conservatif

$$\mathcal{F}^{\alpha}_{,\alpha} = 0, \quad \widetilde{\mathcal{Q}}^{\alpha}_{,\alpha} = 0. \tag{47}$$

Les équations d'évolution des Q s'expriment par des relations à caractère conservatif des composantes

$$\mathcal{F}^{\alpha} - \mathcal{J}^{\alpha} = 0, \quad \widetilde{\mathcal{Q}}^{\alpha} - \widetilde{\mathcal{J}}^{\alpha} = 0. \tag{48}$$

($\mathcal{J}^{\alpha}, \widetilde{\mathcal{J}}^{\alpha}$ sont des 4-vecteurs conservatifs qui expriment les densités des sources des composantes). Aux composantes $\vec{U}$, $\vec{\tilde{U}}$ on peut associer les grandeurs [liées au concept de force]

$$f^{\alpha} = (+\vec{p} \times \vec{\tilde{U}} + q\vec{U}, \quad i\vec{p}\,\vec{U}), \quad g^{\alpha} = (-\vec{p} \times \vec{U} + q\vec{\tilde{U}}, \quad i\vec{p}\vec{\tilde{U}}) \tag{49}$$

qui proviennent de $\mathcal{F}^{\alpha}$, $\widetilde{\mathcal{Q}}^{\alpha}$ par la substitution formelle $\frac{\partial}{\partial x^{\alpha}} \to \vec{p}_{\alpha} \equiv (\vec{p}, iq)$ qui ne change pas le caractère de la variance. Le formalisme des Q-vecteurs se montre, de cette manière, capable de donner un modèle mathématique approprié pour décrire quelques situations physiques remarquables.

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Supliment la tomul XXVII, s. I a, 1981

FINSLER GEOMETRY ON VECTOR BUNDLES

BY

FRANCISC C. KLEPP

To the memory of Professor Mendel Haimovici

The notion of Finsler bundle and its properties have been given by M. Matsumoto [1]. The purpose of the present paper is to introduce the concept, called *Finsler vector bundle* associated to a pair of vector bundles, and to define a Finsler connection, using this Finsler vector bundle. If the pair of vector bundles is a pair of tangent bundles, we obtain the results of M. Matsumoto. Another particular case is studied by D. Opriş [3]. In the following we find the curvature of a Finsler connection on a Finsler vector bundle. The notations and terminology are those of M. Matsumoto [1] with few modifications.

§ 1. The Finsler vector bundle associated to a pair of vector bundles. Let $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ be two vector bundles over a differentiable n-manifold M, having the vector spaces $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$ respectively as standard fibres.

Definition 1.1. *A Finsler vector bundle associated to the vector bundles* $\pi_i : E_i \to M (i = 1, 2)$ *is the induced bundle* $\pi : F = \pi_1^* E_2 \to E_1$.

The standard fibre of the bundle $\pi : F \to E_1$ is the same as the standard fibre of the bundle $\pi_2 : E_2 \to M$. Let $(\pi_1!, \pi_1)$ be the canonical morphism of $\pi : F \to E_1$ and $\pi_2 : E_2 \to M$. The exact sequence of vector bundles, associated to the Finsler vector bundle $\pi : F \to E_1$, is

$$0 \to VF \xrightarrow{i} TF \xrightarrow{j} FX_{E_1}TE_1 \to 0. \tag{1.1}$$

A map of $\pi : F \to E_1$ obtained by the trivialisation $\pi_1^{-1}(U) = \widetilde{U}_1$ of $\pi_1 : E_1 \to M$, is $(\widetilde{U}_1, \varphi)$, where φ is the diffeomorphism $\varphi : \pi^{-1}(\widetilde{U}_1) \to \widetilde{U}_1 \times \mathbb{R}^{m_2}$.

$$\begin{tikzcd} F = \pi_1^* E_2 \arrow[r, "\pi_1 !"] \arrow[d, "\pi"'] & E_2 \arrow[d, "\pi_2"] \\ E_1 \arrow[r, "\pi_1"'] & M \end{tikzcd}$$

We define the following subspaces of $T_{\hat{x}}F$ at a point $\hat{x} \in F$:

the vertical subspace: $\overset{v}{F}_{\hat{x}} = \{X \in T_{\hat{x}}F,\ \pi' X = 0\}$;

the induced vertical subspace: $\overset{i}{F}_{\hat{x}} = \{X \in T_{\hat{x}}F,\ (\pi_1!)' X = 0\}$;

the quasi-vertical subspace: $\overset{q}{F}_{\hat{x}} = \{X \in T_{\hat{x}}F,\ \pi_2' \circ (\pi_1!)' X = 0\}$.

Using these subspaces we define the following vector bundles:

$$p_1 : VF = \bigcup_{\hat{x} \in F} \overset{v}{F}_{\hat{x}} \to F\ ;\ p_2 : (VF)^i = \bigcup_{\hat{x} \in F} \overset{i}{F}_{\hat{x}} \to F\ ;\ p_3 : (VF)^q = \bigcup_{\hat{x} \in F} \overset{q}{F}_{\hat{x}} \to F.$$

Then the vector bundles $p_1 : VF \to F$; $p_2 : (VF)^i \to F$ are subbundles of the vector bundle $p_3 : (VF)^q \to F$, and the sequence:

$$0 \to VF \to (VF)^q \to (VF)^i \to 0 \tag{1.2}$$

of vector bundles is exact and splits.

Definition 1.2. *The vector bundle $\pi_3 : F(x) \to \pi_1^{-1}(x)$, where $\pi_1^{-1}(x) \subset \subset E$; $F(x) = F \mid \pi_1^{-1}(x)$, is called a Finsler vector subbundle at the point $x \in M$. The exact sequence associated to $\pi_3 : F(x) \to \pi_1^{-1}(x)$ is*

$$0 \to VF(x) \to TF(x) \to F(x) X_{\pi_1^{-1}(x)} T(\pi_1^{-1}(x)) \to 0. \tag{1.3}$$

Let $\hat{x} \in F$, $\pi \hat{x} = \tilde{x}$ and $\pi_1 \tilde{x} = x \in M$. Since $T_{\hat{x}} F(x) = (VF)^q_{\hat{x}}$ $V_{\hat{x}} F(x) = V_{\hat{x}} F$ and $F(x) X_{\pi_1^{-1}(x)} T(\pi_{1(x)}^{-1}) = (FX_{E_1} VE)_{\hat{x}}$, by (1.3) it follows the exact sequence of vector bundles

$$0 \to VF \to (VF)^q \to FX_{E_1} VE \to 0. \tag{1.4}$$

A left-split β^F of the exact sequence (1.1) is called a connection on $\pi : F \to E_1$. We denote by $i_{VF} : VF \to FX_{E_1}F$ the canonical isomorphism and by $p_2 : FX_E F \to F$ the second projection. The connection map is given by [2]

$$D^F = p_2 \circ i_{VF} \circ \beta^F : TF \to F. \tag{1.5}$$

The map D^F is linear on the fibres of $TF \to F$, but it is non-linear on the fibres of $\pi^T : TF \to TE_1$. If D^F is also linear on the fibers of $\pi^T : TF \to TE_1$, then the connection D^F is called linear. Using the restriction of the sequence (1.1) to $\pi^{-1}(\tilde{U}_1)$, we obtain the following connection map

$$D^F(x, z, t\ ;\ s, p, q) = (x, z\ ;\ q + \overset{1}{\omega}(x, z, t)s + \overset{2}{\omega}(x, z, t)p) \tag{1.6}$$

where $\overset{1}{\omega} : U \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^{m_2})$ and

$$\overset{2}{\omega} : U \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathcal{L}(\mathbb{R}^{m_1}, \mathbb{R}^{m_2})$$

are differentiable maps called the local components of the connection $\mathcal{C}^F$ in the map $(\tilde{U}_1, \varphi)$. Let us denote

$$\begin{cases} \overset{1}{\omega}(x, z, t)(e_i) = \Gamma_i^\alpha(x, z, t) e_\alpha, & i = 1, 2, \ldots, n\ ;\ a = 1, 2, \ldots, m_1\ ; \\ \overset{2}{\omega}(x, z, t)(e_a) = C_a^\alpha(x, z, t) e_\alpha, & \alpha = 1, 2, \ldots, m_2\ ; \end{cases} \tag{1.7}$$

where (e_i) is the canonical base in $\mathbb{R}^n$, (e_a) is the canonical base in $\mathbb{R}^{m_1}$ and (e_α) is the canonical base in $\mathbb{R}^{m_2}$.

The functions $(\Gamma_i^\alpha, C_a^\alpha)$ are called the coefficients of the connection $\mathcal{C}^F$. If the connection $\mathcal{C}^F$ is linear, then

$$\Gamma_i^\alpha(x, z, t) = \Gamma_{i\beta}^\alpha(x, z) t^\beta\ ;\ C_a^\alpha(x, z, t) = C_{a\beta}^\alpha(x, z) t^\beta$$

and $(\Gamma_{i\beta}^\alpha, C_{a\beta}^\alpha)$ are called the coefficients of the linear connection $\mathcal{C}^F$.

Let $X \in \mathcal{X}(E_1)$, $\hat{Y} \in \mathrm{Sect}\,(F)$ and $\mathcal{C}^F$ a connection on $\pi : F \to E_1$. The covariant derivative of Y relative to the field X is the section $D_X^F \hat{Y} \in \mathrm{Sect}(F)$ given by

$$D_X^F \hat{Y}(x) = D^F(\hat{Y}^T X(\tilde{x}))\ ;\ \tilde{x} \in E_1. \tag{1.8}$$

Let (x^i, z^a, t^α) be the local coordinates of $\tilde{x} \in F$ in the mapping $(\tilde{U}_1, \varphi)$ and let $A \in \mathcal{X}\tilde{U}_1(F)$ be the form

$$A = A^i \frac{\partial}{\partial x^i} + \bar{A}^a \frac{\partial}{\partial z^a} + \bar{\bar{A}}^\alpha \frac{\partial}{\partial t^\alpha}.$$

By (1.8) we have:

$$D^F A = (A^\alpha + \Gamma_i^\alpha A^i + C_a^\alpha \bar{A}^a) E_\alpha. \tag{1.9}$$

Let $(\tilde{U}_1', \varphi')$ be another mapping of $\pi : F \to E_1$ such that $\tilde{U}_1 \cap \tilde{U}_1' \neq \emptyset$ and let $(x^{i'}, z^{a'}, t^{\alpha'})$ be the local coordinates on $(\tilde{U}_1', \varphi')$. Then

$$x^{i'} = x^{i'}(x^i)\ ;\ z^{a'} = P_a^{a'} z^a\ ;\ t^{\alpha'} = M_\alpha^{\alpha'} t^\alpha;\ E_\alpha(\tilde{x}) = M_\alpha^{\alpha'} E_{\alpha'}(\tilde{x}), \tag{1.10}$$

where

$$P : U \cap U' \to GL(m_1, \mathbb{R}) \text{ and } M : U \cap U' \to GL(m_2, \mathbb{R})$$

are differentiable maps. The coordinates of the transformation on F is

$$\left\| \begin{matrix} \dfrac{\partial x^{i'}}{\partial x^i} & 0 & 0 \\ \dfrac{\partial P_a^{a'}}{\partial x^i} z^a & P_a^{a'} & 0 \\ \dfrac{\partial M_\alpha^{\alpha'}}{\partial x^i} t^\alpha & 0 & M_\alpha^{\alpha'} \end{matrix} \right\|. \tag{1.12}$$

Using (1.8) and (1.11), we see that transformation laws of the functions $(\Gamma_i^\alpha, C_a^\alpha)$ are

$$(1.12)\qquad \begin{cases} \Gamma^{\alpha'}_{i'}\dfrac{\partial x^{i'}}{\partial x^{i}} M^{\alpha}_{\alpha'} + C^{\alpha'}_{a'}\dfrac{\partial P^{a'}_{a}}{\partial x^{i}} z^{a} M^{\alpha}_{\alpha'} + M^{\alpha}_{\alpha'}\dfrac{\partial M^{\alpha'}_{\gamma}}{\partial x^{i}} t^{\gamma} = \Gamma^{\alpha}_{i}\,, \\ C^{\alpha'}_{a'} M^{\alpha}_{\alpha'} P^{a'}_{a} = C^{\alpha}_{a}. \end{cases}$$

Let $X \in \mathscr{X}_{\overline{U}_1}(E_1)$ be given by $X = X^{i}\dfrac{\partial}{\partial x^{i}} + \overline{X}^{a}\dfrac{\partial}{\partial z^{a}}$. From (1.12) we have

$$(1.13)\qquad D^{F}_{X}\hat{Y} = \left[X^{i}\left(\frac{\partial \hat{Y}^{\gamma}}{\partial x^{i}} + \Gamma^{\gamma}_{i}(\bar{x},\hat{Y})\right) + \overline{X}^{a}\left(\frac{\partial \hat{Y}^{\gamma}}{\partial z^{a}} + C^{\gamma}_{a}(x,\,Y)\right)\right] E_{\gamma}.$$

To a vector field $X \in \mathscr{X}(E_1)$ we can associate another vector field $l^{F}X \in \mathscr{X}(F)$ called the *horizontal lift* of X relative to the connection $\mathscr{C}^{F}$. For $X \in \mathscr{X}(E_1)$ given in the map U_1 by $X = X^{i}\dfrac{\partial}{\partial x^{i}} + \overline{X}^{a}\dfrac{\partial}{\partial z^{a}}$ the horizontal lift $l^{F}X$ is

$$l^{F}X = X^{i}\left(\frac{\partial}{\partial x^{i}} - \Gamma^{\alpha}_{i}\frac{\partial}{\partial t^{\alpha}}\right) + \overline{X}^{a}\left(\frac{\partial}{\partial z^{a}} - C^{\gamma}_{a}\frac{\partial}{\partial t^{\gamma}}\right).$$

Let $\mathscr{C}E_1$ be a connection on the vector bundle $\pi_1 : E_1 \to M$ having $\overline{\omega}$ as local component, then this connection induces the connection ${}^{t}\mathscr{C}^{F}$ on $\pi : F \to E_1$ with $t^{1}_{\omega} = \overline{\omega}$, $t^{2}_{\omega} = 0$ as local components. We will call the connection ${}^{t}\mathscr{C}^{F}$ the *trivial connection* obtained from $\mathscr{C}^{E_1}$.

A vertical connection $(C^{F})^{v}$ in the Finsler vector bundle $\pi : F \to E_1$ is a left split $(\beta^{F})^{v}$ of the exact sequence (1.4). The connection map $(D^{F})^{v}$ of the connection $(\mathscr{C}^{F})^{v}$ is

$$(D^{F})^{v}(x,\,z,\,t,\,p,\,q) = (x,\,z,\,q + \overset{2}{\omega}(x,\,z,\,t)p),$$

where $\overset{2}{\omega}(x,\,z,\,t) \in \mathscr{L}(\mathbb{R}^{m_1},\,\mathbb{R}^{m_2})$. If we denote: $\overset{2}{\omega}(x,\,z,\,t)(e_a) = C^{\alpha}_{a}(x,\,z,\,t)e_{\alpha}$, and use the local coordinates $(x^{i},\,z^{a},\,t^{\alpha})$ on F, then the functions C^{α}_{a} become

$$(1.14)\qquad C^{\alpha'}_{a'}\,P^{a'}_{a}\,M^{\alpha}_{\alpha'} = C^{\alpha}_{a},$$

which represent a generalized tensor, called the *Cartan tensor* associated to the vertical connection $(\mathscr{C}^{F})^{v}$.

Since the exact sequence of vector bundles (1.2) splits, it follows that on the Finsler vector bundle $\pi : F \to E_1$ there exists a particular vertical connection $(C^{F})^{f}$. The corresponding Cartan tensor vanishes identically and this connection is called the *flat vertical connection*.

Proposition 1.1. *To every connection $\mathscr{C}^{F}$ on $\pi : F \to E_1$ there is associated a subordinate vertical connection $(\mathscr{C}^{F})^{v}$.*

If $\overset{1}{\omega}$ and $\overset{2}{\omega}$ are the local components of $\mathscr{C}^{F}$, then $\overset{2}{\omega}$ is the local component of $(\mathscr{C}^{F})^{v}$.

§ 2. **Finsler connections on Finsler vector bundles. Definition 2.1.** *A pair $F\mathscr{C} = (\mathscr{C}^{F},\,\mathscr{C}^{E_1})$, where $\mathscr{C}^{F}$ is a connection on $\pi : F \to E_1$ and $\mathscr{C}^{E_1}$ is a connection on $\pi_1 : E_1 \to M$, is called a Finsler connection on the Finsler vector bundle associated to the vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$.*

If $\mathscr{C}^{F}$ is a linear connection, then the Finsler connection $F\mathscr{C}$ is called a linear Finsler connection.

Definition 2.2. *A pair of differentiable distributions : $(HF)^{h} : \hat{x} \in F \to (HF)^{h}_{\hat{x}} \subset T_{\hat{x}}F$; $(HF)^{v} : \hat{x} \in F \to (HF)^{v}_{\hat{x}} \subset T_{\hat{x}}F$ such that :*

$$(2.1)\qquad T_{\hat{x}}F = (HF)^{h}_{\hat{x}} \oplus (HF)^{v}_{\hat{x}} \oplus (VF)_{\hat{x}}\ ;\ \ \forall \hat{x} \in F,$$

$$(2.2)\qquad \pi'(HF)^{v}_{\hat{x}} = (VE_1)_{\bar{x}}\ ;\ \ \pi(\hat{x}) = \bar{x}\ ;\ \ \forall \hat{x} \in F\ ;\ \ \forall \bar{x} \in E_1$$

is called a Finsler pair on the Finsler vector bundle $\pi : F \to E_1$.

Theorem 2.1. *There exists a one-to-one correspondence between the set of all Finsler connections on the Finsler vector bundle $\pi : F \to E_1$ and the set of all Finsler pairs on $\pi : F \to E_1$.*

Definition 2.3. *A family of C^{∞}-differential distributions $HE_{2(z)}$ on E_2 parametrized by $z \in \mathbb{R}^{m_1}$, and satisfying the condition :*

$$(2.3)\qquad T_{\bar{x}}E_2 = (HE_{2(z)})_{\bar{x}} \oplus (VE_{2(z)})_{\bar{x}}\ ;\ \ \forall \bar{x} \in E_2,$$

is called a $\mathbb{R}^{m_1}$-connection of $\pi_2 : E_2 \to M$ and is denoted by $\mathscr{C}^{E_2}_{\mathbb{R}^{m_1}}$.

Proposition 2.1. *To every Finsler connection $F\mathscr{C} = (\mathscr{C}^{F},\,\mathscr{C}^{E})$ on $\pi : F \to E_1$ it can be canonically associated a $\mathbb{R}^{m_1}$-connection $\mathscr{C}^{E_2}_{\mathbb{R}^{m_1}}$ on $\pi_2 : E_2 \to M$.*

Definition 2.4. *The triad $(\mathscr{C}^{E_2}_{\mathbb{R}^{m_1}},\,\mathscr{C}^{E_1},\,(\mathscr{C}^{F})^{v})$, where $\mathscr{C}^{E_2}_{\mathbb{R}^{m_1}}$ is an $\mathbb{R}^{m_1}$-connection on $\pi_2 : E_2 \to M$, $\mathscr{C}^{E_1}$ is a connection on $\pi_1 : E_1 \to M$ and $(\mathscr{C}^{F})^{v}$ is a vertical connection on $\pi : F \to E_1$, is called a Finsler triad on the Finsler vector bundle $\pi : F \to E_1$.*

Theorem 2.2. *There exists a one-to-one correspondence between the set of all Finsler connections on the Finsler vector bundle $\pi : F \to E_1$ and the set of all Finsler triades on $\pi : F \to E_1$.*

The linear Finsler connections can be defined by the coefficients $(\Gamma^{\alpha}_{i\beta},\,C^{\alpha}_{a\beta},\,N^{a}_{i})$; $\alpha,\,\beta = 1,\,2,\,\ldots,\,m_2$; $a = 1,\,2,\,\ldots,\,m_1$; $i = 1,\,2,\,\ldots,\,n$, or by the coefficients $(F^{\alpha}_{i\beta},\,C^{\alpha}_{a\beta},\,N^{a}_{i})$, where $F^{\alpha}_{i\beta} = \Gamma^{\alpha}_{i\beta} - C^{\alpha}_{a\beta}N^{a}_{i}$.

Let $(\Gamma^{\alpha}_{i\beta},\,C^{\alpha}_{a\beta},\,N^{a}_{i})$ be a linear Finsler connection on $\pi : F \to E_1$. The non-linear connection N on $\pi_1 : E_1 \to M$ admits the local base $\left(\dfrac{\delta}{\delta x^{i}},\ \dfrac{\partial}{\partial z^{a}}\right)$ $\begin{matrix} i = 1,\,2,\ldots,\,n \\ a = 1,\,2,\,\ldots,\,m_1 \end{matrix}$, where $\dfrac{\delta}{\delta x^{i}} = \dfrac{\partial}{\partial x^{i}} - N^{a}_{i}\dfrac{\partial}{\partial z^{a}}$. Let E_{α}, $\alpha = 1,\,\ldots,\,m_2$, be the base of the local sections on $\pi : F \to E_1$, then

$$(2.4)\qquad \overset{F}{D}_{\frac{\delta}{\delta x^{i}}} E_{\alpha} = (\Gamma^{\beta}_{i\alpha} - N^{a}_{i}C^{\beta}_{a\alpha})E_{\beta} = F^{\beta}_{i\alpha}E_{\beta}\,;\ \ \overset{F}{D}_{\frac{\partial}{\partial z^{a}}} E_{\alpha} = C^{\beta}_{a\alpha}E_{\beta}.$$

We define the curvature of the linear Finsler connection on $\pi : F \to E_1$ by

$$R^F(X,Y)\hat{Z} = D^F_X D^F_Y \hat{Z} - D^F_Y D^F_X \hat{Z} - D^F_{[X,Y]}\hat{Z};\quad \forall X, Y \in \mathcal{X}(E_1), \tag{2.5}$$
$$\forall z \in \mathrm{Sect}\,(F).$$

The linear Finsler connection has three curvatures

$$\left\{\left[D^F_{\frac{\delta}{\delta x^j}};\ D^F_{\frac{\delta}{\delta x^k}}\right] - D^F_{\left[\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\right]}\right\}E_\alpha = R^\beta_{\alpha kj}E_\beta \tag{2.6}$$

$$\left\{\left[D^F_{\frac{\delta}{\delta z^a}},\ D^F_{\frac{\delta}{\delta x^k}}\right] - D^F_{\left[\frac{\delta}{\delta z^a},\frac{\delta}{\delta x^k}\right]}\right\}E_\alpha = P^\beta_{\alpha ka}E_\beta \tag{2.7}$$

$$\left\{\left[D^F_{\frac{\delta}{\delta z^a}},\ -D^F_{\frac{\delta}{\delta z^b}}\right] - D^F_{\left[\frac{\delta}{\delta z^a},\frac{\delta}{\delta z^b}\right]}\right\}E_\alpha = S^\beta_{\alpha ba}E_\beta. \tag{2.8}$$

It results that

$$\left[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right] = R^a_{jk}\frac{\partial}{\partial z^a};\quad \left[\frac{\partial}{\partial z^a}, \frac{\delta}{\delta x^k}\right] = -\frac{\partial N^b_k}{\partial z^a}\frac{\partial}{\partial z^b}, \tag{2.9}$$

where

$$R^a_{jk} = \frac{\delta N^a_j}{\delta x^k} - \frac{\delta N^a_k}{\delta x^j}.$$

By (2.9) we have

$$R^\beta_{\alpha kj} = \frac{\delta F^\beta_{k\alpha}}{\delta x^j} - \frac{\delta F^\beta_{j\alpha}}{\delta x^k} + F^\gamma_{k\alpha}F^\beta_{j\gamma} - F^\gamma_{j\alpha}F^\beta_{k\gamma} + C^\beta_{a\alpha}R^a_{kj}, \tag{2.10}$$

$$P^\beta_{\alpha ka} = \frac{\partial F^\beta_{k\alpha}}{\partial z^a} - \left(\frac{\delta C^\beta_{a\alpha}}{\delta x^k} + F^\beta_{k\gamma}C^\gamma_{a\alpha} - F^\lambda_{k\alpha}C^\beta_{a\lambda}\right) + \frac{\partial N^b_k}{\partial z^a}C^\beta_{b\alpha}, \tag{2.11}$$

$$S^\beta_{\alpha ba} = \frac{\partial C^\beta_{\alpha b}}{\partial z^a} - \frac{\partial C^\beta_{\alpha a}}{\partial z^b} + C^\gamma_{\beta a}C^\beta_{a\gamma} - C^\gamma_{a\alpha}C^\beta_{b\gamma}. \tag{2.12}$$

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Supliment la tomul XXVII, s. I a, 1981

APPLICATIONS OF THE DIFFERENTIAL FORM'S METHOD IN THE GEOMETRY OF REAL Δ^α-STRUCTURES ON A MANIFOLD

BY

ELENA MACOVEI

To the memory of Professor Mendel Haimovici

The development of the theory of P f a f f systems, initiated by E. C a r t a n [1], gave rise to researches in differential geometry of manifolds with given structures, through differential systems (differentiable distributions in the sense of C. C h e v a l l e y). Romanian mathematicians: M. H a i m o v i c i [3; 4], Gh. Al. P a n t a z i, G. V r ă n c e a n u [10], G h e o r g h i e v [2], R. M i r o n [7; 8], have important contributions in this field.

An improved form of Cartan's equivalence method elaborated by M. H a i m o v i c i [3], the method of differential forms for the intrinsic geometry of nonholonomic manifolds, was applied by R. M i r o n [8] in the geometrization of nonholonomic mechanical systems.

This Note contains a study of the integrability of real Δ^α - structures by a suitable adaptation of the methods of differential forms.

§. 1. **Derived differential systems associated to a real Δ^α-structure on a manifold.** Let M be a differentiable n-dimensional manifold of class C^∞ endowed with a real Δ^α - structure ([5]) defined by two differential systems:

$$\Delta^\alpha : x \in M \to \Delta^\alpha_x \subset T_xM,\ \forall x \in M$$
$$n_\alpha = \dim \Delta^\alpha_x,\ 1 \leqslant n_\alpha < n;\ \alpha = 1, 2 \tag{1.1}$$

We suppose the existence of differential system $\Delta^\alpha (\alpha = 1, 2)$ and we consider the associated differential systems:

$$K : x \in M \to K_x = \Delta^1_x \cap \Delta^2_x \subset T_xM,\ \forall x \in M,$$
$$0 \leqslant \varkappa = \dim K_x \leqslant \min(n_1, n_2),$$
$$\Delta : x \in M \to \Delta_x = \Delta^1_x + \Delta^2_x \subset T_xM,\ \forall x \in M \tag{1.2}$$
$$2 \leqslant p = \dim \Delta_x \leqslant n.$$

The differential systems K, Δ^α, Δ are locally defined by the Pfaff systems:

$$(K): \omega^{j_1} = \omega^{j_2} = \omega^\mu = 0,$$
$$\varkappa + 1 \leqslant j_\alpha \leqslant n_\alpha,\ \alpha = 1, 2;\ n_1 + n_2 - \varkappa + 1 \leqslant \mu \leqslant n, \tag{1.3}$$
$$(\Delta^\alpha): \omega^{j_\beta} = \omega^\mu = 0,\ (\alpha \neq \beta;\ \alpha, \beta = 1, 2),$$
$$(\Delta): \omega^\mu = 0$$