$$R^F(X,Y)\hat{Z} = D^F_X D^F_Y \hat{Z} - D^F_Y D^F_X \hat{Z} - D^F_{[X,Y]}\hat{Z};\ \forall X, Y \in \mathcal{H}(E_1), \tag{2.5}$$

$$\forall z \in \text{Sect}\,(F).$$

The linear Finsler connection has three curvatures

$$\left\{\left[\overset{F}{D}_{\frac{\delta}{\delta x^j}};\ \overset{F}{D}_{\frac{\delta}{\delta x^k}}\right] - \overset{F}{D}_{\left[\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^k}\right]}\right\} E_\alpha = R^\beta_{\alpha kj} E_\beta \tag{2.6}$$

$$\left\{\left[\overset{F}{D}_{\frac{\delta}{\delta z^a}},\ \overset{F}{D}_{\frac{\delta}{\delta x^k}}\right] - \overset{F}{D}_{\left[\frac{\delta}{\delta z^a},\frac{\delta}{\delta x^k}\right]}\right\} E_\alpha = P^\beta_{\alpha ka} E_\beta \tag{2.7}$$

$$\left\{\left[\overset{F}{D}_{\frac{\delta}{\delta z^a}},\ -\overset{F}{D}_{\frac{\delta}{\delta z^b}}\right] - \overset{F}{D}_{\left[\frac{\delta}{\delta z^a},\frac{\delta}{\delta z^b}\right]}\right\} E_\alpha = S^\beta_{\alpha ba} E_\beta. \tag{2.8}$$

It results that

$$\left[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right] = R^a_{jk}\frac{\partial}{\partial z^a};\ \left[\frac{\partial}{\partial z^a}, \frac{\delta}{\delta x^k}\right] = -\frac{\partial N^b_k}{\partial z^a}\frac{\partial}{\partial z^b}, \tag{2.9}$$

where

$$R^a_{jk} = \frac{\delta N^a_j}{\delta x^k} - \frac{\delta N^a_k}{\delta x^j}.$$

By (2.9) we have

$$R^\beta_{\alpha kj} = \frac{\delta F^\beta_{k\alpha}}{\delta x^j} - \frac{\delta F^\beta_{j\alpha}}{\delta x^k} + F^\gamma_{k\alpha}F^\beta_{j\gamma} - F^\gamma_{j\alpha}F^\beta_{k\gamma} + C^\beta_{a\alpha}R^a_{kj}, \tag{2.10}$$

$$P^\beta_{\alpha ka} = \frac{\partial F^\beta_{k\alpha}}{\partial z^a} - \left(\frac{\delta C^\beta_{a\alpha}}{\delta x^k} + F^\beta_{k\gamma}C^\gamma_{a\alpha} - F^\lambda_{k\alpha}C^\beta_{a\lambda}\right) + \frac{\partial N^b_k}{\partial z^a}C^\beta_{b\alpha}, \tag{2.11}$$

$$S^\beta_{\alpha ba} = \frac{\partial C^\beta_{\alpha b}}{\partial z^a} - \frac{\partial C^\beta_{\alpha a}}{\partial z^b} + C^\gamma_{\beta a}C^\beta_{a\gamma} - C^\gamma_{a\alpha}C^\beta_{b\gamma}. \tag{2.12}$$

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APPLICATIONS OF THE DIFFERENTIAL FORM'S METHOD IN THE GEOMETRY OF REAL Δ^α-STRUCTURES ON A MANIFOLD

BY

ELENA MACOVEI

To the memory of Professor Mendel Haimovici

The development of the theory of Pfaff systems, initiated by E. Cartan [1], gave rise to researches in differential geometry of manifolds with given structures, through differential systems (differentiable distributions in the sense of C. Chevalley). Romanian mathematicians: Al. Pantazi, G. Vrănceanu [10], M. Haimovici [3; 4], Gh. Gheorghiev [2], R. Miron [7; 8], have important contributions in this field.

An improved form of Cartan's equivalence method elaborated by M. Haimovici [3], the method of differential forms for the intrinsic geometry of nonholonomic manifolds, was applied by R. Miron [8] in the geometrization of nonholonomic mechanical systems.

This Note contains a study of the integrability of real Δ^α - structures by a suitable adaptation of the methods of differential forms.

§. 1. **Derived differential systems associated to a real Δ^α-structure on a manifold.** Let M be a differentiable n-dimensional manifold of class C^∞ endowed with a real Δ^α - structure ([5]) defined by two differential systems:

$$\Delta^\alpha : x \in M \to \Delta^\alpha_x \subset T_xM,\ \forall x \in M$$
$$n_\alpha = \dim \Delta^\alpha_x,\ 1 \leqslant n_\alpha < n\ ;\ \alpha = 1, 2 \tag{1.1}$$

We suppose the existence of differential system $\Delta^\alpha (\alpha = 1, 2)$ and we consider the associated differential systems:

$$K : x \in M \to K_x = \Delta^1_x \cap \Delta^2_x \subset T_xM,\ \forall x \in M,$$
$$0 \leqslant \varkappa = \dim K_x \leqslant \min (n_1, n_2),$$
$$\Delta : x \in M \to \Delta_x = \Delta^1_x + \Delta^2_x \subset T_xM,\ \forall x \in M \tag{1.2}$$
$$2 \leqslant p = \dim \Delta_x \leqslant n.$$

The differential systems K, Δ^α, Δ are locally defined by the Pfaff systems:

$$(K) : \omega^{j_1} = \omega^{j_2} = \omega^\mu = 0,$$
$$\varkappa + 1 \leqslant j_\alpha \leqslant n_\alpha,\ \alpha = 1, 2\ ;\ n_1 + n_2 - \varkappa + 1 \leqslant \mu \leqslant n, \tag{1.3}$$
$$(\Delta^\alpha) : \omega^{j_\beta} = \omega^\mu = 0,\ (\alpha \neq \beta\ ;\ \alpha, \beta = 1, 2),$$
$$(\Delta) : \omega^\mu = 0$$

where $(\omega^i)=(\omega^u, \omega^{j_1}, \omega^{j_1}, \omega^{\mu})$ are differential forms locally defined by the property that $(\omega_x^{j_1}, \omega_x^{j_2}, \omega_x^{\mu})$, $(\omega_x^{j_\beta}, \omega_x^{\mu})$, (ω_x^{μ}) are linearly independent in each point $x\in M$.

The differential system K (as well as Δ^α and Δ) determines a vector subbundle V_K of rank $\varkappa$ of the tangent bundle $T(M)$ and, in a dual manner, a vector subbundle V_K^* of rank $(n-\varkappa)$ of the cotangent bundle $T^*(M)$([5]).

Also, the differential system K (as well as Δ^α and Δ), determines a G-structure on M, denoted by B_K, with structure group G_K isomorphic with a non-singular quadratic matrix group of the following form :

$$\|c\| = \begin{Vmatrix} c_v^u & 0 \\ c_v^{\bar u} & c_{\bar v}^{\bar u} \end{Vmatrix} \quad 1 \leqslant u,\ v \leqslant \varkappa\ ;\ \varkappa+1 \leqslant \bar u, \bar v \leqslant n, \quad c_v^u \in GL(\varkappa, R), \tag{1.4}$$

$$c_{\bar v}^{\bar u} \in GL(n-\varkappa, R), \quad c_v^{\bar u} \in \mathrm{Hom}(R^\varkappa, R^{n-\varkappa}).$$

In a similar way, taking B_{Δ^α} as G-structures determined by the differential systems $\Delta^\alpha(\alpha=1, 2)$ and assuming that $\Delta_x^1\cap\Delta_x^2$ are of constant dimension in each $x\in M$, then the local fibres of those G-structures have a nonvoid interceection. It follows that we can state :

Propossition 1.1. ([9]) *Let B_{Δ^α} $(\alpha=1, 2)$ be two principal subbundles of the linear frame bundle, $E(M)$. $B_{\Delta^1}\cap B_{\Delta^2}$ is a principal subbundle of $E(M)$ if and only if their local fibres have a nonvoid intersection in each point of M.*

Definition 1.1. *The differential system K (as well as Δ^α and Δ) is derivable (admits a derived system), if the K^1 submodule of $\mathcal{B}(M)$ generated by $[K, K]+K$ satisfies the condition $\dim K_x^1=$ constant in each $x\in M$. The differential system K^1 is, by definition, the first derived system of K and by reccurence, the derived systems of order $q(q\in N)$ are defined by $K^q=(K^{q-1})^1$* ([9]).

It results from definition 1.1 that the subbundle B_{K^q} of the frame bundle $E(M)$ is a G-structure with structure group G_{K^q} of type (1.4). The sequence $K\subset K^1\subset \ldots$ (with $\varkappa=n_0<n_1<\ldots<n_q<\ldots<n$ and $n_q=\dim K^q$) determines inclusions of the same type among suitable vector bundles $V_K\subset \subset V_{K^1}\subset \ldots \subset V_{K^q}\subset \ldots$. Under the hypothesis that the differential system K admits derived systems of any order and as the sequence n_q is increasing and bounded by n, there exists a $q_0\in N$ such that : $K_x^{q_0}=T_xM$, $x\in M$, or $n_{q_0}=n_{q_0+1}=\ldots$ and $K^{q_0+1}=K^{q_0}$. We therefore state the following :

Proposition 1.2. ([9]). *If there exists a natural number q_0 with property that $K_x^{q_0}=T_xM$, $x\in M$ or $K^{q_0+1}=K^{q_0}$, then the intersection $B_K\cap B_{K^1}\cap \ldots \ldots\cap B_{K^{q_0}}=\bar B$ is a G-structure on M with structure group $G_K\cap G_{K^1}\cap \ldots \cap \cap G_{K^{q_0}}=\bar G$.*

§ **2. The integrability of real Δ_K^α-structures.** Let us suppose that M is endowed with a real Δ_K^α-structure defined by two differential systems Δ^1, Δ^2 for which the intersection K is a differential system and $\dim \Delta_x^1 + +\dim\Delta_x^2-\dim K_x<n$, $\forall x\in M$ ([5]).

The G-structures $B_{\Delta^\alpha}(\alpha=1, 2)$ have the property that in each $x\in M$ their local fibres have nonvoid intersection $(K_x=\Delta_x^1\cap\Delta_x^2\neq\{o\}_x,\ \forall x\in M)$

and according to Proposition 1.1 results that $B_{\Delta^1}\cap B_{\Delta^2}$ is a G-structure on M with a structure group $G_{\Delta^1}\cap G_{\Delta^2}$. The group $G_{\Delta^1}\cap G_{\Delta^2}=G_K$ preserves each differential system Δ^α and their intersection K, so that $B_{\Delta^1}\cap B_{\Delta^2}$ is a real Δ_K^α-structure on M (according to definition 1.1 [6]).

The Δ_K^α-structure may be considered as being locally determined by an oppen covering as $\mathcal{U}=\{U\}$ of M, with the property that on each open U, systems of Pfaff forms $\omega_U=(\omega_U^u, \omega_U^{j_1}, \omega_U^{j_2}, \omega_U^{\mu})=(\omega_U^i)$ are linearly independent and on $U\cap U'\neq\varnothing$ we have :

$$\begin{aligned}
\omega'^u &= c_v^u\omega^u + c_{l_1}^u\omega^{l_1} + c_{l_2}^u\omega^{l_2} + c_\lambda^u\omega^\lambda,\\
\omega'^{j_1} &= c_{l_1}^{j_1}\omega^{l_1} + c_\lambda^{j_1}\omega^\lambda,\\
\omega'^{j_2} &= c_{l_2}^{j_2}\omega^{l_2} + c_\lambda^{j_2}\omega^\lambda,\\
\omega'^\mu &= c_\lambda^\mu\omega^\lambda,
\end{aligned} \tag{2.1}$$

$$c_v^u\in GL(\varkappa, R)\ ;\ c_{l_\alpha}^{j_\alpha}\in GL(n_\alpha-\varkappa, R),$$
$$c_\lambda^\mu\in GL(n-n_1-n_2+\varkappa, R)\ ;\ c_{l_\alpha}^u\in \mathrm{Hom}(R^\varkappa, R^{n_\alpha-\varkappa}).$$
$$c_\lambda^u\in \mathrm{Hom}(R^\varkappa, R^{n-n_1-n_2+\varkappa})\ ;\ c_\lambda^{j_\alpha}\in \mathrm{Hom}(R^{n_\alpha-\varkappa}, R^{n-n_1-n_2+\varkappa}).$$

With respect to the cobase defined by the differential forms $\omega_U=(\omega_U^i)$, on each open U, the differential systems of given Δ_K^α-structure are locally determined by Pfaff systems (1.3). The exterior differentials of the Pfaff forms (ω_U^i) are expressed as follows :

$$\begin{aligned}
d\omega^u &= \omega^i\wedge\pi_i^u,\\
d\omega^{j_1} &= \omega^{l_1}\wedge\pi_{l_1}^{j_1} + \omega^{l_2}\wedge\pi_{l_2}^{j_1} + \omega^\lambda\wedge\pi_\lambda^{j_1} + \tfrac{1}{2}W_{uv}^{j_1}\omega^u\wedge\omega^v,\\
d\omega^{j_2} &= \omega^{l_2}\wedge\pi_{l_2}^{j_2} + \omega^{l_1}\wedge\pi_{l_1}^{j_2} + \omega^\lambda\wedge\pi_\lambda^{j_2} + \tfrac{1}{2}W_{uv}^{j_2}\omega^u\wedge\omega^v,\\
d\omega^\mu &= \omega^\lambda\wedge\pi_\lambda^\mu + \tfrac{1}{2}W_{uv}^\mu\omega^u\wedge\omega^v + W_{uh_1}^\mu\omega^u\wedge\omega^{h_1} + W_{uh_2}^\mu\omega^u\wedge\omega^{h_2} +\\
&+ \tfrac{1}{2}W_{l_1h_1}^\mu\omega^{l_1}\wedge\omega^{h_1} + W_{\lambda_1h_2}^\mu\omega^{l_1}\wedge\omega^{h_2} + \tfrac{1}{2}W_{l_2h_2}^\mu\omega^{l_2}\wedge\omega^{h_2},
\end{aligned} \tag{2.2}$$

where the coefficients are functions on each open U. The relations (2.2) are called *structure equations of the given Δ_K^α-structure on M*. A direct computation shows that the quantities :

$$\tfrac{1}{2}W_{uv}^{j_1},\ \tfrac{1}{2}W_{uv}^{j_2},\ \tfrac{1}{2}W_{uv}^\mu,\ W_{uh_2}^\mu,\ W_{uh_1}^\mu,\ \tfrac{1}{2}W_{l_1h_1}^\mu,\ W_{l_1h_2}^\mu,\ \tfrac{1}{2}W_{l_2h_2}^\mu, \tag{2.3}$$

determine the components of a tensor which is, by definition, *the structure tensor of a Δ_K^α-structure* denoted by τ, $\tau=(\tfrac{1}{2}W_{uv}^{j_1}, \ldots, \tfrac{1}{2}W_{l_2h_2}^\mu)$ ([10]).

Theorem 2.1. *The differential system Δ^α is the first derived system of the differential system K if and only if the components: $W_{uv}^{j_\beta}(\alpha\neq\beta\ ;\ \alpha\beta=1, 2)$, W_{uv}^μ of the structure tensor τ vanish in each point $x\in M$.*

According to definition 1.1, we can deduce from the relations (1.3) that K admits Δ^α as derived system of order 1, on Δ^α and since $[K, K]\subset \subset\Delta^\alpha$, it results :

$$\begin{aligned}
d\omega^{j_\beta} &= 0(\mathrm{mod}\ \omega^{j_1}, \omega^{j_2}, \omega^\mu),\\
d\omega^\mu &= 0(\mathrm{mod}\ \omega^{j_1}, \omega^{j_2}, \omega^\mu).
\end{aligned} \tag{2.4}$$

From (2.2) and (2.4) it follows that :

$$W^{j\beta}_{uv} = 0, \quad W^{\mu}_{uv} = 0 (\alpha \neq \beta ; \ \beta = 1, 2), \tag{2.5}$$

and that the tensor τ satisfies the requested condition.

If the tensor τ satisfies the conditions (2.5) in each point $x \in M$ then, from the structure equations (2.2) and the definition system (1.3), it results that Δ^{α} is the first derived system of K.

Theorem 2.2. *The differential system Δ is the second derived system for K, if and only if the components : W^{μ}_{uv}, $W^{\mu}_{uh_1}$, $\frac{1}{2} W^{\mu}_{l_1h_1}$ $W^{\mu}_{uh_2}$, $\frac{1}{2} W^{\mu}_{l_2h_2}$ of the structure tensor τ vanish in each point $x \in M$.*

According to definition 1.1 we have $[\Delta^{\alpha}, \Delta^{\alpha}] \subset \Delta$ and from (2.2) it results $d\omega^{\alpha} = 0 \ (\mathrm{mod}\ \omega^{j\beta}, \omega^{\mu})$. Then in each point $x \in M$ we have :

$$W^{\mu}_{uv} = 0, \quad W^{\mu}_{uh_{\alpha}} = 0, \quad W^{\mu}_{l_{\alpha}h_{\alpha}} = 0 \quad (\alpha = 1, 2). \tag{2.6}$$

Each differential system Δ^{α} is the first derived system for K and from (2.6) it results that the condition in theorem 2.2 is satisfied.

If the structure tensor τ satisfies (2.6), then, from the structure equations (2.2), it follows that Δ is the second derived system for K.

The integrability tensor of the differential systems K, Δ^{α}, Δ are determined, (see G. Vrănceanu [10]), by :

$$\tau_K = (\tfrac{1}{2} W^{j_1}_{uv}, \tfrac{1}{2} W^{j_2}_{uv}, \tfrac{1}{2} W^{\mu}_{uv}),$$

$$\tau_{\Delta^{\alpha}} = (\tfrac{1}{2} W^{j\beta}_{uv}, W^{j\beta}_{uh_{\alpha}}, \tfrac{1}{2} W^{j\beta}_{l_{\alpha}h_{\alpha}}, \tfrac{1}{2} W^{\mu}_{uv}, W^{\mu}_{uh_{\alpha}}, \tfrac{1}{2} W^{\mu}_{l_{\alpha}h_{\alpha}}), \tag{2.7}$$

$$\tau_{\Delta} = (\tfrac{1}{2} W^{\mu}_{uv}, W^{\mu}_{uh_{\alpha}}, W^{\mu}_{l_{\alpha}h_{\beta}}, \tfrac{1}{2} W^{\mu}_{l_{\alpha}h_{\alpha}}),$$

and are the restrictions of the tensor τ for K, Δ^{α}, Δ.

Theorem 2.3. *The differential system K (respectivelly Δ^{α}) is integrable if and only if the tensor τ_K (respectivelly $\tau_{\Delta}\alpha$) has null components in each point $x \in M$.*

The differential system K (respectivelly Δ^{α}) is integrable if and only if we have : $d\omega^{j_1} = d\omega^{j_2} = d\omega^{\mu} = 0 \ (\mathrm{mod}\ \omega^{j_1}, \omega^{j_2}, \omega^{\mu})$ (respectivelly $d\omega^{j\beta} = d\omega^{\mu} = 0 \ (\mathrm{mod}\ \omega^{j\beta}, \omega^{\mu})$).

Under this hypothesis, from the structure equations (2.2), it results that the integrability of the differential system K (respectivelly Δ^{α}) is characterised by the vanishing of the components of the integrability tensor τ_K (respectivelly Δ^{α}).

The Δ^{α}_K-structure on M is integrable, if the differential systems of the structure : K, Δ^1, Δ^2 are integrable [6] and from the previous theorems, the following assertion results :

Corollary 2.1. *The real Δ^{α}_K-structure on M is integrable if and only if the structure tensor τ vanishes in every point $x \in M$.*

§. 3. **The integrability of Δ^{α}-structures subordonate to real Δ^{α}_K-structures.** 1. A real Δ^{α}_0-structure on M is a pair of differential systems (Δ^1, Δ^2) for which $\Delta^1_x \cap \Delta^2_x = \{0\}_x$ and $\dim \Delta^1_x + \dim \Delta^2_x < n$, $\forall x \in M$ ([5]).

The differential systems Δ^{α} and $\Delta(\Delta = \Delta^1 \oplus \Delta^2)$ determine G-structures $B_{\Delta}\alpha$, B_{Δ} which have the property that their local fibres have a nonvoid intersection in each point $x \in M$.

From Proposition 1.1 it follows that $B_{\Delta}\alpha \cap B_{\Delta}$ is a G-structure on M with structure group $G_{\Delta}\alpha \cap G_{\Delta} = G_0$. The group G_0 preserves each differential system Δ^{α} and therefore, $B_{\Delta}\alpha \cap B_{\Delta}$ is a Δ^{α}_0 -structure on M [6].

A Δ^{α}_0-structure on M is locally defined by a Pfaff system $(\omega^t_U) = (\omega^{j_1}_U, \omega^{j_2}_U, \omega^{\mu}_U)$ $(1 \leqslant j_1 \leqslant n_1 ; \ n_1 + 1 \leqslant j_2 \leqslant n_1 + n_2 ; \ n_1 + n_2 + 1 \leqslant \mu \leqslant n)$ such that on $U \cap U' \neq \varnothing$ the conditions (2.1) be fulfilled for $\varkappa = 0$. The differential systems Δ^{α}, Δ are locally determined on every open U, by :

$$\begin{aligned} &(\Delta^{\alpha}) : \omega^{j\beta} = \omega^{\mu} = 0 \quad (\alpha \neq \beta ; \ \beta = 1, 2). \\ &(\Delta) : \omega^{\mu} = 0 \ (1 \leqslant j_1 \leqslant n_1 ; n_1 + 1 \leqslant j_2 \leqslant n_1 + n_2 ; n_1 + n_2 + 1 \leqslant \mu \leqslant n)). \end{aligned} \tag{3.1}$$

The exterior differentials of the Pfaff forms $(\omega^t) = (\omega^{j_1}, \omega^{j_2}, \omega^{\mu})$ define *the structure equations of a Δ^{α}_0*-structure :

$$\begin{aligned} d\omega^{j_1} &= \omega^t \wedge \overset{\circ}{\pi}{}^{j_1}_t, \\ d\omega^{j} &= \omega^{l_2} \wedge \overset{\circ}{\pi}{}^{j_2}_{l_2} + \omega^{\lambda} \wedge \overset{\circ}{\pi}{}^{j_2}_{\lambda} + \tfrac{1}{2} \overset{\circ}{W}{}^{j_2}_{l_1h_1} \omega^{l_1} \wedge \omega^{h_1} \\ d\omega^{\mu} &= \omega^{\lambda} \wedge \overset{\circ}{\pi}{}^{\mu}_{\lambda} + \tfrac{1}{2} \overset{\circ}{W}{}^{\mu}_{l_1h_1} \omega^{l_1} \wedge \omega^{h_1} + \overset{\circ}{W}{}^{\mu}_{l_1h_1} \omega^{l_1} \wedge \omega^{h_2} + \tfrac{1}{2} \overset{\circ}{W}{}^{\mu}_{l_2h_2} \omega^{l_2} \wedge \omega^{h_2}, \end{aligned} \tag{3.2}$$

The coefficients of equations (3.2) determine the components of *the structure tensor of a Δ^{α}_0-structure* :

$$\overset{\circ}{\tau} = (\tfrac{1}{2} \overset{\circ}{W}{}^{j\beta}_{l_{\alpha}h_{\alpha}}, \tfrac{1}{2} \overset{\circ}{W}{}^{\mu}_{l_{\alpha}h_{\alpha}}, \overset{\circ}{W}{}^{\mu}_{l_{\alpha}h_{\beta}}) (\alpha \neq \beta ; \ \alpha, \beta = 1, 2). \tag{3.3}$$

Theorem 3.1. *The differential system Δ is the first derived system of each of the differential systems Δ^{α} $(\alpha = 1, 2)$, if and only, if the components : $\overset{\circ}{W}{}^{\mu}_{\alpha h\alpha}$ of the structure tensor $\overset{\circ}{\tau}$ vanish in each point $x \in M$.*

We have, from definition 1.1, $[\Delta^{\alpha}, \Delta^{\alpha}] \subset \Delta$ and from equations (3.1) it follows that $d\omega^{\mu} = 0 \ (\mathrm{mod}\ \omega^{j\beta}, \omega^{\mu})$ which implies

$$\overset{\circ}{W}{}^{\mu}_{l_{\alpha}h_{\alpha}} = 0. \tag{3.4}$$

and the tensor $\overset{\circ}{\tau}$ satisfies the requested condition.

Conversely, if $\overset{\circ}{\tau}$ has the components $\overset{\circ}{W}{}^{\mu}_{l_{\alpha}h_{\alpha}} = 0$, in each point $x \in M$, then from the structure equations (3.2), it results that Δ is the first derived system of the differential system Δ^{α}.

Theorem 3.2. *The differential system Δ^{α} is integrable if and only if the components $\overset{\circ}{W}{}^{j\beta}_{l_{\alpha}h_{\alpha}}$ $\overset{\circ}{W}{}^{\mu}_{l_{\alpha}h_{\alpha}}$ of the structure tensor $\overset{\circ}{\tau}$ vanish identically in each point $x \in M$.*

The integrability of the differential system Δ^{α} $(\alpha = 1, 2)$ is characterised by the following conditions $d\omega^{j\beta} = d\omega^{\mu} = 0 \ (\mathrm{mod}\ \omega^{j\beta}, \omega^{\mu})$ and from the structure equations (3.2) taking into account definition 1.1 and theorem 3.1 the statement in theorem 3.2 follows directly.

The following statements hold :

Theorem 3.3. *The differential system Δ is integrable, if and only if the components $\overset{\circ}{W}{}^{j\beta}_{l_\alpha h_\alpha}$, $\overset{\circ}{W}{}^{\mu}_{l_\alpha h_\alpha}$, of the structure tensor $\overset{\circ}{\tau}$ vanish identically in each point $x \in M$.*

It is noticed, that the restrictions of the structure tensor $\overset{\circ}{\tau}$ to Δ^α and Δ give the integrability tensors of the differential systems Δ^α and Δ :

$$\overset{\circ}{\tau}_{\Delta}\alpha = \left(\tfrac{1}{2}\overset{\circ}{W}{}^{j\beta}_{l_\alpha h_\alpha},\ \tfrac{1}{2}\overset{\circ}{W}{}^{\mu}_{l_\alpha h_\alpha}\right).$$

$$\overset{\circ}{\tau}_{\Delta} = \left(\tfrac{1}{2}\overset{\circ}{W}{}^{\mu}_{l_\alpha h_\alpha},\ \overset{\circ}{W}{}^{\mu}_{l_\alpha h_\beta},\ \tfrac{1}{2}\overset{\circ}{W}{}^{\mu}_{l_\beta h_\beta}\right). \tag{3.5}$$

The vanishing of the tensors $\overset{\circ}{\tau}_{\Delta\alpha}$, $\overset{\circ}{\tau}_{\Delta}$ represents the integrability condition for the differential systems Δ^α, Δ and we get :

Corollary 3.1. *A real Δ^α_0-structure is integrable, if and only if the structure tensor $\overset{\circ}{\tau}$ vanishes identically on M.*

2. A real Δ^α_μ-structure on M is given by two differential systems Δ^1, Δ^2 with Δ^1 involutive and for $\Delta^1_x \cap \Delta^2_x = \Delta^1_x$, $\dim \Delta^2_x < n$, $\forall x \in M$ ([6]).

A real Δ^α_μ-structure on M coincides with a $B_{\Delta^1} \cap B_{\Delta^2}$-structure and is locally defined on every open U by the Pfaff system $(\omega^i_U) = (\omega^{j_1}_U, \omega^{j_2}_U, \omega^\mu_U)$, with the condition that (2.1) be fulfilled on $U \cap U' \neq \varnothing$ for $\varkappa = n_1$. The differential systems Δ^α are locally determined on U by :

$$(\Delta^1) : \omega^{j_2} = 0,\ \omega^\mu = 0,\ (1 \leqslant j_1 \leqslant n_1 ;\ n_1 + 1 \leqslant j_2 \leqslant n_2),$$
$$(\Delta^2) : \omega^\mu = 0,\ (n_2 + 1 \leqslant \mu \leqslant n). \tag{3.6}$$

The structure equations of a real Δ^α_μ-structure are

$$d\omega^{j_1} = \omega^i \wedge \overset{\bullet}{\pi}{}^{j_1}_i,$$
$$d\omega^{j_2} = \omega^{l_2} \wedge \overset{\bullet}{\pi}{}^{j_2}_{l_2} + \omega^\lambda \wedge \overset{\bullet}{\pi}{}^{j_2}_\lambda + \tfrac{1}{2}\overset{\bullet}{W}{}^{j_2}_{l_1 h_1}\omega^{l_1} \wedge \omega^{h_1}, \tag{3.7}$$
$$d\omega^\mu = \omega^\lambda \wedge \overset{\bullet}{\pi}{}^{\mu}_\lambda + \tfrac{1}{2}\overset{\bullet}{W}{}^{\mu}_{l_1 h_1}\omega^{l_1}_\lambda \wedge \omega^{h_1} + \overset{\bullet}{W}{}^{\mu}_{l_1 h_2}\omega^{l_1} \wedge \omega^{h_2} + \tfrac{1}{2}\overset{\bullet}{W}{}^{\mu}_{l_2 h_2}\omega^{l_2} \wedge \omega^{h_2},$$

The coefficients $\overset{\bullet}{W}$ define the components of the structure tensor :

$$\overset{\bullet}{\tau} = \left(\tfrac{1}{2}\overset{\bullet}{W}{}^{j_2}_{l_1 h_1},\ \tfrac{1}{2}\overset{\bullet}{W}{}^{\mu}_{l_1 h_1},\ \overset{\bullet}{W}{}^{\mu}_{l_1 h_2},\ \tfrac{1}{2}\overset{\bullet}{W}{}^{\mu}_{l_2 h_2}\right) \tag{3.8}$$

and its restrictions for Δ^α represent the integrability tensors $\overset{\bullet}{\tau}_{\Delta}\alpha$ of the differential systems $\Delta^\alpha (\alpha = 1, 2)$.

From the equations (3.6), (3.7), (3.8) and definition 1.1 it follows directly :

Theorem 3.4. *The differential system Δ^2 is the first derived system or Δ^1, if only if the components $\overset{\bullet}{W}{}^{\mu}_{l_1 h_1}$ of the structure tensor $\overset{\bullet}{\tau}$ vanish in each point $x \in M$.*

From the hypothesis that Δ^1 is an involutive differential system it follows :

Corollary 3.2. *In each point $x \in M$, the following conditions are fulfilled.*

$$\overset{\bullet}{W}{}^{j_2}_{l_1 h_1} = 0,\ \overset{\bullet}{W}{}^{\mu}_{l_1 h_1} = 0. \tag{3.9}$$

Theorem 3.5. *The differential system Δ^2 is integrable if and only if, the components $\overset{\bullet}{W}{}^{\mu}_{l_1 h_1}$, $\overset{\bullet}{W}{}^{\mu}_{l_1 h_2}$, $\overset{\bullet}{W}{}^{\mu}_{l_2 h_2}$ of the structure tensor $\overset{\bullet}{\tau}$ vanish identically in each point $x \in M$.*

Corollary 3.3. *A real Δ^α_μ-structure on M is integrable, if and only if, the structure tensor $\overset{\bullet}{\tau}$ vanishes identically on M.*

By the method of differential forms one can determine an invariant canonical coframe, its equations of motion and its integrability conditions, and on this way can build the local differential geometry of real Δ^α-structure on a manifold.

To avoid the difficult and laborions computations, we only studied the integrability of real Δ^α-structures by the method of differential forms.

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