où P^i_j est le champ de tenseurs donné par (2.5).

Théorème 9. *Toutes les connexions linéaires compatibles à la structure Riemann généralisée du type presque 1-horsympllectique métrique hyoperbolique g sont données par la formule* (4.1) *où* $\overset{c}{\Gamma}$ *est la connexion canonique de la proposition* 2, *et* Y *est un champ de tenseurs du type* (1.2) *arbitraire.*

BIBLIOGRAPHIE

1. Atanasiu Gh. — *Sur le problème d'Eisenhart*, Rev. Roum. Math. pures et appl., XVI, 3 (1971), 309—311.
2. Eisenhart L. P. — *General Riemann spaces* I, II, Proc. Nat. Sc., U.S.A. 37 (1951, 5, 38 (1952), 6.
3. Ianuş S. — *Sulla geometria delle varietà consimplectische*, Relajione 18, Università di Napoli, 1972, pg. 42.
4. Kawaguchi A. — *Beziehung zwischen einer metrischen linearen Übertragung und einer nicht — metrischen in einem allgemeinen metrischen Raume*, Akad. Wetensch. Amsterdam Proc., 40 (1937), 596—611.
5. Libermann P. — *Sur les automorphismes infinitésimaux des structures symplectiques et des structures de contact*. Coll. Géom. Diff., Bruxelles (1957), 37—53.
6. Lichnerowicz A. — *Réductivité des algèbres d'automorphismes de structures symplectiques, cosymplectiques et des structures de contact*. Rend. Mat. Univ. Roma, 1—2 (1963), 197—244.
7. Miron R. — *Connexions compatibles aux structures conformes presque symplectiques*. C.R. Acad. Sc. Paris, sér. A, 265 (1967), 685—687.
8. Miron R. and Atanasiu Gh. — *Hyperbolical almost Hermitian structures*. Lucr. Coll. Naţ. Geom. Top., Buşteni-Prahova, 27—30 iunie 1981.
9. Miron R. et Atanasiu Gh. — *Existence et arbitraire des connexions compatibles aux structures Riemann généralisées du type hermitien* (sous presse dans TENSOR, Japan).
10. Miron R. et Oproiu V. — *Connexions compatibles aux structures presque cosymplectiques et conformes presque cosymplectiques*. C.R. Acad. Sc. Paris, sér. A, 268 (1969), 451—454.
11. Obata M. — *Affine connections on manifolds with almost complex, quaternion or Hermitian structure*. Jap. J. Math., 26 (1957) 43—77.
12. Vanstone J. R. — *Connections satisfying a generalized Ricci lemma*. Canadian J. of Math., XVI, 3 (1964).
13. Yano K. — *Differential Geometry on complex and almost complex spaces*. Pergamon Press, New-York (1965).

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A CHARACTERIZATION OF TRANSVERSALITY TO DISTRIBUTIONS

BY

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To the memory of Professor Mendel Haimovici

In this paper we introduce the notion of transversal mapping to a regular distribution (subbundle of the tangent bundle) and we give a characterization of this transversality in terms of the preimage of the distribution by the tangent map.

All manifolds considered are finite dimensional paracompact and smooth. The distributions are always smooth and regular (subbundles of the tangent bundle). As usual, we shall denote by T_xX the tangent space of the smooth manifold X at the point $x\in X$ and by $df_x : T_xX\to T_{f(x)}Y$ the differential of the smooth mapping $f: X\to Y$ at $x\in X$. Also if D is a distribution on a manifold Y, then D_y will denote the fiber of D over $y\in Y$.

Definition 1. *Let X, Y be smooth manifolds and let D be a distribution on Y. A smooth mapping $f: X\to Y$ is said to be transversal to D* (and we denote $f\pitchfork D$) *if for each $x\in X$ we have*

$$df_x(T_xX) + D_{f(x)} = T_{f(x)}Y.$$

The concepts of transversality to a submanifold and to a foliation are well known and are very important for the differential topology (see for example [2]). We think that the notion of transversality to an arbitrary distribution (not necessarily integrable) we have just defined could be also of interest in view of the importance of the distributions for the differential geometry. Certainly, if the distribution is integrable, Definition 1 reduces to the transversality to the corresponding foliation.

Now we give some examples of transversal mappings to distributions.

Example 1. Let $\pi: E\to X$ be a vector bundle and let $s: X\to E$ be a smooth section of E. Then $s\pitchfork TF(E)$, where $TF(E)$ is the vertical tangent bundle $(TF(E)=\ker d\pi)$.

Example 2. Let G, H be Lie groups and let $f: G\to H$ be a homomorphism of Lie groups. Let K be a connected Lie subgroup of H and let D be the Lie algebra of K. Then the following are equivalent :

(i) $f\pitchfork D$ (here D is considered as a distribution on H) ;
(ii) $f\pitchfork K$;
(iii) $f\pitchfork K$ at the unity of G.

Example 3. Let D be the distribution on R^3 generated by the vector fields $\partial/\partial x^1$ and $(\partial/\partial x^2)+(\exp x^1)(\partial/\partial x^3)$ which is clearly not integrable (see [3], p. 195). The mapping $f: R\to R^3$ defined by $f(t)=(0, t, 0)$ is transversal to D.

In the following we shall point out some facts concerning the relationship between the concepts of transversality to a submanifold and to a distribution.

Remark 1. If the mapping $f: X \to Y$ is transversal to the distribution D on Y, then f is transversal to any integral manifold of D. If D is integrable, the converse is also true.

Remark 2. Example 2 illustrates a case when the transversality to only one integral manifold implies the transversality to distribution. The following result shows that always the transversality to an integrable distribution is implied locally by the transversality to one integral manifold for mappings defined on compact manifolds.

Proposition 1. *Let X be a compact manifold (possibly with boundary) Y a manifold without boundary, $f: X \to Y$ a mapping, D an integrable distribution on Y and W a (maximal) integral manifold of D. If $f \pitchfork W$ and $(f|_{\partial X}) \pitchfork W$, then there exists an open neighborhood U of W in Y such that $f \pitchfork (D|_U)$ and $(f|_{\partial X}) \pitchfork (D|_U)$.*

Proof. Let $m = \dim Y$, $r = \dim D$ and let $p: R^m = R^r \times R^{m-r} \to R^{m-r}$ be the projection onto the second factor. For every $x \in W$ let (V, φ) be a chart of Y with $x \in V$ such that

$$W^a = \{y \in V \,;\, p \circ \varphi(y) = a\}$$

is an integral manifold of D for each $a \in R^{m-r}$ belonging to a neighborhood of $0 \in R^{m-r}$ and $W^0 = W \cap V$. To obtain U as union of suitable open sets covering W, it is sufficient to prove that there exists $\varepsilon > 0$ such that $f \pitchfork W^a$ and $(f|_{\partial X}) \pitchfork W^a$ for each $a \in R^{m-r}$ with $\|a\| < \varepsilon$. If there were no such $\varepsilon > 0$, then one would find sequences $(a_i) \subset R^{m-r}$ and $(x_i) \subset X$ such that $a_i \to 0$ and the following properties hold

(i) if $x_i \in \operatorname{Int} X$, then f is not transversal to W^{a_i} at x_i;

(ii) if $x_i \in \partial X$, then $f|_{\partial X}$ is not transversal to W^{a_i} at x_i.

Since X is compact we can assume $x_i \to x$ and thus we deduce $f(x) \in W$. I $x \in \operatorname{Int} X$, then $x_i \in \operatorname{Int} X$ for i sufficiently large and according to the hypothesis $f \pitchfork W$, we derive rank $(p\, \varphi f)_{x_i} = m - r$ for i large enough. This contradicts the choice of the points x_i. If $x \in \partial X$, then by essentially the same argument we contradict the conditions (i) and (ii). This completes the proof.

The following example (obtained with the help of Dr. G h e o r g h e A n i c u l ă e s e i) shows that in the statement of the previous result we cannot drop the hypothesis on M to be compact.

Example 4. Let D be the 0-dimensional distribution on R and let $f: R \to R$, $f(x) = (\exp(-x)) \sin x$. Denoting $W = \{0\}$, it follows $f \pitchfork W$, but for each $\varepsilon > 0$ there exists $x \in f^{-1}(-\varepsilon, \varepsilon)$ such that $df_x = 0$.

It is well known that if $f: X \to Y$ is transversal to a submanifold $W \subset Y$, the pull-back $f^{-1}(W)$ is a submanifold of X of the same codimension as W. The converse is obviously false. In the case of transversality with respect to a distribution the property holds together with its converse. This is the main result of the paper.

Theorem 1. *Let $f: X \to Y$ be a smooth mapping of a manifold (possibly with boundary) X to a manifold without boundary Y and let D be a distribution on Y. Then we have $f \pitchfork D$ and $(f|_{\partial X}) \pitchfork D$ if and only if $D' = (df)^{-1}(D)$ is a distribution on X such that* $\operatorname{codim} D' = \operatorname{codim} D$ *and* $\operatorname{codim} (D'|_{\partial X}) = \operatorname{codim} D$.

Proof. Assume $f \pitchfork D$ and $(f|_{\partial X}) \pitchfork D$. If we take a Riemannian metric on Y then there exists a subbundle E of $T(Y)$ such that $D \oplus E = T(Y)$ and let $p: T(Y) \to E$ denote the projection. By the transversality conditions, it follows that $p \circ df$ and $p \circ d(f|_{\partial X})$ are of maximal rank. Then $\ker(p \circ df) = D'$ and $\ker(p \circ d(f|_{\partial X}))$ are subbundles of $T(X)$ and $T(\partial X)$ respectively (see for example [2]) and clearly are satisfied the codimension properties.

Conversely, assume $D' = (df)^{-1}(D)$ is a distribution on X with the properties from the statement. Then the linear map $f_{*x}: T_x X / D'_x \to T_{f(x)} Y / D_{f(x)}$ induced by $df_x: T_x X \to T_{f(x)} Y$ is an isomorphism of linear spaces for all $x \in X$ and therefore the composition

$$T_x X \xrightarrow{df_x} T_{f(x)} \to T_{f(x)} Y / D_{f(x)}$$

is surjective. It follows $f \pitchfork D$ and in the same way we deduce $(f|_{\partial X}) \pitchfork D$.

Corollary 1. *Let $f: X \to Y$ be a smooth mapping and let D be a distribution on Y. If W is an integral manifold of D, then $f^{-1}(W)$ is an integral manifold of $(df)^{-1}(D)$. Thus if D is integrable, it follows that $(df)^{-1}(D)$ is also integrable.*

Proof. Since $f \pitchfork D$, it follows $f \pitchfork W$. Therefore $f^{-1}(W)$ is a submanifold of X of the same codimension as D. If $x \in f^{-1}(W)$, by [2] we have

$$T_x f^{-1}(W) = (df_x)^{-1}(T_{f(x)} W).$$

By Theorem 1, $(df)^{-1}(D)$ is a distribution on X and consequently $f^{-1}(W)$ is an integral manifold of $(df)^{-1}(D)$.

From Theorem 1 and Corollary 1 we obtain, in particular, a result describing a case when a morphism of manifolds endowed with foliations induces a morphism of the corresponding foliations. We think that it is useful to give a result of this type also for foliations with singularities (in the sense of [5]). Using the terminology from [5], let $F(A)$ be the foliation with singularities on X defined by a family A of arrows. Recall that an arrow $a: R \times X \to X$ is said to preserve the equivalence relation ρ on X if for every $(t, x) \in \operatorname{dom} a$, we have $x \rho a(t, x)$.

Proposition 2. *Let ρ and ρ' be equivalence relations on the manifolds X and Y respectively. Assume $f: X \to Y$ is a smooth mapping which respects the equivalence relations ρ and ρ' (i.e. if $x_1 \rho x_2$, then $f(x_1) \rho' f(x_2)$). Denote by A and B the families of all arrows which preserve ρ and ρ' respectively. Then f is a smooth mapping between the foliations with singularities $F(A)$ and $F(B)$.*

Proof. For a fixed $x \in X$ denote by L and L' the accessible sets of A and B containing x and $f(x)$ respectively. By Theorem 1 from [5], L and L' are connected immersed submanifolds of X and Y respectively. Let $k = \dim L$ and $h = \dim L'$. By [[5], Lemma 3.2] there exist arrows $a_1, \ldots, a_k \in A$ and $b_1, \ldots, b_h \in B$ such that $\{(d/dt)\, a_i(t, x)|_{t=0}\}_{1 \le i \le k}$ and

$\{(d/dt)b_j(t, f(x))|_{t=0}\}_{1 \leqslant j \leqslant k}\}$ are linearly independent. Let $g(t_1, \ldots, t_k) = a_1^{t_1} \ldots a_k^{t_k}(x)$. Then g is a smooth mapping $V \to X$, where V is a neighborhood of 0 in R^k. Clearly we have $g(V) \subset L$ and therefore $f \circ g(V) \subset L'$. Moreover, by the definition of the leaf, the mapping $g : V \to L$ and $f \circ g : V \to L'$ are smooth. Since $(\text{rank } g)_{,} = k$, we can assume, by contracting V if necessary, that g is a diffeomorphism of V onto an open subset U of L containing x. Thus $f : U \to L'$ is a smooth mapping and so $f : F(A) \to F(B)$ is smooth.

Remark 4. Proposition 2 contains as corollary Theorem 2 from [5] stated for diffeomorphisms of foliations with singularities. Other results concerning transversality to foliations with singularities are proved in [4].

Finally, we remark that Theorem 1 gives a general method to obtain new distributions and therefore different structures on manifolds. For example, we shall apply Theorem 1 to obtain CR-structures (introduced by A. Bejancu in [1]). Let X and Y be Kählerian manifolds with Kählerian structures (J, g) and (J', g') respectively. From [1] we recall that a submanifold N of Y is a CR-submanifold if there exist two complementary orthogonal distributions D and $D^\perp$ on N such that $J'(D_y) = D_y$ and $J'(D^\perp_y) \subset (T_y N)^\perp$, for all $y \in N$.

Proposition 3. *Let $f : X \to Y$ be a smooth mapping of the Kählerian manifolds X and Y such that $J' \circ df = df \circ J$ and let N be a CR-submanifold of Y defined by the distributions D and $D^\perp$. Assume f is transversal to the submanifold N and $f|_{f^{-1}(N)}$ is transversal to the distribution D on N. Then $f^{-1}(N)$ is a CR-submanifold of X.*

Proof. Theorem 1 implies that $E = d(f|_M)^{-1}(D)$ is a distribution on the submanifold $M = f^{-1}(N)$. The transversality conditions imply that for each $x \in M$, $E^\perp_x = df_x^{-1}(D^\perp_{f(x)})$ and $(T_x M)^\perp = df_x^{-1}(T^\perp_{f(x)} N)$. Then it is straightforward to verify $J(E_x) = E_x$ and $J(E^\perp_x) \subset (T_x M)^\perp$.

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REFERENCES

1. Bejancu A. — *Une classe de sous-variétés d'une variété kählérienne*. C.R. Acad. Sc. Paris 286 (1978), 597—599.
2. Bourbaki N. — *Variétés différentiables et analytiques. Fascicule de résultats*. Hermann Paris, § 1—7 (1967), § 8—15 (1971).
3. Brickell F. and Clark R.S. — *Differentiable manifolds*. Van Nostrand Reinhold Co., London, 1970.
4. Motreanu D. — *Distributions and Transversality*. Colocviul Naţional de Geometrie şi Topologie „Gh. Vrănceanu", Vaslui, 1980.
5. Stefan P. — *Accessible sets, orbits and foliations with singularities*. Proc. London Math. Soc. 29 (1974), 699—713.

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UNE CLASSE DE COMPLEXES DE CHAINES D'UNE VARIETE FEUILLETEE

PAR

GHEORGHE PITIŞ

Dédié à la mémoire du Professeur Mendel Haimovici

1. Chaines feuilletées. Pour $p \geqslant 0$ et $q \geqslant 0$ soit $\Delta^{p,q} = \Delta^p \times \Delta^q \subset R^{p+1} \times R^{q+1} \equiv R^{(p+1)+(q+1)}$, où Δ^p et Δ^q sont les simplexes standard de dimension p, resp. q.

Soint V une variété feuilletée à $n+m$ dimensions et de codimension n.

Définition 1.1. *Une application feuilletée $\sigma : \Delta^{p,q} \to V$ s'appelle (p, q)-simplexe feuilleté* de V.

Remarque 1.1. Dans la définition 1.1 σ est la restriction à Δ^{pq} d'une application feuilletée de U dans V, U étant un voisinage ouvert de $\Delta^{p,q}$ dans $R^{(p+1)+(q+1)}$, envisagé comme variété feuilletée de codimension $p+1$.

Définition 1.2. *Si G est un groupe abelien, toute combinaison linéaire finie du type $l = \sum_i b_i \sigma_i$, où $g_i \in G$ et les σ_i sont des (p, q)-simplexes feuilletés, s'appelle (p, q)-chaîne feuilletée de V, à coefficients dans G.*

L'ensemble $C_{p,q}(V, G)$ des (p, q)-chaînes feuiletées peut être doté d'une structure naturelle de groupe abelien. Si $f : V \to W$ est une application feuilletée et si σ est un (p, q)-simplexe feuilleté de V, alors $f\sigma$ est un (p, q)-simplexe feuilleté de W. Nous obtenons donc, d'une manière évidente, les homomorphismes

$$f_{\#(p,q)} : C_{p,q}(V, G) \to C_{p,q}(W, G).$$

Soit $U = U_1 \times U_2 \subseteq R^{n_1+n_2}$, où $U_1 \subseteq R^{n_1}$, $U_2 \subseteq R^{n_2}$ sont des ensembles ouverts convexes, et considérons les points $A_i \in U_1$, $B_j \in U_2$, $0 \leqslant i \leqslant p$, $0 \leqslant j \leqslant q$. Si (ξ^i), (η^j) sont les coordonnées barycentriques des simplexes standard Δ^p et Δ^q, l'application $S_{p,q} = (A_0, \ldots, A_p, B_0, \ldots, B_q)$ définie par

$$S_{p,q}((\xi^i), (\eta^j)) = (\sum_j \xi^i A_i, \sum_j \eta^j B_j) \tag{1}$$

est un (p, q)-simplexe feuilleté de la variété feuilletée $R^{n_1+n_2}$ (de codimension n_1), appelé (p, q)-simplexe feuilleté spécial.

Remarque 1.2. i) L'identité de $\Delta^{p,q}$ est un (p, q)-simplexe feuilleté spécial, noté toujours $\Delta^{p,q}$. En effet il suffit de poser $X_i(\delta_i^k) \in \Delta^p$ et $Y_j(\delta_j^l) \in \Delta^q$.

ii) Si $l = \sum_i g_i \lambda^i$ est une (s, t)-chaîne feuilletée spéciale sur un voisinage ouvert de $\Delta^{p,q}$, sur lequel le (p, q)-simplexe feuilleté σ de V soit défini, alors $\sigma l = \sum_i g_i \sigma \lambda^i$ est une (s, t)-chaîne feuilletée de V.