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ON SET FUNCTIONS OF BOUNDED SLOPE p-VARIATION

BY

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The concept of function of bounded slope variation was introduced by F. Riesz [8] and it has been used to characterize the primitives of the functions of bounded variation.

Afterwards, Riesz's result was generalized by F. N. Huggins [3], [4], using the Lane integral and the Lebesgue-Stieltjes integral; in [6], we introduced the notion of function of bounded slope p-variation on a real compact interval with respect to a real-valued increasing function. In the same time, we proved that a necessary and sufficient condition that a function F be the Lebesgue-Stieltjes integral of a function of bounded p-variation (in Wiener-Young's sense), with respect to an increasing function m, is that F be of bounded slope p-variation with respect to m.

In this paper, we extend the notion of slope p-variation to set functions and we establish an integral representation of the set functions of bounded slope p-variation.

Let $\mathcal{F}$ be a σ-algebra over a space X, μ a finite measure defined on $\mathcal{F}$ and $F : \mathcal{F} \to R$ a set function. We consider a non-void set $A \in \mathcal{F}$ and we denote the σ-algebra induced by $\mathcal{F}$ on A by $\mathcal{F}_A$.

Definition 1. *We shall call $\mathcal{F}$-partition of A any finite collection $(B_i)_{i=\overline{1,m}}=\overline{1,m}$ of disjoint sets $B_i \in \mathcal{F}$ which covers A i.e., $\bigcup_{i=1}^{m} B_i = A$.*

Definition 2. *For $p \geq 1$ we say that the function F is of bounded slope p-variation on the set A with respect to μ, if there exists $M \geq 0$ such that*

$$\left\{ \sum_{i=1}^{n-1} \left| \frac{F(A_{i+1})}{\mu(A_{i+1})} - \frac{F(A_i)}{\mu(A_i)} \right|^p \right\}^{1/p} \leq M, \tag{1}$$

for every $\mathcal{F}$-partition $(A_i)_{i=\overline{1,n}} (n > 1)$ of A, where we make the calculation convention $a/b = 0$ if $a = b = 0$.

The least such number M is called *the slope p-variation of F with respect to μ on A.*

We denote by $BSV_p(A\,;\,\mu)$ the class of functions of bounded slope p-variation on the set A with respect to μ.

Theorem 1. *If F is finite additive and $F \in BSV_p(A\,;\,\mu)$, then there exists $K \geq 0$ such that*

$$(2) \qquad |F(B)| \leq K, \quad \forall B \in \mathcal{F}_A.$$

Proof. We observe that for every $B \in \mathcal{F}_A$ the family $\{B,\ A \setminus B\}$ represents a $\mathcal{F}$-partition of A and that

$$\left|\frac{F(B)}{\mu(B)} - \frac{F(A)-F(B)}{\mu(A)-\mu(B)}\right| \leq M, \quad \forall B \in \mathcal{F}_A$$

implies

$$|F(B)\,\mu(A) - F(A)\,\mu(B)| \leq M\mu(B)\,\mu(A \setminus B) \leq M\,[\mu(A)]^2,$$

which proves the theorem.

Theorem 2. *If $F \in BSV_p(A\,;\,\mu)$ and if F is finite additive, then F satisfies a uniform Lipschitz condition with respect to μ on $\mathcal{F}_A$, namely, there exists a nonnegative number L such that*

$$(3) \qquad |F(B)| \leq L\mu(B)$$

for every $B \in \mathcal{F}_A$.

Proof. Note that, if $\mu(A)=0$, then $F(B)=0$ for every $B \in \mathcal{F}_A$ (since we made the convention $a/b=0$ if $a=b=0$) and the theorem is proved.

Let $\mu(A) \neq 0$ and let us suppose that F does not satisfy a uniform Lipschitz condition with respect to μ on $\mathcal{F}_A$. Hence, for every $n \in N$, there exists $B_n \in \mathcal{F}_A$ such that

$$(4) \qquad |F(B_n)| > n\mu(B_n)\,.$$

Since for every $n \in N$ the family $\{B_n,\ A \setminus B_n\}$ represent a $\mathcal{F}_A$-partition of A, it follows, (by hypothesis), that there exists $M \geq 0$ such that

$$\left|\frac{F(B_n)}{\mu(B_n)} - \frac{F(A)-F(B_n)}{\mu(A)-\mu(B_n)}\right| \leq M, \quad \forall n \in N.$$

Therefore

$$\left|\frac{F(B_n)}{\mu(B_n)}\right| \leq M + \left|\frac{F(A)-F(B_n)}{M(A)-\mu(B_n)}\right|, \quad \forall n \in N,$$

from which, by (4), we obtain

$$(5) \qquad \lim_{n\to\infty}\left|\frac{F(A)-F(B_n)}{\mu(A)-\mu(B_n)}\right| = +\infty.$$

The sequence $(\mu(B_n))$ is bounded and we may assume that $(\mu(B_n))$ converges; otherwise, we choose a suitable subsequence. If $\lim\limits_{n\to\infty} \mu(B_n) = \mu(A)$, then, by (4), we obtain

$$\lim_{n\to\infty} |F(B_n)| = \lim_{n\to\infty}\left|\frac{F(B_n)}{\mu(B_n)}\right| \mu(B_n) = +\infty,$$

which is a contradiction because the function F is bounded on $\mathcal{F}_A$. If $\lim\limits_{n\to\infty} \mu(B_n) \neq \mu(A)$, then, from (5), it follows that

$$\lim_{n\to\infty} |F(A)-F(B_n)| = \lim_{n\to\infty}\left|\frac{F(A)-F(B_n)}{\mu(A)-\mu(B_n)}\right| [\mu(A)-\mu(B_n)] = +\infty,$$

which also contradicts the Theorem 1.

Therefore (3) is satisfied and the theorem is proved.

Corollary 1. *Every finite additive set function $F \in BSV_p(A\,;\,\mu)$ is absolutely continuous on the set A with respect to μ.*

Corollary 2. *If $F \in BSV_p(A\,;\,\mu)$ and if F is finite additive, then there exists $K \geq 0$ such that*

$$\left\{\sum_{i=1}^{n} |F(A_i)|^p\right\}^{1/p} \leq K$$

for every $\mathcal{F}$-partition $(A_i)_{i=\overline{1,n}}$ of A.

Corollary 3. *Every finite additive set function $F \in BSV_p(A\,;\,\mu)$ is of bounded p-variation on the set A with respect to μ, that is, there exists $M \geq 0$ such that*

$$\left\{\sum_{i=1}^{n} \frac{|F(A_i)|^p}{[\mu(A_i)]^{p-1}}\right\}^{1/p} \leq M,$$

for every $\mathcal{F}$-partition $(A_i)_{i=\overline{1,n}}$ of A.

Definition 3. *We say that a function $f : X \to \overline{R}$ is of bounded p-variation (in Wiener-Young's sense) almost everywhere if there exists a number $K \geq 0$ and a set $A \subset X$ of null measure such that*

$$(6) \qquad \left\{\sum_{i=1}^{n} |f(x_i)-f(y_i)|^p\right\}^{1/p} \leq K$$

for every F-partition $(A_i)_{i=\overline{1,n}}$ with non-void sets $X \setminus A$ and for every $x_i,\ y_i \in A_i,\ i=\overline{1,n}$.

We denote by $BV_p(X\,;\,\mu)$ the class of functions of bounded p-variation almost everywhere on X.

Now we want to establish an integral representation of the set functions of bounded slope p-variation. For this, we suppose the following hypotheses:

(i) X is a metric space;
(ii) $Y \subset X$ is a separable subspace;
(iii) $\mathcal{F} = \mathcal{B}(Y)$ is the τ-algebra of Borel subsets of Y;
(iv) $F : \mathcal{F} \to R$ is a finite additive set function;
(v) $\mu : \mathcal{F} \to R_+$ is a measure.

From the Definition 3 and the above hypotheses we easily obtain

Theorem 3. *A function $f \in BV_p(Y\,;\,\mu)$ if and only if there exists a number $K \geq 0$ and a set $A \subset Y$ of null measure such that*

(7) $$\left\{\sum_{i=1}^{n-1} |f(x_{i+1})-f(x_i)|^p\right\}^{1/p} \leqslant K$$

for every finite subset $\{x_1, x_2, \ldots, x_n\}$ *of* $Y \setminus A$.

Theorem 4. *The finite additive function* F *belongs to* $BSV_p(Y;\mu)$ *if and only if there exists* $f \in BV_p(Y;\mu)$ *such that*

(8) $$F(A)=\int_A f d\mu, \quad \forall A \in F.$$

Proof. Let f be the function for which (8) is satisfied, (7) is satisfied on $Y \setminus A_0$ with $A_0 \subset Y$ and $\mu(A_0)=0$ and let $(A_i)_{i=\overline{1,n}}$ $(n>1)$ be an arbitrary $\mathcal{F}$-partition of Y. We may assume (from the Theorem 2 and from the considerations below) without loss of generality that A_i are non-void and $A_i \neq A_0$ for every $i=\overline{1,n}$.

By virtue of Minkowski's inequality, we have

$$\sum_{i=1}^{n-1}\left|\frac{F(A_{i+1})}{\mu(A_{i+1})}-\frac{F(A_i)}{\mu(A_i)}\right|^p\Bigg\}^{1/p} = \left\{\sum_{i=1}^{n-1}\left|\frac{\int_{A_{i+1}}[f(x)-f(x_{i+1})]\,d\mu}{\mu(A_{i+1})} - \frac{\int_{A_i}[f(x)-f(x_i)]\,d\mu}{\mu(A_i)}+f(x_{i+1})-f(x_i)\right|^p\right\}^{1/p} \leqslant$$

$$\leqslant \left\{\sum_{i=1}^{n-1}\left[\frac{\int_{A_{i+1}}|f(x)-f(x_{i+1})|\,d\mu}{\mu(A_{i+1})}\right]^p\right\}^{1/p} + \left\{\sum_{i=1}^{n-1}\left[\frac{\int_{A_i}|f(x)-f(x_i)|\,d\mu}{\mu(A_i)}\right]^p\right\}^{1/p} +$$

$$+\left[\sum_{i=1}^{n-1}|f(x_{i+1})-f(x_i)|^p\right]^{1/p} \leqslant 2\left[\sum_{i=1}^{n}(M_i-m_i)^p\right]^{1/p}+K,$$

where $x_i \in A_i \setminus A_0$, $M_i=\sup\limits_{x\in A_i\setminus A_0} f(x)$, $m_i=\inf\limits_{x\in A_i\setminus A_0} f(x)$ for every $i=\overline{1,n}$.

Therefore $F \in BSV_p(Y;\mu)$.

Conversely, let us suppose that $F \in BSV_p(Y;\mu)$. By the Corollary 1 it follows that F is absolutely continuous with respect to μ. Hence (cf. [7], p. 198) there exists a function $f: Y \to \overline{R}$, $\mathcal{F}$-measurable, finite almost everywhere on Y and determined up to a set of null measure such that

(9) $$F(A)=\int_A f d\mu, \quad \forall A \in \mathcal{F}.$$

We shall prove that $f \in BV_p(Y;\mu)$.

Indeed, if $\mathcal{M}=\{\mathcal{M}_k\}_{k\in N}$ is a regular sequence of networks in Y (see [9], p. 153 and [5], p. 255), let us consider the sequence $f_n: Y \to \overline{R}$ defined by

(10) $$f_n(x)=\begin{cases}\dfrac{F(M)}{\mu(M)} & \text{if } \mu(M)\neq 0,\\ +\infty & \text{if } \mu(M)=0 \text{ and } F(M)>0,\\ -\infty & \text{if } \mu(M)=0 \text{ and } F(M)<0,\\ 0 & \text{if } \mu(M)=F(M)=0,\end{cases}$$

if $x \in M$, where M runs through $\mathcal{M}_n$. We have

(11) $$\lim_{n\to\infty} f_n(x)=f(x)=\frac{dF}{d\mu}(x) \ \mu\text{.a.e.}$$

We denote by

$$A_0=\{x\in Y : x \text{ does not satisfy (11) or } |f(x)|=+\infty\}$$

and let $\{x_1, x_2, \ldots, x_m\}$ $(m>1)$ be an arbitrary finite subset of $Y \setminus A_0$.

If $\mathcal{M}_n=\{Y_1^n, Y_2^n, \ldots, Y_{p_n}^n\}$ we have

(12) $$\left\{\sum_{i=1}^{m-1}|f(x_{i+1})-f(x_i)|^p\right\}^{1/p}=\lim_{n\to\infty}\left\{\sum_{i=1}^{m-1}|f_n(x_{i+1})-f_n(x_i)|^p\right\}^{1/p}\leqslant$$

$$\leqslant \lim_{n\to\infty}\left\{\sum_{k=1}^{p_n-1}\left|\frac{F(Y_{k+1}^n)}{\mu(Y_{k+1}^n)}-\frac{F(Y_k^n)}{\mu(Y_k^n)}\right|^p\right\}^{1/p}\leqslant M.$$

It is clear that this conclusion holds if the network is infinitely countable. Using the Theorem 3, it follows that (12) implies $f \in BV_p(Y;\mu)$ and the theorem is proved.

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