$$\begin{aligned}\Delta_l X^i_u &= \frac{\partial X^i_u}{\partial x^l} + L^h_{ul} X^{l'}_h ,\\ \Delta_l X^i_{j_1} &= \frac{\partial X^{l'}_{j_1}}{\partial x^l} + L^h_{j_1 l} X^{l'}_h ,\\ \Delta_l X^i_{j_1} &= \frac{\partial X^{j_1}_{l'}}{\partial x^l} + L^h_{j_2 l} X^{l'}_h .\end{aligned} \tag{4.4}$$

La connexion linéaire $L(L^i_{jk})$ est compatible avec une Δ^α-structure sur M si et seulement si nous avons :

$$\begin{gathered} L^{l_1}_{ul}(x) = L^{l_2}_{ul}(x) = L^{\mu}_{ul}(x) = L^{l_2}_{j_1 l}(x) = L^{\mu}_{j_1 l}(x) = L^{l_1}_{j_2 l}(x) = L^{\mu}_{j_2 l}(x) = 0,\\ \forall x \in M (1 \leqslant l \leqslant n\ ;\ 1 \leqslant u \leqslant \varkappa\ ;\ \varkappa + 1 \leqslant l_\alpha\ ;\ j_\alpha \leqslant n_\alpha,\ \alpha = 1,2\\ n_1 + n_2 - \varkappa + 1 \leqslant \mu \leqslant n)\end{gathered} \tag{4.5}$$

En utilisant la définition 4.3 et les relations (4.4) et (4.5) on peut démontrer sous difficultés :

Théorème 4.1. *Une connexion linéaire L sur une variété M munie d'une Δ^α-structure s'identifie à une Δ^α-connexion, si et seulement si, elle est compatible à la Δ^α-structure donnée.*

Théorème 4.2. *La connexion symmétrique, associée à une connexion linéaire L compatible à une Δ^α-structure sur M est compatible à cette structure si et seulement si les distributions K, $\Delta^\alpha(\alpha = 1,2)$ sont involutives.*

Théorème 4.3. *Une Δ^α-structure réelle sur M est intégrable si et seulement si, il y a une connexion linéaire et symmetrique compatible à cette structure.*

BIBLIOGRAPHIE

1. B e r n a r d, D. — *Sur la géométrie différentielle des G-structures*, Ann. Inst. Fourier, 1960 T. 10 ; 151—270
2. C a r t a n, E. — *La théorie des groupes finis et la géométrie différentielle traitées par la méthode de repère mobile*, Gauthier-Villars, Paris, 1937
3. C h e v a l l e y, C. — *Theory of Lie Groups*, I. Princeton, 1946
4. G h e o r g h i e v, Gh., O p r o i u, V. — *Geometrie diferențială*, București EDP, 1977
5. K o b a y a s h i, S., N o m i z u, K. — *Foundation of Differential Geometry*, vol. I, II 1963, 1969. Willey Interscience, Publ. New York, London
6. M i r o n, R. — *Introducere în geometrie diferențială*, vol. I, Universitatea „Al. I. Cuza"-Iași (litografiat), 1971
7. M a c o v e i, E. — *Geometria diferențială a Δ^α-structurilor pe o varietate diferențială*, Teză de doctorat, Iași, 1980
8. N e a g u, A. — *On the intersection of principal fibre subbundles*, Atti Acad. Naz. Lincei 57, 1978 ; 350—354
9. P a p u c, D. — *Sur la géométrie différentielle des champs de drapeaux des variétés différentiables*. An. Univ. Timișoara, T. VI, 1968, 275—284
10. P a p u c, D., A l b u, A. — *Elemente de geometrie diferențială globală*. EDP, București, 1973
11. S i r o c o v, A. P. — *Structuri pe varietăți diferențiabile* (l. rusă) Itoghi nauk — *Algebra — Topologia — Geometria*, Moscova 1969, 127—188
12. V r ă n c e a n u, Gh. — *Asupra invarianților intrinseci ai spațiilor neolonome*, An. Univ. București, T. 11, 1956, 9—23

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ON THE FIRST STRAIN-GRADIENT THEORY OF ELASTODYNAMICS

BY

D. IEȘAN

To Professor Adolf Haimovici on the occasion of his 70-th birthday

1. Introduction. The elastic energy density of non-simple bodies depends on the derivatives of the displacement vector up to the n-th order, $n > 1$ (see e.g. [1]). In the case that $n = 2$, we speak of the first strain-gradient theory. For detailed analysis of this case see [2, 3].

This paper is concerned with the boundary-initial-value problem in the first strain-gradient theory. Using the method established in [4, 5], in the first part of the paper we derive the reciprocal theorem and a variational theorem of Gurtin type. Variational theorems in the static case were established in [3]. In the second part of the paper we establish two complete solutions of the field equations for isotropic solids and derive the fundamental solution in the case of steady vibrations.

2. Preliminaries. We refer the motion of the continuum to a fixed system of rectangular Cartesian axes $Ox_i (i = 1, 2, 3)$. We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers (1, 2, 3), summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate. A superposed dot denotes partial derivation with respect to time t. We assume that the body occupies a bounded region B with Lipschitz boundary ∂B which consists of a finite number of smooth surfaces. Let γ_α be intersections of two adjointed smooth surfaces and $\Gamma = \cup \gamma_\alpha$.

Let $T = [0, t_0)$ or $[0, \infty)$, and let u and v be scalar fields on $B \times T$ that are continuous in time. We denote by $u * v$ the convolution of u and v

$$[u * v](x, t) = \int_0^t u(x, t - \tau) v(x, \tau) d\tau .$$

The basic equations in the linear theory of anisotropic elastic solids are [2]:

— the equations of motion

$$\tau_{ji,j} - \mu_{sji,sj} + F_i = \rho \ddot{u}_i , \tag{2.1}$$

— the constitutive equations

$$\begin{aligned}\tau_{ij} &= A_{ijrs} e_{rs} + B_{ijpqr} \varkappa_{pqr} ,\\ \mu_{ijk} &= B_{rsijk} e_{rs} + C_{ijkpqr} \varkappa_{pqr} ,\end{aligned} \tag{2.2}$$

— the strain-displacement relations

$$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}),\quad \varkappa_{ijk}=u_{k,ij}. \tag{2.3}$$

In these equations we have used the following notations:
u_i — the components of the displacement vector, τ_{ij} — the stress tensor, μ_{ijk} — the couple-stress tensor, F_i — the body force per unit volume, ρ — the density in the reference configuration, e_{ij}, $\varkappa_{ijk}$ — the strain tensors, A_{ijrs}, B_{ijpqr}, C_{ijkpqr} — the elastic coefficients. The elastic coefficients satisfy the symmetry conditions.

$$\begin{aligned}&A_{ijrs}=A_{rsij}=A_{jirs},\ B_{ijpqr}=B_{ijqpr}=B_{jipqr}\ ,\\ &C_{ijkpqr}=C_{pqrijk}=C_{jikpqr}\ .\end{aligned} \tag{2.4}$$

Let $\partial B=\Sigma_1\cup\Sigma_2\cup\Gamma\cup N$ be a disjoint decomposition of ∂B, Σ_1 and Σ_2 being either open in ∂B or empty, Γ and N are sets of zero surface measure. Suppose that Γ has a finite one-dimensional measure. We consid er the following boundary conditions [1]—[3]

$$\begin{aligned}&u_i=\tilde{u}_i\ ,\ \nu_i\equiv u_{i,j}n_j=\tilde{\nu}_i \quad \text{on } \Sigma_1\times T,\\ &P_i\equiv[\tau_{ji}-\mu_{sji,s}+(\mu_{pji}n_pn_r-\mu_{pri}n_pn_j)_{,r}]n_j=\tilde{P}_i\\ &R_i\equiv\mu_{rsi}n_rn_s=\tilde{R}_i \qquad \text{on } \Sigma_2\times T,\\ &Q_i\equiv<\mu_{pji}n_p\eta_j>=\tilde{Q}_i \ \text{ on } \Gamma\times T,\end{aligned} \tag{2.5}$$

where $\eta_j=\varepsilon_{jrs}s_rn_s$ and we have used the notations: n_i — the components of the outward unit normal to ∂B, $<>$ denotes the difference of limits from both sides of γ_α, s_i — the components of the unit tangent vector to γ_α, ε_{ijk} — the alternating symbol, $\tilde{u}_i$, $\tilde{\nu}_i$, $\tilde{P}_i$, $\tilde{R}_i$, $\tilde{Q}_i$ — prescribed functions.

The initial conditions are

$$u_i(x,0)=a_i(x),\ \dot{u}_i(x,0)=b_i(x),\ x\in\overline{B}, \tag{2.6}$$

where a_i, b_i are prescribed functions.

We assume that: i) A_{ijkl}, B_{ijpqr}, C_{ijkpqr}, ρ are smooth on $\overline{B}$; ii) F_i is continuous on $\overline{B}\times T$; iii) a_i, b_i are continuous on $\overline{B}$; iv) $\tilde{u}_i$ is continuous on $\Sigma_1\times T$; v) $\tilde{\nu}_i$ is continuous in time and piecewise regular on $\Sigma_1\times T$; vi) $\tilde{P}_i$, $\tilde{R}_i$ are continuous in time and piecewise regular on $\Sigma_2\times T$; vii) $\tilde{Q}_i$ is continuous in time and piecewise regular on $\Gamma\times T$.

Let us introduce the notations

$$g(t)=t,\ f_i=g*F_i+\rho(tb_i+a_i). \tag{2.7}$$

Following [6] we have

Theorem 2.1. *The functions u_i, τ_{ij}, μ_{ijk} satisfy the equations* (2.1) *and the initial conditions* (2.6), *if and only if*

$$g*(\tau_{ji,j}-\mu_{sji,sj})+f_i=\rho u_i. \tag{2.8}$$

This theorem enables us to give an alternate formulation of the boundary-initial-value problem in which the initial conditions are incorporated into the field equations. Thus, $S=\{u_i, e_{ij}, \varkappa_{ijk}, \tau_{ij}, \mu_{ijk}\}$ is a solution of the boundary-initial-value problem if and only if S satisfies the equations (2.2), (2.3), (2.8) and the boundary conditions (2.5).

3. **Variational Theorems.** In order to derive variational theorems, we will establish a reciprocal theorem by using the method given in [5].

Let us consider the body subjected to two different systems of loadings $L^{(\alpha)}=\{F_i^{(\alpha)}, \tilde{u}_i^{(\alpha)}, \tilde{\nu}_i^{(\alpha)}, \tilde{P}_i^{(\alpha)}, \tilde{R}_i^{(\alpha)}, \tilde{Q}_i^{(\alpha)}, a_i^{(\alpha)}, b_j^{(\alpha)}\}$, $(\alpha=1,\ 2)$ and the two corresponding states $C^{(\alpha)}=\{u_i^{(\alpha)}, e_{ij}^{(\alpha)}, \varkappa_{ijk}^{(\alpha)}, \tau_{ij}^{(\alpha)}, \mu_{ijk}^{(\alpha)}\}$. We denote by $f_i^{(\alpha)}$, $\nu_i^{(\alpha)}$, $P_i^{(\alpha)}$, $R_i^{(\alpha)}$, $Q_i^{(\alpha)}$ the values of f_i, ν_i, P_i, R_i, Q_i corresponding to the states $C^{(\alpha)}$.

Theorem 3.1. *If an elastic solid is subjected to two systems of loadings $L^{(\alpha)}(\alpha=1,\ 2)$, then between the corresponding states $C^{(\alpha)}$ there is the following relation*

$$\begin{aligned}&\int_B f_i^{(1)}*u_i^{(2)}\,dV+\int_{\partial B}g*[P_i^{(1)}*u_i^{(2)}+R_i^{(1)}*\nu_i^{(2)}]dA+\int_\Gamma g*Q_i^{(1)}*u_i^{(2)}\,ds=\\ &=\int_B f_i^{(2)}*u_i^{(1)}\,dV+\int_{\partial B}g*[P_i^{(2)}*u_i^{(1)}+R_i^{(2)}*\nu_i^{(1)}]dA+\int_\Gamma g*Q_i^{(2)}*u_i^{(1)}ds.\end{aligned} \tag{3.1}$$

Proof. On the basis of the relations (2.4), we conclude from the constitutive equations that

$$\tau_{ij}^{(1)}*e_{ij}^{(2)}+\mu_{ijk}^{(1)}*\varkappa_{ijk}^{(2)}=\tau_{ij}^{(2)}*e_{ij}^{(1)}+\mu_{ijk}^{(2)}*\varkappa_{ijk}^{(1)}. \tag{3.2}$$

If we introduce the notations

$$L_{\alpha\beta}=\int_B g*(\tau_{ij}^{(\alpha)}*e_{ij}^{(\beta)}+\mu_{ijk}^{(\alpha)}*\varkappa_{ijk}^{(\beta)})\,dV, \tag{3.3}$$

then from (3.2), we have

$$L_{12}=L_{21}. \tag{3.4}$$

Using the equations (2.3), (2.8) and the divergence theorem we obtain

$$\begin{aligned}L_{\alpha\beta}=&\int_{\partial B}g*[P_i^{(\alpha)}*u_i^{(\beta)}+R_i^{(\alpha)}*\nu_i^{(\beta)}]\,dA+\int_\Gamma g*Q_i^{(\alpha)}*u_i^{(\beta)}\,ds+\\ &+\int_B[f_i^{(\alpha)}*u_i^{(\beta)}-\rho\ddot{u}_i^{(\alpha)}*u_i^{(\beta)}]dV.\end{aligned} \tag{3.5}$$

From (3.4) and (3.5) we obtain (3.1).

Let us consider now the equations (2.8). By using (2.2) and (2.3), the equations (2.8) can be expressed in terms of components of displacement vector. If we introduce the notation

$$A_i\mathbf{u}=-g*[(A_{ijrs}u_{r,s}+B_{ijpqr}u_{r,pq})_{,j}-(B_{pqsji}u_{p,q}+C_{sjipqr}u_{r,pq})_{,sj}]+\rho u_i,$$

then the equations (2.8) become

$$A_i \mathbf{u} = f_i.$$

We write these equations in the form of one vector equation

$$\mathbf{Au} = \mathbf{f}.$$

Let us consider the case of homogeneous boundary conditions. In this case, the relation (3.1) reduces to

$$\int_B f_i^{(1)} * u_i^{(2)} dV = \int_B f_i^{(2)} * u_i^{(1)} dV. \tag{3.6}$$

Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors with components $u_i^{(1)}$ and $u_i^{(2)}$, respectively. If we denote by $\mathbf{f}^{(\alpha)}$ the vectors with the components $f_i^{(\alpha)}$, then $\mathbf{Au} = \mathbf{f}^{(1)}$, $\mathbf{Av} = \mathbf{f}^{(2)}$ and the relation (3.6) can be written in the form

$$\int_B \mathbf{v} * \mathbf{Au} dV = \int_B \mathbf{u} * \mathbf{Av} dV.$$

This relation shows that for homogeneous boundary condition the operator $\mathbf{A}$ is symmetric in convolution. Let us denote by D_A the domain of the definition of the operator $\mathbf{A}$.

Following [5], [8] we have

Theorem 3.2. *Let $M \subset D_A$ be the set of all displacement fields which satisfy the homogeneous boundary conditions, and for each $t \in T$ define the functional $\Lambda_t\{.\}$ on M by*

$$\Lambda_t\{\mathbf{u}\} = \int_B [\mathbf{u} * \mathbf{Au} - 2\mathbf{u} * \mathbf{f}](x, t) dV_x,$$

for every $u \in M$. Then

$$\delta\Lambda_t\{\mathbf{u}\} = 0, \ t \in T,$$

at $\mathbf{u} \in M$ if and only if $\mathbf{u}$ is a solution of the boundary-initial-value problem with homogeneous boundary condition.

With the help of (2.2)–(2.4), (3.3) and (3.5) we can express the functional $\Lambda_t\{.\}$ in the form

$$\Lambda_t\{\mathbf{u}\} = \int_B [g * (A_{ijrs} u_{r,s} * u_{i,j} + 2B_{ijpqr} u_{i,j} * u_{r,pq} +$$

$$+ C_{ijkpqr} u_{k,ij} * u_{r,pq})](x, t) dV_x + \int_B [\rho u_i * u_i](x, t) dV_x -$$

$$- 2\int_B [f_i * u_i](x, t) dV_x.$$

As in [5] we can prove

Theorem 3.3. *Let $K \subset D_A$ be the set of all displacement fields which satisfy the displacement boundary conditions $(2.5)_1$, and for each $t \in T$ define the functional $\Phi_t\{.\}$ on K by*

$$\Phi_t\{\mathbf{u}\} = \int_B [g * (A_{ijrs} u_{i,j} * u_{r,s} + 2B_{ijpqr} u_{i,j} * u_{r,pq} +$$

$$+ C_{ijkpqr} u_{k,ij} * u_{r,pq})](x, t) dV_x + \int_B [\rho u_i * u_i](x, t) dV_x -$$

$$- 2\int_B [f_i * u_i](x, t) dV_x - 2\int_{\Sigma_1} [g * (\tilde{P}_i * u_i + \tilde{R}_i * v_i)](x, t) dA_x -$$

$$- 2\int_\Gamma [g * \tilde{Q}_i * u_i](x, t) ds_x.$$

Then

$$\delta\Phi_t\{\mathbf{u}\} = 0, \ t \in \mathbf{T},$$

at $\mathbf{u} \in K$ if and only if $\mathbf{u}$ is a solution of the boundary-initial-value problem.

The method can be used to derive other variational theorems, similar to those established in [5], [7].

4. Complete Solutions of the Field Equations. Fundamental Solution for Steady Vibrations. Throughout this section, B is homogeneous and isotropic so that the constitutive equations are

$$\begin{aligned} \tau_{ij} &= \lambda e_{rr}\delta_{ij} + 2\mu e_{ij}, \\ \mu_{ijk} &= \frac{1}{2}\alpha_1(\varkappa_{rri}\delta_{jk} + 2\varkappa_{krr}\delta_{ij} + \varkappa_{rrj}\delta_{ik}) + \alpha_2(\varkappa_{irr}\delta_{jk} + \\ &+ \varkappa_{jrr}\delta_{ik}) + 2\alpha_3\varkappa_{rrk}\delta_{ij} + 2\alpha_4\varkappa_{ijk} + \alpha_5(\varkappa_{kji} + \varkappa_{kij}), \end{aligned} \tag{4.1}$$

where α, μ, α_1, α_2, ..., α_5 are the elastic coefficients, and δ_{ij} is Kronecker's delta.

Let us introduce the notations

$$\begin{aligned} &\Box_\alpha = c_\alpha^2 (1 - l_\alpha^2 \triangle)\triangle - \frac{\partial^2}{\partial t^2}, \qquad (\alpha = 1, 2), \\ &c_1^2 = \frac{\lambda + 2\mu}{\rho}, \qquad c_2^2 = \frac{\mu}{\rho}, \\ &l_1^2 = \frac{2}{\lambda + 2\mu}\sum_{i=1}^{5} \alpha_i, \qquad l_2^2 = \frac{2}{\mu}(\alpha_3 + \alpha_4), \end{aligned} \tag{4.2}$$

where Δ is the Laplacian.

From (2.1), (2.3) and (4.1) we obtain the displacement equation of motion in the form

$$\Box_2 \mathbf{u} + [c_1^2 - c_2^2 - (c_1^2 l_1^2 - c_2^2 l_2^2)\Delta] \text{grad div } \mathbf{u} = -\frac{1}{\rho}\mathbf{F}. \tag{4.3}$$

The equation (4.2) may be written in the alternative form

$$c_1^2(1-l_1^2\Delta)\,\mathrm{grad\,div}\,\mathbf{u}-c_2^2(1-l_2^2\Delta)\,\mathrm{curl\,curl}\,\mathbf{u}-\ddot{\mathbf{u}}=-\frac{1}{\rho}\mathbf{F}. \tag{4.4}$$

We assume that $\mathbf{F}/\rho$ admits the Helmholtz decomposition

$$\frac{1}{\rho}\mathbf{F}=-\mathrm{grad}\,Q-\mathrm{curl}\,\mathbf{H}, \qquad \mathrm{div}\,\mathbf{H}=0,$$

with Q and $\mathbf{H}$ of class C^2 on B.

A solution $\mathbf{u}$ of class C^4 on $B\times T$ will be referred to as an elastic motion corresponding to $\mathbf{F}$.

i) *Green-Lamé solution.* Let

$$\mathbf{u}=\mathrm{grad}\,\varphi+\mathrm{curl}\,\Psi, \tag{4.5}$$

where φ and Ψ are class C^5 fields on $B\times T$ that satisfy

$$\Box_1\varphi=Q, \qquad \Box_2\Psi=\mathbf{H}. \tag{4.6}$$

Then $\mathbf{u}$ is an elastic motion corresponding to $\mathbf{F}$.

This result follows from (4.3), (4.6).

ii) *Cauchy-Kovalevski-Somigliana solution.* Let

$$\mathbf{u}=\Box_1\mathbf{G}-[c_1^2-c_2^2-(c_1^2l_1^2-c_2^2l_2^2)\Delta]\,\mathrm{grad\,div}\,\mathbf{G}, \tag{4.7}$$

where $\mathbf{G}$ is a class C^6 vector field on $B\times T$ that satisfies

$$\Box_1\Box_2\mathbf{G}=-\frac{1}{\rho}\mathbf{F}. \tag{4.8}$$

Then $\mathbf{u}$ is an elastic motion corresponding to $\mathbf{F}$.

This result follows from (4.2) and (4.7).

iii) *Fundamental solution for steady vibrations.* In what follows we use the solution (4.7) in order to determine the fundamental solution of the displacement equation of motion in the case of steady vibrations. We assume that

$$\mathbf{u}=\mathrm{Re}[\mathbf{u}^*(x)e^{-i\omega t}], \qquad \mathbf{F}=\mathrm{Re}\,[\mathbf{F}^*(x)e^{-i\omega t}], \qquad i=\sqrt{-1}. \tag{4.9}$$

If we introduce the notations

$$\mathbf{G}=\mathrm{Re}[\mathbf{G}^*(x)e^{-i\omega t}], \qquad \Box_\alpha^*=c_\alpha^2(1-l_\alpha^2\Delta)\Delta+\omega^2, \tag{4.10}$$

then from (4.7), (4.8) and (4.9) we obtain

$$\mathbf{u}^*=\Box_1^*\mathbf{G}^*-[c_1^2-c_2^2-(c_1^2l_1^2-c_2^2l_2^2)\Delta]\,\mathrm{grad\,div}\,\mathbf{G}^*, \tag{4.11}$$

$$\Box_1^*\Box_2^*\mathbf{G}^*=-\frac{1}{\rho}\mathbf{F}^*. \tag{4.12}$$

Let us investigate the effect of the body force acting in an infinite body along the axis Ox_1. In this case $F_i^*=f^*\delta_{i1}$ so that we can take $G_2^*=$

$=G_3^*=0$. The function G_1^* satisfies the equation

$$\Box_1^*\Box_2^*\chi=-\frac{1}{\rho}f^*. \tag{4.13}$$

The equation (4.13) can be written in the form

$$D_1D_2D_3D_4\chi=-af^*, \tag{4.14}$$

where

$$\begin{aligned} &D_i=\Delta+k_i^2\,, \qquad a^{-1}=\rho c_1^2c_2^2l_1^2l_2^2,\\ &k_{1,2}^2=[-c_1^2\pm(c_1^4+4c_1^2l_1^2)^{1/2}]/2c_1^2l_1^2,\\ &k_{3,4}^2=[-c_2^2\pm(c_2^4+4c_2^2l_2^2)^{1/2}]/2c_2^2l_2^2. \end{aligned} \tag{4.15}$$

If the functions g_s satisfy the equations

$$D_jg_j=-af^* \text{ (no sum ; } j=\overline{1,4}), \tag{4.16}$$

then we have

$$\chi=\sum_{s=1}^{4}a_sg_s\,, \tag{4.17}$$

where

$$a_1^{-1}=(k_1^2-k_2^2)(k_1^2-k_3^2)(k_1^2-k_4^2), \qquad a_2^{-1}=(k_2^2-k_1^2)(k_2^2-k_3^2)(k_2^2-k_4^2),$$
$$a_3^{-1}=(k_3^2-k_1^2)(k_3^2-k_2^2)(k_3^2-k_4^2), \qquad a_4^{-1}=(k_4^2-k_1^2)(k_4^2-k_2^2)(k_4^2-k_3^2).$$

Let us note that we have

$$\begin{aligned} &k_1^2=\varkappa_1^2, \quad \varkappa_1>0\,; \qquad k_2^2=-\sigma_1^2, \quad \sigma_1>0,\\ &k_3^2=\varkappa_2^2, \quad \varkappa_2>0\,; \qquad k_4^2=-\sigma_2^2, \quad \sigma_2>0. \end{aligned} \tag{4.18}$$

If we assume that $f^*=\delta(x-y)$, where δ is the Dirac measure, we get

$$g_1=\frac{a}{4\pi r}\,e^{i\varkappa_1 r}, \; g_2=\frac{a}{4\pi r}\,e^{-\sigma_1 r}, \; g_3=\frac{a}{4\pi r}\,e^{i\varkappa_2 r}, \; g_4=\frac{a}{4\pi r}\,e^{-\sigma_2 r},$$
$$r^2=(x_i-y_i)(x_i-y_i),$$

and (4.17) leads to

$$\chi(x,y\,;\omega)\equiv E=\frac{a}{4\pi r}\,(a_1e^{i\varkappa_1 r}+a_2e^{-\sigma_1 r}+a_3e^{i\varkappa_2 r}+a_4e^{-\sigma_2 r}). \tag{4.19}$$

If we take $G_i^*=E\delta_{1i}$ in (4.11), we obtain the corresponding displacement vector, denoted by $\mathbf{u}^{(1)}$. In general, if the force has the form $F_i^*=\delta_{ij}\delta(x-y)$ then $G_i^*=E\delta_{ij}$ and the corresponding displacement vector is denoted by $u^{(j)}$. From (4.11) we obtain

$$u_i^{*(j)}=\delta_{ij}\Box_1^*E-[c_1^2-c_2^2-(c_1^2l_1^2-c_2^2l_2^2)\Delta]E_{,ij}. \tag{4.20}$$

The components of the vector $\mathbf{u}^{(j)}$ are $u_k^{(j)} = \mathrm{Re}[u_k^{*(j)} e^{-i\omega t}]$. The solution (4.20) can be used in order to obtain the representation of Somigliana type.

In the case of elastostatics the fundamental solution was established in [9].

REFERENCES

1. Toupin, R. A. — *Theories of elasticity with couple stresses.* Arch. Rational Mech. Anal., 17 (1964), 85—112
2. Mindlin, R. D., Eshel, N. N. — *On first strain-gradient theories in linear elasticity.* Int. J. Solids and Structures, 4 (1968), 75—95
3. Hlavaček, I., Hlavaček, M. — *On the existence and uniqueness of solution and some variational principles in linear theories of elasticity with couple-stresses.* Applikace matematiky, 14 (1969), 411—427
4. Ieşan, D. — *Sur la théorie de la thérmoélasticité micropolaire couplée.* C.R. Acad. Sc. Paris, 265 (1967), 271—275
5. Ieşan, D. — *On reciprocity theorems and variational theorems in linear elastodynamics.* Bull. Acad. Polon. Sci., Sér. Sci. techn, 22 (1974), 273—281
6. Ignaczak, J. — *A completness problem for stress equation of motion in the linear elasticity theory.* Arch. Mech. Stos., 15 (1963), 225—236
7. Gurtin, M. E. — *Variational principles for linear elastodynamics.* Arch. Rational Mech. Anal., 16 (1964), 34—50
8. Hlavaček, I. — *Variational formulation of the Cauchy problem for equations with operator coefficients.* Aplikace matematiky, 16 (1971)
9. Rogula, D. — *Some basic solutions in strain gradient elasticity theory of arbitrary order.* Arch. Mech., 25 (1973), 43—68

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ON CERTAIN ASPECTS FOR A SIMPLE RANDOM WALK WITH A REFLECTING AND A RANDOM ABSORBING BARRIER

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In the present we study the probability distribution of the maximum distance upwards run by the barrier in a simple random walk with a reflecting and a random absorbing barrier. A theorem is provided where the above-probability distribution is expressed in terms of the probabilities of absorption and non-absorption of two other interesting random walk processes. The first one is the probability of the process being in a non-absorbing state at time n in a simple random walk between an absorbing and a reflecting barrier. The second is the probability of absorption at time n in a simple random walk process in the presence of a reflecting, a random absorbing and a non-random absorbing barrier.

1. Introduction. Certain aspects of a simple random walk with a reflecting and a random absorbing barrier are presented in the present paper. A stochastic process $\{X_n\}$, $n \geq 0$ is a simple random walk if the state space of the process is the integers and the process executes independent steps $+1, 0, -1$ with stationary (constant) probabilities. The random absorbing barrier is assumed to undergo a simple random walk $\{Y_n\}$, $n \geq 0$ which is independent of $\{X_n\}$. In Vassiliou and Karakostas (1978) for the first time the idea of the random absorbing barrier was introduced. This generalization introduces a certain element of realism into the random walk process. A simple application of the idea of the random absorbing barrier to an insurance problem could be found in Vassiliou (1980a). The absorption probabilities of the simple random walk process between an absorbing (non-random) and a reflecting barrier were given in a correct form by Blassi (1976). This was done for the special case when the steps of the process are $(-1, 1)$. Vassiliou (1980b) studied the probability distribution of the absorption probabilities in a simple random walk between a random absorbing barrier and a reflecting barrier. This was done in two ways: (i) difference equations were derived and an algorithm for their numerical solution was given and (ii) these absorption probabilities were expressed in terms of other known probabilities.

In the present, in section 2, we study the probability distribution of the maximum distance upwards run by the barrier at time n. The probability distributions of the maximums are interesting aspects of any stochastic process in probability theory (see Cox and Miller 1972, p. 39). A theorem is provided where the wanted probability distribution is expressed in terms of the probabilities of absorption and non-absorption of two other interesting random walk processes. Such relations have found a wide range of applications in probability theory. Here among other applications they could lead to analytic expressions for the required pro-