$$n_4=[1/2(a+n-1)],\ n_5=[1/2(n+i-j)],\ c_1=\max\{a-s,0\},$$
$$c_2=\min\{s+j,m-1\}\ c_3=\max\{0,i+s-n\},\ c_4=\min\{m-1,n+i-s\}.$$

We have that

Theorem 2. *Let* $n^*=r/2$, if r *even and* $n^*=(r+2)/2$ if r *odd, then*

a) *If* $2i>a+j$

$$q^{(n)}(r,0,j,i)=p_*^{(n)}(j,i)p_{**}^{(n)}(a,i)-\sum_{s=n^*}^{n_4}\sum_{c=c_1}^{c_2}V(s,c)-\sum_{s=n_4+1}^{n_5}\sum_{c=c_3}^{c_2}V(s,c)-$$
$$-\sum_{s=n_5+1}^{n-1}\sum_{c=c_3}^{c_4}V(s,c);$$

b) *If* $2i<a+j$

$$q^{(n)}(r,0,j,i)=p_*^{(n)}(j,i)p_{**}^{(n)}(a,i)-\sum_{s=n^*}^{n_5}\sum_{c=c_1}^{c_2}V(s,c)-\sum_{s=n_5+1}^{n_4}\sum_{c=c_1}^{c_4}V(s,c)-$$
$$-\sum_{s=n_4+1}^{n-1}\sum_{c=c_3}^{c_4}V(s,c).$$

Letting successively $n=n^*, n^*+1,\dots$ we get from theorem 2 the probabilities $q^{(n)}(r,0,j,i)$. Note that a similar relation could be obtained for the probability $q^{(n)}(r,-1,j,i)$ and thus all the necessary probabilities have been evaluated.

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Received 20.VI.1981

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Supliment la tomul XXVIII, s. I a, 1982

THE (m,p)-BARGAINING SET FOR THE THREE PERSON GAMES

BY

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This paper is concerned with a new concept of „stability" of the coalitional structure of a three-person game. Starting from the ideas of a paper written by Aumann and Maschler [1] and considering a game $G=G(n,v)$, we introduce the parameters m_i and p_i; $i=1,2,3$, in order to define an m-objection of the player i against the player j and a p-counter-objection of the player j against the player i (see Definitions 1.1 and 1.3). Our parameters correspond to the discrete case, because a player i could change the partners (especially j) provided that in the new coalition he will get with m_i more than in the old coalition (such that his request m_i is satisfied), while the payer j could have a p-counter objection against the player i.

Using the notions of m-objections and p-counter-objections, we define the notion of (m,p)-stability (see Definition 1.6). The set of all (m,p)-stable outcomes, called the (m,p)-bargaining set, is defined in Section 2. We give some criteria that a coalitionally rational payoff configuration be (m,p)-stable in a three-person game.

We do not consider here the case of a n-person game, but the extension for this case is clear, although difficult in writing.

1. The (m,p)-stability. Let $G=(N,v)$ be a cooperative n-person game, where $N=\{1,2,\dots,n\}$ is the set of players and $v:\mathcal{P}(N)\setminus\{\emptyset\}\to R$ is the characteristic function of the game.

First we recall some definitions from [1].

We denote the set of all permissible coalitions B of the game by $\mathcal{B}$. A coalition structure is a partition of N. A coalitionally rational payoff configuration (c.r.p.c.) is a pair $(x;C)$, where C is a coalition structure and $x=(x_1,x_2,\dots,x_n)$, called payoff vector, is assumed to satisfy:

(1.1) $$x_i\geq 0,\ \text{for all } i,\ i\in N,$$

(1.2) $$\sum_{i\in S}x_i=v(S),\quad S\in C.$$

In the following we consider a three-person game, given by:

$$(1.3)\qquad \begin{cases} v(\{i\})=0, & i=1,2,3, \\ v(12)=a, & v(23)=b,\ v(13)=c,\ v(123)=d, \end{cases}$$

where a, b, c, d are four non-negative parameters.

Now, we define the notion of *m-objection* and *p-counter objection*. Let m and p be two parameters, $m, p \in R_+^n$.

Definition 1.1. *Let $(x\,;\,C)$ be a c.r.p.c. in a game G. Let i and j be two distinct players belonging to the same coalition C_0, in C. A m-objection of i against j in $(x\,;\,C)$ is a c.r.p.c. $(y\,;\,D)$ for which there exists $D_0 \in D$ such that $i \in D_0$, $j \notin D_0$ and the following inequalities hold:*

a) $y_i > x_i + m_i$, b) $y_k \geq x_k + m_k$, for all $k \in D_0 \setminus \{i\}$.

The c.r.p.c. $(y\,;\,D)$ is also called m-objection in $(x\,;\,C)$, with respect to D_0. Verbally, in his m-objection, the player i claims that, without the aid of a player j, he can get with m_i more in another c.r.p.c. (condition a)), and the new situation is reasonable, because his new partners get with m_k more than they have got in the previous payoff configuration (condition b).

Remark 1.2. Using Definition 1.1 and relation $\sum_{i \in D_0} y_i = v(D_0)$, one can see that:

$$(1.4)\qquad \sum_{i \in D} m_i < v(D_0),$$

therefore a necessary condition for the existence of a m-objection is:

$$(1.5)\qquad v(D_0) > 0.$$

Definition 1.3. *Let $(x\,;\,C)$ be a c.r.p.c. in a game G and let i and j be two distinct players belonging to the same coalition C_0 in C. Let $(y\,;\,D)$ be a m-objection of i against j in $(x\,;\,C)$ with respect to D_0, $D_0 \in D$. A p-counter objection of j against i is a c.r.p.s. $(x\,;\,E)$ for which there exists $E_0 \in E$ such that $j \in E_0$, $i \notin E_0$ and the following conditions are satisfied:*

a') $z_j \geq x_j + p_j$,

b') $z_k \geq y_k + p_k$, *for all* $k \in E_0 \cap D_0 \setminus \{j\}$,

c') $z_l \geq x_l$, *for all* $l \in E_0 \setminus D_0$.

Verbally, in his *p-counter objection*, the player j claims that he can get with minimum p_j more than what he has got in the previous configuration (condition a')). his partners, who are not partners of j in the m-objection, will get not less than what they have got in the previous configuration (condition c')) and those partners of i who become partners of j will get with minimum p_k more than what they have got from the player i (condition b')).

Remark 1.4. One can easily obtain from Definition 1.3 that:

$$(1.6)\qquad \sum_{k \in E_0 \cap D_0} p_k \leq v(E_0 \cap D_0).$$

Remark 1.5. Taking in Definition 1.1 and Definition 1.3 $m_i = 0$ and $p_i = 0$, for all i, $i \in N$, we obtain the definitions of objections and counter objections in [1] (Definitions 2.1 and 2.2). In other words, a 0-objection is an objection and a o-counter objection is a counter objection.

Now, by using the notions of m-objection and p-counter objection, we shall define the notion of *(m, p)-stability*.

Definition 1.6. *A c.r.p.c. $(x\,;\,C)$ is called (m, p)-stable if for each m-objection of a player i against a player j in $(x\,;\,C)$ there exists a p-counter objection of j against i. for any two distinct players i and j belonging to the same coalition of C. Then the (m, p)-bargaining set $\mathcal{M}$ of a game G is the set of all (m, p)-stable c.r.p.c.'s.*

Example. Consider the three-person game (1.3). Let $(x_1, x_2, 0\,;\,12, 3)$ be a c.r.p.c. and $m \in R_+^3$, $m = (m_1, m_2, m_3)$ and let $i = 1$, $j = 2$, be two players, An *m-objection* of the player 1 against the player 2 is a c.r.p.c. $(y_1, 0, y_3\,;\,13, 2)$ satisfying:

$$(1.7)\qquad \begin{aligned} y_1 &> x_1 + m_1, \\ y_3 &\geq m_3, \end{aligned}$$

with the condition (resulting from Remark 1.2):

$$(1.8)\qquad m_1 + m_3 < c.$$

A p-counter objection, $p \in R$, $p = (p_1, p_2, p_3)$, of the player 2 against the player 1 is a c.r.p.c. $(0, z_2, z_3\,;\,1, 23)$ satisfying:

$$(1.9)\qquad \begin{aligned} z_2 &\geq x_2 + p_2, \\ z_3 &\geq y_3 + p_3, \end{aligned}$$

with the condition:

$$(1.10)\qquad p_2 + p_3 \leq b.$$

Remark 1.7. Making the necessary permutations in relations (1.8) and (1.10), we obtain:

$$(1.11)\qquad a > 0,\ b > 0,\ c > 0.$$

In the following, we shall use the notions of m-objection, p-counter objection and (m, p)-stability to obtain a characterization of the (m, p)-bargaining set $\mathcal{M}$ in a three-person game.

2. The (m, p)-bargaining set for a three-person game. Under the hypotheses that $\mathcal{B} = \mathcal{P}(N) \setminus \{N\}$, we can consider the coalition $(1, 2, 3)$ with a unique c.r.p.c. $(0, 0, 0\,;\,1, 2, 3)$, or the coalitions $(12, 3)$, $(1, 23)$, $(13, 2)$ with c.r.p.c.'s of the form:

$(x_1, x_2, 0\,;\,12, 3)$ with $x_1 + x_2 = a$, $x_1 \geq 0$, $x_2 \geq 0$,

$(0, x_2, x_3\,;\,1, 23)$ with $x_2 + x_3 = b$, $x_2 \geq 0$, $x_3 \geq 0$,

$(x_1, 0, x_3\,;\,13, 2)$ with $x_1 + x_3 = c$, $x_1 \geq 0$, $x_3 \geq 0$.

Next, let us examine under what circumstances a c.r.p.c. $(x_1, x_2, 0;\ 12, 3)$ may belong to the (m, p)-bargaining set $\mathcal{M}$. For the other two payoff configurations, we can make the necessary permutations in order to obtain the desired result.

Lemma 2.1. *Let $(x_1, x_2, 0;\ 12, 3)$ be a c.r.p.c., i.e., we have $x_1+x_2=a$, $x_1\geq 0$, $x_2\geq 0$. A necessary and sufficient condition for the existence of the m-objection of the player 1 against the player 2 in $(x; C)$, $m\in R^3_+$, $m=(m_1, m_2, m_3)$, $m_1+m_3<c$, is:*

$$x_1\leq \min\{a, c-m_1-m_3\}. \tag{2.1}$$

Proof. First, we prove that a necessary and sufficient condition for the existence of the m-objection $(y_1, 0, y_3;\ 13, 2)$ of the player 1 against the player 2, is that there exists ε which satisfies:

$$0<\varepsilon\leq c-x_1-m_1-m_3. \tag{2.2}$$

Indeed, if there exists an m-objection, then, denoting $\varepsilon=y_1-x_1-m_1$, from the first relation (1.7), it results that $\varepsilon>0$. From the second relation (1.7), taking into account that $y_3=c-y_1$, we obtain $c-y_1\geq m_3$ or $c-\varepsilon-x_1-m_1\geq m_3$, therefore $\varepsilon\leq c-x_1-m_1-m_3$.

Conversely, since ε must satisfy inequalities (2.2), we denote $y_1=\varepsilon+x_1+m_1$, and $y_3=c-y_1$. From $\varepsilon>0$, we have that $y_1>x_1+m_1$, i.e., the first relation (1.7), and relations $\varepsilon\leq c-m_1-m_3-x_1$ and $y_3=c-y_1$ imply that $y_3\geq m_3$, i.e., the second relation (1.7). On the other hand, relations (1.7) are satisfied if and only if the inequalities (2.2) are satisfied.

Further, if the inequalities (2.2) hold, then $x_1<c-m_1-m_3$. But $x_1+x_2=a$, therefore (2.1) holds.

Conversely, if (2.1) holds, then denoting $\varepsilon=c-m_1-m_3-x_1$ and taking $y_1=x_1+m_1+\varepsilon$, it follows that $y_1>x_1+m_1$, and, from $y_3=c-y_1=c-x_1-m_1-\varepsilon$, it results $y_3\geq m_3$, therefore (1.7) hold.

Finally, we conclude that the inequality (2.1) is satisfied if and only if the inequalities (1.7) are satisfyied.

Remark 2.2. It is easy to see that the minimum can be reached only if $a<c-m_1-m_3$ and $x_1=a$, $x_2=0$.

Remark 2.3. From the previous proof, it follows that, if $m_1+m_3\geq c$, then the player 1 has not any m-objection against the player 2.

Lemma 2.4. *Let us suppose that there exists a m-objection of the player 1 against the player 2. Then a necessary and sufficient condition for the existence of a p-counterobjection of the player 2, $p\in R^3_+$, $p=(p_1, p_2, p_3)$, $p_2+p_3\leq b$, is*

$$x_1\geq \max\left\{0,\ \frac{a-b+c}{2}+\frac{p_2+p_3-m_1}{2}\right\}. \tag{2.3}$$

Proof. The necessary and sufficient condition for the existence of the p-counter-objection $(0, z_2, z_3;\ 1, 23)$ of the player 2 against the player 1, is to find a δ satisfying:

$$0\leq\delta\leq b-a-c-p_2-p_3+m_1+2x_1, \tag{2.4}$$

for all ε which satisfy (2.2).

Indeed, if there exists a p-counter objection, then, denoting $\delta=z_2-x_2-p_2$ it yields, from the first relation (1.9) that $\delta\geq 0$. Let $z_3=b-z_2$. Then, from the second relation (1.9) we obtain $b-z_2\geq y_3+p_3$ or $\delta\leq b-a-c-p_2-p_3+m_1+2x_1+\varepsilon$, for ε satisfying (2.2) whence it results that

$$\delta\leq b-a-c-p_2-p_3+m_1+2x_1.$$

Conversely, for ε satisfying (2.2), the player 1 has a m-objection which can be (see Lemma 2.1) only of the form:

$$(x_1+m_1+\varepsilon, 0, c-x_1-m_1-\varepsilon;\ 13, 2),$$

and we assume that there exists δ with (2.4).

Let us consider $z_2=x_2+p_2+\delta$ and $z_3=b-z_2=b-x_2-p_2-\delta$. From $\delta\geq 0$, we obtain $z_2\geq x_2+p_2$, which is the first relation (1.9). From $\delta\leq b-a-c-p_2-p_3+m_1+2x_1$ and $y_3=c-x_1-m_1-\varepsilon$ we can write $z_3\geq y_3+p_3+\varepsilon$, for all ε which satisfy (2.2), whence it results the [second relation (1.9). Further if (2.4) hold, then

$$x_1\geq \frac{a-b+c}{2}+\frac{p_2+p_3-m_1}{2},$$

and taking into account that $x_1\geq 0$, we obtain (2.3).

Conversely, if (2.3) holds, then we take

$$\delta'=b-a-c-p_2-p_3+m_1+2x_1+c,$$

for all ε which satisfy (2.2). Consider again the m-objection from Lemma 2.1: $(x_1+m_1+\varepsilon, 0, c-x_1-m_1-\varepsilon;\ 13\ 2)$. From $z_2=x_2+p_2+\delta$ and (2.4), it results $z_2\geq x_2+p_2$, the first relation (1.9).

Now, take $z_3=b-z_2=b-x_2-p_2-\delta$, whence $z_3\geq x-m_1-x_1-\varepsilon+p_3=x_3+p_3$ the second relation (1.9).

Thus, we find shat the inequality (2.3) is satisfied if and only if the inequalities (1.9) are satisfied.

Remark 2.5. From the previous proofs, it follows that if $p_2+p_3\leq b$, then, the player 2 has no p-counter objections against the player 1.

Lemma 2.6. *Let $(x_1, x_2, 0;\ 12, 3)$ be a c.r.p.c., $x_1+x_2=a$ $x_1\geq 0$, $x_2\geq 0$. A necessary and sufficient condition for the existence of a, m-objection of the player 2 against the player 1 din $(x; C)$, $m\in R^3_+$, $m=(m_1, m_2, m_3)$, $m_2+m_3<b$, is:*

$$x_1\geq \max\{0, a-b+m_2+m_3\}. \tag{2.5}$$

Proof. Indeed, with the necessary permutations in Lemma 2.1 we obtain:

$$x_2\leq \min\{a, b-m_3-m_2\}, \tag{2.6}$$

and because $x_1+x_2=a$, the relation (2.6) is equivalent to the following inequality

$$a-x_1\leq \min\{a,\ b-m_2-m_3\}$$

or

$$x_1\geq \max\{0,\ a-b+m_2+m_3\}$$ (the relation 2.5)).

Remark 2.7. It is easy to see that the maximum can be reached only if $a-b+m_2+m_3<0$ and $x_1=0$, $x_2=2$.

The following lemma is obtained by making the necessary permutations in Lemma 2.4.

Lemma 2.8. *Let us suppose that there exists a m-objection of the player 2 against the player 1; then a necessary and sufficient condition that there exists a p-counterobjection the player 1, $p \in R^{3}_{+}$, $p=(p_1, p_2, p_3)$, $p_1+p_3 \leqslant c$, is:*

$$x_1 \leqslant \min\left\{a, \frac{a-b+c}{2} + \frac{m_2-p_1-p_3}{2}\right\}. \tag{2.7}$$

Lemma 2.9. *Let $(x_1, x_2, 0\,;\,12, 3)$ be a c.r.p.c., $x_1+x_2=a$, $x_1 \geqslant 0$, $x_2 \geqslant 0$. A necessary and sufficient condition that the player 1 has no m-objection is*

$$x_1 \geqslant \max\{0, c-m_1-m_3\}. \tag{2.8}$$

Proof. First, assume that the player 1 has no m-objections and let us suppose that the relation (2.8) is not true. In these conditions, esentially two cases arise.

Case A. If $m_1+m_3<c$ and relation (2.8) is not satisfied, then:

a) $x_1<0$ which it is not possible because $x_1 \geqslant 0$,

b) $x_1<c-m_1-m_3$, but since $x_1 \leqslant a$, it results that $x_1 \leqslant \min\{a, c-m_1-m_3\}$. From Lemma 2.1, this implies that there is a m-objection of player 1 against the player 2. This is a non-sense.

Case B. If $m_1+m_3 \geqslant c$, then from Remark 2.3, it results that the player 1 has no m-objections. Therefore, if the player 1 has no m-objections, then the relation (2.8) is satisfied.

Conversely, we suppose that the relation (2.8) is checked. If there exists a m-objections of the player 1 against the player 2 then, from Lemma 2.1, we obtain:

$$x_1 \leqslant \min\{a, c-m_1-m_3\}, \text{ that is } x_1<c-m_1-m_3,$$

which contradicts the relation (2.8). Hence the player 1 has no m-objections.

Making the necessary permutations in Lemma 2.9. and taking into account $x_2=a-x_1$, we get the following lemma.

Lemma 2.10. *Let $(x_1, x_2, 0\,;\,12, 3)$ be a c.r.p.c., $x_1+x_2=a$, $x_1 \geqslant 0$, $x_2 \geqslant 0$. A necessary and sufficient condition that the player 2 has no m-objections is:*

$$x_1 \leqslant \min\{a, a-b+m_2+m_3\}. \tag{2.9}$$

Summing up the above six lemmas, we obtain the conditions under which configuration $(x_1, x_2, 0\,;\,12, 3)$ is a (m, p)-stable c.r.p.c.

Theorem 2.11. *The set of the c.r.p.c.'s of the form $(x_1, x_2, 0\,;\,12, 3)$ which are (m, p)-stable is determined by the inequalities*

$$\begin{cases} \max\{0, c-m_1-m_3\} \leqslant x_1 \leqslant \min\{a, a-b+m_2+m_3\}, \\ x_2=a-x_1 \end{cases} \tag{2.10}$$

If $m_1+m_3<c$, $m_3+m_2<b$, $p_2+p_3<b$, $p_1+p_3 \leqslant c$, then this set is determined by:

$$\begin{cases} \max\left\{0, \frac{a-b+c}{2} + \frac{p_2+p_3-m_1}{2}, a-b+m_2+m_3\right\} \leqslant x_1 \leqslant \\ \leqslant \min\left\{a, c-m_1-m_3+\frac{a-b+c}{2}+\frac{m_2+p_1-p_3}{2}\right\}, \\ x_2=a-x_1, \end{cases} \tag{2.11}$$

$$\begin{cases} \max\left\{0, \frac{a-b+c}{2} + \frac{p_2+p_3-m_1}{2}\right\} \leqslant x_1 \leqslant \min\{a, c-m_1-m_3, \\ a-b+m_2+m_3\}, \; x_2=a-x_1, \end{cases} \tag{2.12}$$

$$\begin{cases} \max\{0, a-b+m_2+m_3, c-m_1-m_3\} \leqslant x_1 \leqslant \min a, \\ \frac{a-b+c}{2} + \frac{m_2-p_1-p_3}{2}, \\ x_2=a-x_1 \end{cases} \tag{2.13}$$

Proof. From Definition 1.6, the only mutually disjoint possible situations are:

1. There is no m-objection of the player i against the player j, where i and j are any of the players 1 and 2, $i \neq j$. This case corresponds to the Lammes 2.9 and 2.10, from which we obtain the relation (2.10).

2. To any m-objection of the player i against the player j, there corresponds a p-counterobjection of the player j against the player i, where i and j are any of the players 1 and 2, $i \neq j$. This case corresponds to the Lemmas 2.1, 2.4, 2.6 and 2.8, from which we obtain the relation (2.11).

3. There is a m-objection of the player i against the player j, but there is no m-objections of the player j against the player i. For $i=1$, $j=2$, from Lemmas 2.1, 2.4 and 2.10, we get the relation (2.12), and for $i=2$, $j=1$, from Lemmas 2.6, 2.8 and 2.9, we get the relation (2.13).

We have now the proof of Theorem 2.11.

Remark 2.12. Making the necessary permutations in this theorem we find also the set of these c.r.p.c.'s having the form:

$$(x_1, 0, x_3\,;\,13, 2),\; x_1+x_3=c,\; x_1 \geqslant 0,\; x_3 \geqslant 0 \text{ and } (0, x_2, x_3\,;\,1, 23),\; x_2+x_3=b,$$
$$x_2 \geqslant 0,\; x_3 \geqslant 0, \text{ which are } (m, p)\text{-stable.}$$

The (m, p)-bargaining set $\mathscr{M}$ consists of the (m, p)-stable c.r.p.c.'s having the form:

$$(x_1, x_2, 0\,;\,12, 3),\; x_1+x_2=a,\; x_1 \geqslant 0,\; x_2 \geqslant 0,$$
$$(0, x_2, x_3\,;\,1, 23),\; x_2+x_3=b,\; x_2 \geqslant 0,\; x_3 \geqslant 0,$$
$$(x_1, 0, x_3\,;\,13, 2),\; x_1+x_3=c,\; x_1 \geqslant 0,\; x_3 \geqslant 0$$

and $(0, 0, 0\,;\,1, 2, 3)$.

Remark 2.13. If we take $m_i=p_i=0$, for all i, $i=1, 2, 3$, Theorem 2.11 is just Theorem 4.1 in [1].

Let us add to our game the big coalition (123) with its value $v(123)=d$, $d \geqslant 0$. Thus, we have to find out under what conditions does a c.r.p.c. $(x_1, x_2, x_3\,;\,123)$ belong to the (m, p)-bargaining set $\mathscr{M}$.

Theorem 2.14. *In the three-person game (1.3), the (m, p)-bargaining set $\mathscr{M}$ consists of the c.r.p.c.'s given by the Theorem 2.11, and also of the c.r.p.c.'s $(x_1, x_2, x_3\,;\,123)$ which satisfy:*

(2.14) $x_1+x_2 \geqslant a-m_1-m_2,\ x_2+x_3 \geqslant b-m_2-m_3,\ x_1+x_3 \geqslant c-m_1-m_3.$

The latter c.r.p.c.'s exists if and only if we have:

$$d \geqslant \max\{a-m_1-m_2,\ b-m_2-m_3, c-m_1-m_3,$$

$$\frac{a+b+c}{2}-\frac{m_1+m_2+m_3}{2}\}. \tag{2.15}$$

Proof. The big coalition has no effects on the previous configurations of the (m, p)-bargaining set $\mathscr{M}$, because this coalition cannot be used for m-objection and p-counterobjections, since it contains all the players. So, we have to see in what conditions a c.r.p.c. $(x_1, x_2, x_3;\ 123)$ belongs to the new (m, p)-bargaining set. As it has to be coalitionally rational it is necessary that x_1, x_2, x_3 satisfy the inequalities:

$$x_1+x_2 \geqslant a,\ x_2+x_3 \geqslant b,\ x_1+x_3 \geqslant c,$$

but, as $a \geqslant a-m_1-m_2$, $b \geqslant b-m_2-m_3$, $c \geqslant c-m_1-m_3$, we have the relations (2.14). On the other hand, if conditions (2.14) are satisfied, then there is no m-objections.

For instance, if the player 1 has a m-objection $(y_1, 0, y_3;\ 13, 2)$ against the player 2, we have $y_1 > x_1+m_1$, $y_3 \geqslant x_3+m_3$, $y_1+y_3 \leqslant c$, therefore $x_1+x_3 \geqslant c-m_1-m_3$. But, from the relations (2.14), we have $x_1+x_3 \geqslant c-m_1-m_3$, so the player 1 cannot have any m-objections. In order that the inequalities (2.14) have at least one solution, the necessary and sufficient conditions are:

$$d \geqslant a-m_1-m_2,\ d \geqslant b-m_2-m_3,\ d \geqslant c-m_1-m_3 \text{ and}$$

$$d \geqslant \frac{a+b+c}{2}-\frac{m_1+m_2+m_3}{2}$$

i.e. relation (2.15).

Remark 2.15. If we take $m_i=0$, for all i, $i=1, 2, 3$, in Theorem 2.14 we obtain Theorem 5.1 in [1].

As one can see the conditions of (m, p)-stability are sufficiently complicated just in the case of a three-person game. In order to simplify them we shall introduce the notions of m-kernels. But this is for a future paper.

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Received 26.IX 1981

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Supliment la tomul XXVIII, s. I a, 1982

ANALYSIS OF CERTAIN POSSIBILITY FOR PARALLEL TREATMENT OF THE ROMBERG QUADRATURE ALGORITM

BY

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I. **The general theory.** In order to compute approximatively the integral $\int_a^b f(x)dx$, Romberg gave a method of numerical quadrature, using the trapezoidal rule and accelerating the convergence. In his method, one builds a triangular table:

$$\begin{array}{l} A_{1,1} \\ A_{2,1}A_{2,2} \\ A_{3,1}A_{3,2}A_{3,3} \\ A_{4,1}A_{4,2}A_{4,3}A_{4,4} \\ \cdots\cdots\cdots\cdots \end{array}$$

such that the first column is obtained by using the generalized trapezoidal method for $m_k=2^{k-1}$, where the values of the stepsize are: $h_k=(b-a)/m_k$; $k=1, 2, 3, \ldots$; hence:

$$A_{k,1}=\frac{h_k}{2}\left\{f(a)+f(b)+2\left[\sum_{i=1}^{2^{k-1}-1} f(a+ih_k)\right]\right\}. \tag{1}$$

Therefore, $\int_a^b f(x)dx=A_{k,1}-\frac{b-a}{12}h_k^2 f''(\xi_k)$, where $\xi_k \in [a, b]$.

The elements of the columns 2, 3, 4,... are calculated by the formula:

$$A_{k,j}=\frac{4^{j-1}A_{k,j-1}-A_{k-1,j-1}}{4^{j-1}-1} \tag{2}$$

with $k=2, 3, 4, \ldots$ and $j=2, 3, \ldots, k$.

Taking into account the way in which the second column was built, one could expect to have a faster convergence to the exact value of the integral on the second column than on the first one. It is proved (see [4]) that, if the function f has derivatives of any order on the closed interval $[a, b]$, then a olumns of Romberg's table converge to $\int_a^b f(x)dx$, and, in some additional conditions (from the extrapolation procedure), every column