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GENERATION OF THE PROBABILISTIC QUASI-PSEUDO-METRIC SPACES (II)

BY

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1. **Introduction.** In this paper we continue our study initiated in [1] on q-p-PM spaces. All notations and terminology are preserved.

A *probabilistic topological space* (PT space) is a couple (S, θ), where S is an arbitrary set and θ is a mapping from $\mathcal{P}(S)\times[0,1]$ into $\mathcal{P}(S)$ satisfying the following conditions (we denote $\theta(A,\lambda)=A^\lambda$ and $\{p\}^\lambda=p^\lambda$, $p\in S$):

C1. $A^0=S$, C2. $\Phi^\lambda=\Phi$, for every $\lambda\in(0,1]$,

C3. $A\subset A^\lambda$, C4. $(A\cup B)^\lambda=A^\lambda\cup B^\lambda$,

C5. $A^\lambda\subset A^\mu$ if $\lambda\geq\mu$, LC. $A^\lambda=\bigcap\limits_{\mu<\lambda}A^\mu$ for every $\lambda\in(0,1]$,

T. $q\in p^1$ and $r\in q^1\Rightarrow r\in p^1$.

A *PT space of Menger type* is an ordered triple (S,θ,T), where (S,θ) is a PT space and T is a t-norm such that the following „triangle inequality":

(TM) $\qquad q\in p^\lambda$ and $r\in q^\mu\Rightarrow r\in p^{T(\lambda,\mu)}$, $\lambda,\mu\in[0,1]$

be satisfied. A q-p-PM space (S,F) induces in a natural way two structures:

a) a PT space (S,θ), where θ is defined by

$$(1)\qquad A^\lambda=\begin{cases}\bigcap\limits_{\varepsilon>0}\overline{S}(A,\varepsilon,1-\lambda), & A\neq\varnothing \text{ and } \lambda\in[0,1],\\ \varnothing, & A=\varnothing \text{ and } \lambda\in(0,1],\\ S, & A=\varnothing \text{ and } \lambda=0;\end{cases}$$

b) a classical structure (S,ρ), where ρ is an extended $q-p-M$ on S defined by

$$(2)\qquad \rho(p,q)=\inf\{\varepsilon\,;F_{pq}(\varepsilon)=1\}.$$

If (S,F,T) is a q-p-PM space of Menger type, then the induced PT space is of Menger type.

Let now be a PT space (S,θ) or a PT space of Menger type (S,θ,T) and a q-p-M space (S,ρ) (we note that ρ may be $+\infty$). In [1] we have introduced the function $F: S\times S\to\Delta$ by

(F) $$F_{pq}(\varepsilon)=\begin{cases} 0, & \varepsilon\leqslant 0,\\ \sup\{\lambda\,;\ q\in(S_\rho(p,\varepsilon))^\lambda\}, & \varepsilon>0.\end{cases}$$

In Th. 1–5 [1] there are given sufficient conditions in order that the space (S,F) or (S,F,T) be a q-p-PM space, or a q-p-PM space of Menger type, respectively; in this case we say that the q-p-PM space (S,F) *is generated by the spaces* $(S,0)$ *and* (S,ρ) or that the q-p-PM space of Menger type (S,F,T) *is generated by the spaces* $(S,0,T)$ *and* (S,ρ). For the space $(S,0,T)$ we shall need some stronger than (TM) „triangle inequalities" namely:

(3) $$(p^\lambda)^\mu\subset p^{T(\lambda,\mu)},\ p\in S\,;\ \lambda,\mu\in[0,1],$$

or

(4) $$(A^\lambda)^\mu\subset A^{T(\lambda,\mu)},\ A\subset S\,;\ \lambda,\mu\in[0,1].$$

On the other hand, we shall use the following compatibility conditions between the spaces $(S,0)$ and (S,ρ).

(I) $$(S_\rho(p,\varepsilon))^1\subset\overline{S}_\rho(p,\varepsilon),\ p\in S,\ \varepsilon>0\,;$$

(II) $$S=\bigcup_{\varepsilon>0}(S_\rho(p,\varepsilon))^\lambda,\ p\in S,\ \lambda\in[0,1)\,;$$

(III) $$(S_\rho(p,\varepsilon))^\lambda=\bigcap_{\mu>\lambda}\bigcup_{\eta<\varepsilon}(S_\rho(p,\eta))^\mu,\ p\in S,\ \varepsilon>0,\ \lambda\in(0,1]\,;$$

(IV) $$(S_\rho(A,\varepsilon))^\lambda=\overline{S}_\rho(A^\lambda,\varepsilon),\ A\subset S,\ \varepsilon>0,\ \lambda\in[0,1]\,;$$

(V) $$(S_\rho(A,\varepsilon))^\lambda=\overline{S_\rho(A^\lambda,\varepsilon)}^{\tau_\rho},\ A\subset S,\ \varepsilon>0,\ \lambda\in[0,1].$$

The purpose of the present paper is to establish conditions in which the space (S,F,T) generated by the spaces $(S,0,T)$ and (S,ρ) induces them in his turn. In Section 2 we make some considerations on topologies of a PT space; they are necessary in the next section. In Section 3 we are dealing with the principal problem of the paper. In Section 4 we obtain necessary and sufficient conditions in order that a q-p-PM space (S,F) generated by the spaces $(S,0)$ and (S,ρ) admit some prescribed properties. In Section 5 we give two examples illustrate our general results obtained in the preceding sections.

2. **Topologies in the probabilistic topological spaces.** In a PT space, as in a PM space, it is possible to introduce some generalized topologies. One of these in the strongest and in the same time very important in the following section:

Let $(S,0)$ be a PT space and let λ_0 be a fixed number in $(0,1]$. The mapping $\tau:\mathcal{P}(S)\to\mathcal{P}(S)$ defined by $A\to A^{\lambda_0}$, $A\subset S$, has the following properties: a) $\tau(\varnothing)=\varnothing$, b) $A\subset\tau(A)$, and c) $\tau(A)\subset\tau(B)$ if $A\subset B$. Then, in view of Th. 9[6, p. 20] it results that the mapping τ determines a generalized topology (g-topology) on S (for the last term see, for example, [4, Definition 2.3]). We denote by τ_0 the g-topology obtained for $\lambda_0=1$, i.e., the g-topology determined by the mapping $A\to A^1$, $A\subset S$.

Next, we establish that, if the PT space $(S,0)$ is induced by a PM space, then these g-topologies coincide with the φ-g-topologies generated by the profile functions of the form $\varphi(x)=(1-\lambda_0)H(x-\varepsilon_0)$, which have been introduced and examined by R. Fritsche in [4].

Given a q-p-PM space (S,F) and a profile function φ, the φ-P-topology on S is the g-topology determined by the family of all sets of the form

$$N_p(\varphi\,;\ \varepsilon,\lambda)=\{q\,;\ F_{pq}(\varepsilon)>\varphi(\varepsilon)-\lambda\}\,;$$

analogously, we can define φ-Q-topology on S. The following result is a generalization of Prop. 2, 3) [1].

Lemma. *In a q-p-PM space (S,F), we have:*

$$\overline{A}^{\varphi-Q}=\{p\,;\ d(A,p)\geqslant\varphi\}.$$

Proposition 1. *Let: (S,F) be a q-p-PM space, $\varepsilon_0\geqslant 0$, and $\lambda_0\in[0,1)$ be given; let φ be the profile function of the form*

$$\varphi(x)=(1-\lambda_0)H(x-\varepsilon_0),\ x\geqslant 0.$$

Then the g-topology on S determined by the mapping $\tau:\mathcal{P}(S)\to\mathcal{P}(S)$ defined by

$$\tau(A)=\begin{cases}\bigcap_{\varepsilon>\varepsilon_0}\overline{S}(A,\varepsilon,\lambda_0) & A\neq\varnothing\\ \varnothing & A=\varnothing\end{cases}$$

coincides with the φ-Q-topology of the space.

Proof. It is easy to verify that the mapping τ has the properties a), b), and c) listed above. Therefore, τ determines a g-topology on S. It only remains to show that

$$\bigcap_{\varepsilon>\varepsilon_0}\overline{S}(A,\varepsilon,\lambda_0)=\overline{A}^{\varphi-Q}.$$

Indeed, we have

$$p\in\bigcap_{\varepsilon>\varepsilon_0}\overline{S}(A,\varepsilon,\lambda_0)\Leftrightarrow p\in\overline{S}(A,\varepsilon,\lambda_0)\ \text{for every}\ \varepsilon>\varepsilon_0\Leftrightarrow$$
$$d(A,p)(\varepsilon)\geqslant 1-\lambda_0\ \textit{for every}\ \varepsilon>\varepsilon_0\Leftrightarrow$$
$$d(A,p)(\varepsilon)\geqslant(1-\lambda_0)H(\varepsilon-\varepsilon_0)\ \textit{for every}\ \varepsilon>0\Leftrightarrow$$
$$d(A,p)\geqslant\varphi\Leftrightarrow p\in\overline{A}^{\varphi-Q},$$

and with this the proof is complete.

Corollary. *Let (S,F) be a q-p-PM space and let $(S,0)$ be the induced PT space. Then the g-topology on S, determined by the mapping*

$$A\to A^\lambda,\ A\subset S,$$

where $\lambda\in(0,1]$ is fixed, coincides with the φ-Q-topology on S determined by the profile function defined by $\varphi(x)=\lambda H(x)$, $x\geqslant 0$.

Proof. If we observe that $A^\lambda=\bigcap_{\varepsilon>0}\overline{S}(A,\varepsilon,1-\lambda)$ and if in Prop. 1 we take $\varepsilon_0=0$ and $\lambda_0=1-\lambda$, then we obtain the desired result.

Proposition 2. *If (S,F,T) is a q-p-PM space of Menger type satisfying the condition*

$$(^*)\qquad \lim_{x\to 1} T(a,x)=a, \text{ for every } a\in[0,1],$$

and if (S,θ) is the induced PT space, then Q and τ_θ are topologies on S and $\tau_\theta=Q$.

Proof. The fact that Q is a topology follows from Prop. 1, a) [1]. On the other hand, if we consider $\lambda=1$ in the preceding corollary we have that $\tau_\theta=Q$. This completes the proof.

3. We now approach the question of determining some conditions under which the space (S,F,T), generated by the spaces (S,θ,T) and (S,ρ), induces them.

To this end, it will be of interest to consider in PT spaces the following *convexity condition* :

$$(5)\qquad p\in A^\lambda \Rightarrow \text{there exists } q\in A^1 \text{ such that } p\in q^\lambda,$$

for every $A\subset S$, $A\neq\emptyset$, and $\lambda\in[0,1]$.

On the other hand, we make use of the following compatibility condition stronger than (I) :

$$(\mathrm{I}')\qquad (S_\rho(A,\varepsilon))^1\subset \overline{S}_\rho(A,\varepsilon),\ A\subset S,\ \varepsilon>0.$$

Theorem 1. *Suppose there are given : 1) a PT space of Menger type (S,θ,T) satisfying the „triangle inequality" (4) and the convexity condition (5), T being a t-norm verifying $(^*)$, 2) a space (S,ρ) with ρ an extended p-M such that τ_ρ is a compact topology on S. If the compatibility conditions (I'), (II), (III), (IV), and $\tau_\theta\subset\tau_\rho$ hold, then (S,F,T), the generated q-p-PM space of Menger type, induces the initial spaces (S,θ,T) and (S,ρ).*

Proof. By Th. 2 or 4 [1] the generated space (S,F,T) induces the space (S,ρ), that is (2) holds. Since ρ is a p-M on S, it is easy to verify that $P\subset\tau_\rho$ and $Q\subset\tau_\rho$. Indeed, to prove the second inclusion, it is sufficient to show that

$$S_\rho(p,\varepsilon)\subset\{q\,;\,F_{qp}(\varepsilon)>1-\lambda\} \text{ for every } \varepsilon>0,\ \lambda>0.$$

But

$$q\in S_\rho(p,\varepsilon)\Rightarrow\rho(p,q)<\varepsilon\Rightarrow\rho(q,p)<\varepsilon\Rightarrow$$

$$\inf\{\alpha\,;\,F_{qp}(\alpha)=1\}<\varepsilon\Rightarrow \text{ there exists } \alpha<\varepsilon$$

$$\text{such that } F_{qp}(\alpha)<1\Rightarrow F_{qp}(\varepsilon)=1\Rightarrow$$

$$F_{qp}(\varepsilon)>1-\lambda\Rightarrow q\in\{q\,;\,F_{qp}(\varepsilon)>1-\lambda\}.$$

Consequently, $Q\subset\tau_\rho$.

In order to show that the PT space induced by (S,F,T) is (S,θ,T), we first show

$$(6)\qquad (S_\rho(A,\varepsilon))^1=\overline{S}_\rho(A,\varepsilon).$$

Indeed, because $\tau_\theta\subset\tau_\rho$ it follows $\overline{A}\subset\overline{A}^{\tau_\theta}=A^1$. Using these relations and that obtained from (IV) for $\lambda=1$, we have

$$(S_\rho(A,\varepsilon))^1=S_\rho(A^1,\varepsilon)\supset\overline{S}_\rho(A,\varepsilon)=\overline{S}_\rho(A,\varepsilon),$$

hence $(S_\rho(A,\varepsilon))^1\supset\overline{S}_\rho(A,\varepsilon)$. From (I') and the last inclusion it follows (6).

We now establish the following implication :

$$(7)\qquad p\in(S_\rho(A,\varepsilon))^\lambda\Rightarrow \text{there exists } q\in\overline{A} \text{ so that } p\in(S_\rho(q,\varepsilon))^\lambda.$$

Indeed, from the convexity condition (5) and the compatibility condition (I') it results :

$$p\in(S_\rho(A,\varepsilon))^\lambda\Rightarrow \text{there exists } q\in(S_\rho(A,\varepsilon))^1 \text{ so that}$$

$$p\in q^\lambda\Rightarrow \text{there exists } q\in\overline{S}_\rho(A,\varepsilon)=\overline{S}_\rho(\overline{A},\varepsilon) \text{ so that } p\in q^\lambda.$$

But $\overline{A}$ is compact because (S,ρ) is compact. Hence

$$p\in(S_\rho(A,\varepsilon))^\lambda\Rightarrow \text{there exists } q\in S \text{ so that } \rho(\overline{A},q)\leqslant\varepsilon \text{ and}$$

$$p\in q^\lambda\Rightarrow \text{there exist } q\in S \text{ and } r\in\overline{A} \text{ so that } \rho(r,q)\leqslant\varepsilon \text{ and } p\in q^\lambda.$$

From this and (6), *we have*

$$p\in(S_\rho(A,\varepsilon))^\lambda\Rightarrow \text{there exist } q\in S \text{ and } r\in\overline{A} \text{ so that}$$

$$q\in\overline{S}_\rho(r,\varepsilon)=(S_\rho(r,\varepsilon))^1 \text{ and } p\in q^\lambda.$$

Taking into account the „triangle inequality" (4), we obtain :

$$p\in(S_\rho(A,\varepsilon))^\lambda_a\Rightarrow \text{there exists } r\in\overline{A} \text{ so that}$$

$$p\in((S_\rho(r,\varepsilon))^1)^\lambda\subset(S_\rho(r,\varepsilon))^{T(1,\lambda)}=(S_\rho(r,\varepsilon))^\lambda,$$

hence we have (7).

In the sequel we need the relation

$$(8)\qquad d(A,p)(\varepsilon)=\sup\{\lambda\,;\,p\in(S_\rho(A,\varepsilon))^\lambda\}$$

which we prove now. We have :

$$(9)\qquad \begin{aligned} d(A,p)(\varepsilon)&=\sup_{q\in A}F_{qp}(\varepsilon)=\sup_{q\in A}\sup\{\lambda\,;\ p\in(S_\rho(q,\varepsilon))^\lambda\}=\\ &=\sup\bigcup_{q\in A}\{\lambda\,;\ p\in(S_\rho(q,\varepsilon))^\lambda\}\leqslant\sup\{\lambda\,;\ p\in(S_\rho((A,\varepsilon))^\lambda\},\end{aligned}$$

and from (7) we obtain

$$\begin{aligned}\sup\{\lambda\,;\,p\in(S_\rho(A,\varepsilon))^\lambda\}&\leqslant\sup\{\lambda\,;\,p\in\bigcup_{q\in\overline{A}}(S_\rho(q,\varepsilon))^\lambda\}=\\ &=\sup\big(\bigcup_{q\in\overline{A}}\{\lambda\,;\,p\in(S_\rho(q,\varepsilon))^\lambda\}\big)=d(\overline{A},p)(\varepsilon),\end{aligned}$$

(the last equality may be verified by proceeding as in (9)). Therefore,

$$(10)\qquad d(A,p)(\varepsilon)\leqslant\sup\{\lambda\,;\,p\in(S_\rho(A,\varepsilon))^\lambda\}\leqslant d(\overline{A},p)(\varepsilon).$$

Since $Q\subset\tau_\rho$ we have $A\subset\overline{A}\subset\overline{A}^Q$. From this and Prop. 3[1] (T being a t-norm verifying (*)), it results

(11) $$d(A,p)(\varepsilon)\leqslant d(\overline{A},p)(\varepsilon)\leqslant d(\overline{A}^Q,p)(\varepsilon)=d(A,p)(\varepsilon).$$

Combining (10) and (11) we obtain the relation (8).

We show that

(12) $$(S_\rho(A,\varepsilon))^\lambda=\{p\,;\ d(A,p)(\varepsilon)\geqslant\lambda\},$$

for every $A\subset S$, $A\neq\varnothing$, $\varepsilon>0$, and $\lambda\in[0,1]$. Evidently, (12) holds for $\lambda=0$. Let now $\lambda\in(0,1]$. The first side of (12) is included in the second because we have

$$p\in(S_\rho(A,\varepsilon))^\lambda\Rightarrow\lambda\in\{\mu\,;\ \ p\in(S_\rho(A,\varepsilon))^\mu\}\Rightarrow$$
$$d(A,p)(\varepsilon)\geqslant\lambda\Rightarrow p\in\{p\,;\ d(A,p)(\varepsilon)\geqslant\lambda\},$$

in view of equality (8). The converse inclusion results again from (8), as follows :

$$p\in\{p\,;\ d(A,p)(\varepsilon)\geqslant\lambda\}\Rightarrow d(A,p)(\varepsilon)\geqslant\lambda\Rightarrow$$
$$\sup\{\lambda\,;\ p\in(S_\rho(A,\varepsilon))^\lambda\}\geqslant\lambda\Rightarrow p\in(S_\rho(A,\varepsilon))^\mu\ \textit{for}$$
$$\textit{every}\ \ \mu,\ 0\leqslant\mu<\lambda\Rightarrow p\in\bigcap_{\mu<\lambda}(S_\rho(A,\varepsilon))^\mu\Rightarrow$$
$$p\in(S_\rho((A,\varepsilon))^\lambda\Rightarrow p\in(S_\rho(A,\varepsilon))^\lambda.$$

Finally, we show that

(13) $$A^\lambda=\bigcap_{\varepsilon>0}\overline{S}(A,\varepsilon,1-\lambda),$$

for every $A\subset S$, $A\neq\varnothing$, and every $\lambda\in[0,1]$. Making use of the relation (12) and the compatibility conditions (IV) and $\tau_\theta\subset\tau_\rho$, we obtain

$$\bigcap_{\varepsilon>0}\overline{S}(A,\varepsilon,1-\lambda)=\bigcap_{\varepsilon>0}\{p\,;\ d(A,p)(\varepsilon)\geqslant\lambda\}=\bigcap_{\varepsilon>0}(S_\rho(A,\varepsilon))^\lambda=$$
$$=\bigcap_{\varepsilon>0}\overline{S_\rho}(A^\lambda,\varepsilon)=\overline{A}^\lambda\subset\overline{A}^{\lambda_\tau}=(A^\lambda)^1\subset A^{T(\lambda,1)}=A^\lambda,$$

that is $\bigcap_{\varepsilon>0}\overline{S}(A,\varepsilon,1-\lambda)=\bigcap_{\varepsilon>0}\overline{S}_\rho(A^\lambda,\varepsilon)\subset A^\lambda$. From this and $A^\lambda\subset\overline{S_\rho}(A^\lambda,\varepsilon)$ for every $\varepsilon>0$, it follows the desired equality (13).

The relation (13) shows that the space (S,θ,T) is the induced PT space and the proof is complete.

Remark. Taking into account Prop. 2 and the inclusion $Q\subset\tau_\rho$, the presence among our hypotheses of the condition $\tau_\theta\subset\tau_\rho$ is justified.

Theorem 2. *Suppose there are given : 1) a PT space of Menger type (S,θ,T) satisfying the „triangle inequality" (4) and the convexity condition (5), 2) a space (S,ρ) with ρ an extended p-M. If the compatibility conditions $\tau_\theta=\tau_\rho$, (II), (III), and (V) are satisfied, then the conclusion of Th. 1 holds.*

Proof. The compatibility condition (I) follows directly from the condition $\tau_\theta=\tau_\rho$. Then, by Th. 3 or 5 [1], the generated space (S,F,T) induces

the space (S,ρ). It only remains to show that, in our new hypotheses, the equality (13) is again true.

Since (9) is again valid, we have

$$d(A,p)\,(\varepsilon)\leqslant\sup\{\lambda\,;\ p\in(S_\rho(A,\varepsilon))^\lambda\}.$$

With the aid of this inequality we may establish, as in proof of Th. 1, that

(14) $$\{p\,;\ d(A,p)(\varepsilon)\geqslant\lambda\}\subset(S_\rho(A,\varepsilon))^\lambda,$$

for every $A\subset S$, $A\neq\varnothing$, and every $\varepsilon>0$, $\lambda\in[0,1]$.

From (14) and the compatibility conditions $\tau_\theta=\tau_\rho$, (V), we obtain

$$\bigcap_{\varepsilon>0}\overline{S}(A,\varepsilon,1-\lambda)=\bigcap_{\varepsilon>0}\{p\,;\ d(A,p)(\varepsilon)\geqslant\lambda\}\subset\bigcap_{\varepsilon>0}(S_\rho(A,\varepsilon))^\lambda=$$
$$\bigcap_{\varepsilon>0}\overline{S_\rho(A^\lambda,\varepsilon)}=\overline{A}^\lambda\subset\overline{A}^{\lambda_\tau}=(A^\lambda)^1\subset A^{T(\lambda,1)}=A^\lambda,$$

hence

(15) $$\bigcap_{\varepsilon>0}\overline{S}(A,\varepsilon,1-\lambda)\subset A^\lambda.$$

We now show that for every $\varepsilon>0$ $A^\lambda\subset\overline{S}(A,\varepsilon,1-\lambda)$ holds. Indeed, using (5), $\tau_\theta=\tau_\rho$, and the fact that ρ is symmetric, we have

$$p\in A^\lambda\Rightarrow\textit{there exists}\ q\in A^1=\overline{A}\ \textit{so that}\ p\in q^\lambda\Rightarrow$$
$$\textit{there exist}\ q\in S\ \textit{and}\ r\in A\ \textit{so that}\ q\in S_\rho(r,\varepsilon)$$
$$\textit{and}\ p\in q^\lambda\Rightarrow\textit{there exists}\ r\in A\ \textit{so that}\ p\in(S_\rho(r,\varepsilon))^\lambda$$
$$\Rightarrow\textit{there exists}\ r\in A\ \textit{so that}\ F_{rp}(\varepsilon)\geqslant\lambda\Rightarrow$$
$$d(A,p)(\varepsilon)\geqslant\lambda\Rightarrow p\in\overline{S}(A,\varepsilon,1-\lambda).$$

Consequently, we have

(16) $$A^\lambda\subset\bigcap_{\varepsilon>0}\overline{S}(A,\varepsilon,1-\lambda).$$

From (15) and (16) it results the equality (13) and the proof is complete.

Remark. If the compatibility condition (V) in Th. 2 is replaced by (IV), then the resulting theorem is true.

We now give a slight improvement of Th. 2. To this end, we begin with the following.

Lemma. *Let (S,θ,T) be a PT space of Menger type satisfying the convexity condition (5) and the condition*

$$(p^\lambda)^1\subset p^\lambda_1,\ p\in S,\ \lambda\in[0,1].$$

Then the „triangle inequality" (3) holds.

Proof. We have

$$q\in(p^\lambda)^\mu\ \Rightarrow\textit{there exists}\ r\in(p^\lambda)^1\ \textit{so that}\ q\in r^\mu$$
$$\Rightarrow\textit{there exists}\ r\in p^\lambda\ \textit{so that}\ q\in r^\mu$$
$$\Rightarrow\ q\in p^{T(\lambda,\mu)}.$$

Hence $(p^\lambda)^\mu \subset p^{T(\lambda,\mu)}$. In view of this Lemma and by an argument similar to one used in Th. 2, the following result be may proved:

Theorem 2'. *Suppose there are given: 1) a PT space of Menger type (S, θ, T) satisfying the convexity condition (5) and the condition*

$$(A^\lambda)^1 \subset A^\lambda,\ A \subset S, \lambda \in [0,1];$$

2) a space (S, ρ) with ρ an extended $p-M$. If the compatibility conditions $\tau_\theta = \tau_\rho$, (II), (III), and (IV) (or(V)) are satisfied, then the conclusion of Th. 1 holds.

4. The next theorems show that, under certain further hypotheses upon the given spaces (S, θ) and (S, ρ), the generated $q-p-PM$ space may have some new and important properties.

Theorem 3. *Let (S, F) be the $q-p-PM$ space generated by the spaces (S, θ) and (S, ρ). Then:*

a) (S, F) is a $q-PM$ space if and only if ρ is an extended $q-M$;

b) (S, F) is a $p-PM$ space if and only if the following condition is satisfied:

$$\text{(Q)} \qquad q \in (S_\rho(p, \varepsilon))^\lambda \Rightarrow p \in (S_\rho(q, \varepsilon))^\lambda.$$

Proof. In view of the assertion b) of Th. 1 [1], we have

$$F_{pq} = H \Leftrightarrow F_{pq}(\varepsilon) = 1 \text{ for every } \varepsilon > 0 \Leftrightarrow \rho(p, q) = 0,$$

whence it results the first part of the theorem.

We prove now the second part of the theorem. From (Q) we obtain

$$\{\lambda;\ q \in (S_\rho(p, \varepsilon))^\lambda\} = \{\lambda;\ p \in (S_\rho(q, \varepsilon))^\lambda\},$$

for every $\varepsilon > 0$. Consequently, $F_{pq}(\varepsilon) = F_{qp}(\varepsilon)$, $\varepsilon > 0$, that is $F_{pq} = F_{qp}$.

Conversely, in view of the conclusion a) of Th. 1 [1], we have

$$q \in (S_\rho(p, \varepsilon))^\lambda \Leftrightarrow F_{pq}(\varepsilon) \geq \lambda \Leftrightarrow F_{pq}(\varepsilon) \geq \lambda \Leftrightarrow p \in (S_\rho(q, \varepsilon))^\lambda,$$

whence (Q) is satisfied.

Theorem 4. *Let (S, F) be the q-p-PM space generated by the spaces (S, θ) and (S, ρ). Then, the functions F_{pq}, $p, q \in S$, are continuous on $R - \{0\}$ if and only if the condition*

$$\text{(C)} \qquad (S_\rho(p, \varepsilon))^\lambda = \bigcap_{\eta > \varepsilon} (S_\rho(p, \eta))^\lambda$$

is satisfied for every $p \in S$, $\varepsilon > 0$, and $\lambda \in [0, 1]$.

Proof. Suppose that the functions F_{pq} are continuous on $R - \{0\}$. Let $p \in S$, $\varepsilon > 0$, and $\lambda \in (0,1]$ be given. Evidently,

$$(S_\rho(p, \varepsilon))^\lambda \subset \Big(\bigcap_{\substack{\eta > \varepsilon \\ \mu < \lambda}} (S_\rho(p, \eta))^\mu = \bigcap_{\eta > \varepsilon} \Big(\bigcap_{\mu < \lambda} (S_\rho(p, \eta))^\mu\Big) = \bigcap_{\eta > \varepsilon} (S_\rho(p, \eta))^\lambda.$$

The converse inclusion follows from the conclusion a) of Th. 1 [1] and our hypothesis on the function F_{pq}. Indeed,

$$q \notin (S_\rho(p, \varepsilon))^\lambda \Rightarrow F_{pq}(\varepsilon) < \lambda \Rightarrow \text{ there exists } \eta > \varepsilon$$

$$\text{so that } F_{pq}(\eta) < \lambda \Rightarrow \text{there exists } \eta > \varepsilon \text{ so that } q \notin (S_\rho(p, \eta))^\lambda$$

$$\Rightarrow q \notin \bigcap_{\eta > \varepsilon} (S_\rho(p, \eta))^\lambda.$$

Consequently, the condition (C) holds.

Conversely, let $p, q \in S$, and $\varepsilon > 0$, be given and $\lambda = F_{pq}(\varepsilon + 0)$. If $\lambda = 0$, it is evident that F_{pq} is continuous at ε. For $\lambda \in (0,1]$ we have:

$$F_{pq}(\varepsilon+0) = \lambda \Rightarrow F_{pq}(\eta) \geq \lambda \text{ for every } \eta > \varepsilon \Rightarrow$$

$$q \in (S_\rho(p, \eta))^\lambda \text{ for every } \eta > \varepsilon \Rightarrow q \in \bigcap_{\eta > \varepsilon} (S_\rho(p, \eta))^\lambda.$$

In view of (C) it follows

$$F_{pq}(\varepsilon+0) = \lambda \Rightarrow q \in (S_\rho(p, \varepsilon))^\lambda \Rightarrow F_{pq}(\varepsilon) \geq \lambda.$$

Hence $\lambda \leq F_{pq}(\varepsilon) \leq F_{pq}(\varepsilon + 0) = \lambda$, that is $F_{pq}(\varepsilon) = F_{pq}(\varepsilon+0)$ and, consequently, F_{pq} is continuous at ε.

5. We give now two applications of the above theorems in order to provide examples in which the hypotheses of these theorems can be satisfied.

Application 1. Given a topological space (S, τ) and a real number $a \in [0,1)$, one may define on S, a structure of PT space by

$$A = \begin{cases} \bar{A}^\tau, & \lambda > a \\ S, & \lambda \leq a \text{ and } A \neq \emptyset, \end{cases}$$

$$\emptyset^\lambda = \begin{cases} \emptyset, & \lambda \neq 0, \\ S, & \lambda = 0. \end{cases}$$

A PT space (S, θ) is said to be a *simple PT space* if there is a topology τ on S and a number $a \in [0,1)$ such that the mapping θ is defined by the above relations. In this case we say that (S, θ) is the simple PT space generated by the topological space (S, τ) and the number a.

It can easily be verified that the simple PT spaces has the following properties:

1) $\tau_\theta = \tau$;

2) the „triangle inequality" (4) relative to any t-norm T is verified;

3) the convexity condition (5) holds;

4) if (S, F) is the simple q-p-PM space generated by the q-p-M space (S, ρ) and the distribution function G, $G \neq H$ (that is, $F: S \times S \to \Delta$ is defined, in the same spirit as in [9, p. 322], by $F_{pq}(x) = H(x)$, if $\rho(p, q) = 0$ and $F_{pq}(x) = G(x/\rho(p,q))$ if $\rho(p, q) \neq 0$), then the PT space induced by (S, F) is a simple PT space; furthermore, this spaces is generated by the number $G(0+)$ and the topological spa ce (S, τ_ρ^*), where τ_ρ^* is the conjugate topology of τ_ρ.

Proposition 3. *Let S be an arbitrary set, $a \in [0,1)$, and let ρ be a p-M on S such that*

$$\text{(17)} \qquad S_\rho(p, \varepsilon) = \overline{S_\rho(p, \varepsilon)}^{\tau_\rho} \text{ for every } p \in S \text{ and } \varepsilon > 0.$$

Let (S, θ) be the simple PT space generated by (S, τ_ρ) and a and let (S, F) be the q-p-PM space generated by (S, θ) and (S, ρ). Then (S, F) is a simple *space, namely, the simple space generated by* (S, ρ) *and the distribution function* G *given by*

$$G(x)=\begin{cases}0, & x\leqslant 0,\\ a, & 0<x\leqslant 1,\\ 1, & x>1.\end{cases} \tag{18}$$

Furthermore, (S,F) *induces the spaces* $(S, 0)$ *and* (S, ρ).

Proof. The hypotheses of Th. 2, regarding the spaces $(S, 0)$ and (S, ρ) are verified with respect to an arbitrary t-norm. The compatibility conditions $\tau_0=\tau_\rho$, (I), and (V) are trivially verified. Since ρ is a mapping with finite values, it follows that (II) holds. From the hypothesis (17), it results (III). In view of Th. 2, the generated space (S,F) induces the spaces $(S, 0)$ and (S, ρ). It only remains to show that (S,F) is the simple space generated by (S, ρ) and the function G given by (18). Indeed, for every $p, q\in S$ and $\varepsilon>0$, we have

$$F_{pq}(\varepsilon)= \sup\ \{\lambda;\ q\in(S_\rho(p,\varepsilon))^\lambda\}=$$

$$\sup\ (\{\lambda;\ \lambda\leqslant a\}\cup\{\lambda;\ \lambda>a, q\in\overline{S_\rho(p,\varepsilon)}^{\tau_\rho}\})=$$

$$\max\ (a;\ \sup\ \{\lambda;\ \lambda>a,\ q\in\overline{S_\rho(p,\varepsilon)}^{\tau_\rho}\}).$$

In view of (17), we obtain:

$$F_{pq}(\varepsilon)=\begin{cases}a, & q\notin S_\rho(p,\varepsilon),\\ 1, & q\in S_\rho(p,\varepsilon)\end{cases}=\begin{cases}a, & \rho(p,q)\geqslant\varepsilon\\ 1, & \rho(p,q)<\varepsilon.\end{cases}$$

Consequently, for every $\varepsilon\in R$, we have

$$F_{pq}(\varepsilon)=\begin{cases}G(\varepsilon/\rho(p,q)), & \rho(p,q)\neq 0,\\ H(\varepsilon), & \rho(p,q)=0,\end{cases}$$

and the proof is complete.

Application 2. Let (S, ρ) be a compact metric space with the diameter $d(S)\leqslant 1$. Furthermore, we suppose that (S, ρ) is convex in Menger's sense [7], i.e., for any two distinct points p and r there exists at last one point q such that $p\neq q\neq r$ and $\rho(p,r)=\rho(p,q)+\rho(q,r)$. Let $0:\mathcal{P}(S)\times[0,1]\to\mathcal{P}(S)$ be the mapping defined by

$$A^\lambda=\begin{cases}\overline{S}_\rho(A, 1-\lambda), & A\neq\varnothing,\\ S, & A=\varnothing,\ \lambda=0,\\ \varnothing, & A=\varnothing,\ \lambda\neq 0.\end{cases}$$

It is easy to verify that $(S, 0)$ is a PT space. In addition, $(S, 0)$ has the following properties:

1) $\tau_0=\tau_\rho$, because $A^1=\overline{S}_\rho(A,0)=\overline{A}$ for every $A\subset S$.

2) The „triangle inequality" (4) with the t-norm T_m given by $T_m(a, b)=\max\ (a+b-1,\ 0)$ holds. Indeed, because $d(S)\leqslant 1$, we have

$$(A^\lambda)^\mu=\overline{S}_\rho(\overline{S}_\rho(A, 1-\lambda), 1-\mu)\subset\overline{S}_\rho(A, 1-(\lambda+\mu-1))=$$

$$=\overline{S}_\rho(A, 1-T(\lambda,\mu))=A^{T(\lambda,\mu)}.$$

3) The convexity condition (5) holds. Since (S, ρ) is compact and $\tau_0=\tau_\rho$, it follows that

$$p\in A^\lambda\Rightarrow p\in\overline{S}_\rho(A, 1-\lambda)=\overline{S}_\rho(\overline{A}, 1-\lambda)\Rightarrow\rho(\overline{A}, p)\leqslant 1-\lambda\Rightarrow$$

there exists $q\in\overline{A}$ *so that* $\rho(q,p)\leqslant 1-\lambda\Rightarrow$

there exists $q\in A^1$ *so that* $p\in\overline{S}_\rho(q, 1-\lambda)=q^\lambda$.

One can apply Th. 1 for the spaces $(S, 0, T_m)$ and (S, ρ). Indeed, from $\tau_0=\tau_\rho$ it follows that (I') holds. Since $d(S)\leqslant 1$, it results that (II) holds. The compatibility conditions (III) and (IV) follow from the compactness and convexity of (S, ρ), by direct calculation. Thus, using these hypotheses and taking into account the theorems in [2, p. 24 and p. 46], we have for (IV):

$$(S_\rho(A,\varepsilon))^\lambda=\overline{S}_\rho(S_\rho(A,\varepsilon),\ 1-\lambda)=\overline{S}_\rho(A,\varepsilon+1-\lambda)=\overline{S}_\rho(\overline{S}_\rho(A,\ 1-\lambda),\ \varepsilon)=$$

$$=\overline{S}_\rho(A^\lambda,\ \varepsilon).$$

A similar argument works for (III).

In view of Th. 1, the spaces $(S,0,T_m)$ and (S, ρ) generate a *q-p-PM* space (S, F, T_m) of Menger type and this space induces the initial spaces. Furthermore, by Th. 3 and Th. 4, the space (S, F, T_m) is a *PM* space of Menger type and the functions F_{pq} are continuous on $R-\{0\}$ for every $p, q\in S$.

Finally it is not difficult to establish that F_{pq} is the function given by

$$F_{pq}(\varepsilon)=\begin{cases}0 & \varepsilon\leqslant 0,\\ \varepsilon+1-\rho(p,q), & 0<\varepsilon\leqslant\rho(p,q),\\ 1, & \varepsilon\geqslant\rho(p,q).\end{cases}$$

We omite this routine calculation.

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