Définition 8. *Une série,* $\bigcup_1^\infty F_n$, *qui satisfait aux propriétés (a) et (b) de la proposition 12, s'appelle (a)-normalement convergente sur X.*

Proposition 13. *Soient* $\mathcal{C}\subset P(Y)$ *un clan Cauchy, et* $m:\mathcal{C}\to[0,+\infty)$ *une mesure sur Y. On considère une série de multi-applications définies sur X à valeurs appartenant à* $\mathcal{C}$, $\bigcup_1^\infty F_n$. *Si elle est analytiquement convergente sur X vers la somme (X, Y, S), alors la série des fonctions* $\sum_1^\infty m\circ F_n$, *est simplement convergente sur X et on a,* $\sum_1^\infty (m\circ F_n)(x)=m\left(\bigcup_1^\infty F_n(x)\right)=m(S(x))$.

Démonstration. Par hypothèse, la suite, (S_n), des sommes partielles de la série, $\bigcup_1^\infty F_n$, est analytiquement convergente sur X. Selon la proposition 5 du §2, on peut écrire : $\lim_n (m\circ S_n)(x)=m((a)-\lim_n S_n(x))=m(S(x))$, pour tout $x\in X$, ce qui achève la démonstration.

Corollaire, *Si, de plus, la série* $\bigcup_1^\infty F_n$ *est (a)-uniformément convergente sur X, alors la série,* $\sum_1^\infty m\circ F_n$, *est uniformément convergente sur X.*

Démonstration. Par hypothèse il existe une suite d'ensembles $(D_n)\subset\mathcal{C}$ $D_n\downarrow\emptyset$, et un indice $n_0\in N$, tels que $S_n(x)\Delta S(x)\subseteq D_n$, $\forall n\geq n_0$ et $\forall x\in X$. Il résulte, $S_n(x)\setminus S(x)\subseteq D_n$ et $S(x)\setminus S_n(x)\subseteq D_n$, d'où $|m(S_n(x))-m(S(x))|\leq m(D_n)$, $\forall x\in X$ et $n\geq n_0$. Donc, on a : $|(m\circ F_1)(x)+(m\circ F_2)(x)+\ldots+(m\circ F_n)(x)-m(S(x))|=|m(F_1(x)\cup F_2(x)\cup\ldots\cup F_n(x))-m(S(x))|=|m(S_n(x))-m(S(x))\leq m(D_n)$, $\forall n\geq n_0$ et $\forall x\in X$. Puisque la suite numérique $(m(D_n))$ converge à zéro, la suite de fonctions $(m\circ S_n)$ est uniformément convergente à $m\circ S$ sur X.

Définition 9. *Soit* $\mathcal{C}\subset P(Y)$ *un clan Cauchy et* $m:\mathcal{C}\to[0,+\infty)$, *une mesure sur Y. Une série de multi-applications,* $\bigcup_1^\infty F_n$, *avec* $F_n(x)\in\mathcal{C}$, $\forall(n,x)\in N\times X$, *s'appelle m-convergente sur X, si la série numérique* $\sum_1^\infty (m\circ F_n)(x)$ *est convergente pour tout* $x\in X$.

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CONDITIONAL STABILITY OF ASYMPTOTICALLY SELF p-INVARIANT SETS

BY
OLUSOLA AKINYELE

1. Introduction. In [1], the concept of self-p-invariance of a set relative to a differential system was introduced, generalizing the notion of self-invariance and consequently, weaker notions of q-stability of the set arose naturally. However, in many concrete problems, the stability of sets that are not self-invariant are usually considered and the concept of asymptotically self invariant sets which form a special subclass of self invariant sets, has been used to describe such situations [3—6]. Naturally one may wish to consider stability of sets which are not self p-invariant as defined in [1]. The weaker concept of q-stability introduced in [1] assumes the existence of a self p-invariant set relative to the differential system and therefore results obtained become inadequate if we wish to consider stability properties of sets which are not self p-invariant.

In order to cope with the situations described above, we introduce in this paper, the new concept of asymptotically self p-invariance of a set. We then carry out investigations of q-conditional stability properties with respect to this kind of invariant sets, which are naturally weaker stability properties than one obtains in [1].

2. Basic definitions. Let R^n, R^m be the Euclidean n-space and m-space respectively and suppose $\Phi\in C(R^n, R^m)$ - the space of continuous functions from R^n into R^m. We define

$$\|\Phi(x)\|=\left[\sum_{i=1}^m \Phi_i^2(x)\right]^{1/2}.$$

Define $B=\{x\in R^n : \|\Phi(x)\|=0\}$, $S(B,\varepsilon)=\{x\in R^n : \|\Phi(x)\|<\varepsilon\}$, and $\overline{S(B,\varepsilon)}=\{x\in R^n : \|\Phi(x)\|\leq\varepsilon\}$.

We shall consider the differential system

$$\frac{dx}{dt}=f(t,x)\quad x(t_0)=x_0 \tag{1}$$

where $f\in C(R^+\times S(B,\rho), R^n)$, $\rho>0$. Here R^+ is the non-negative real line, B a subset of R^n and $C(R^+\times S(B,\rho), R^n)$ is the class of continuous functions from $R^+\times S(B,\rho)$ into R^n.

Definition 2.1. *Denote by* $\mathcal{K}$, *the set* $\{\alpha \in C([0, \rho), R^+) : \alpha(0) = 0$ *and* $\alpha(r)$ *is increasing in* $r\}$ *and by* $\mathcal{L}$ *the set* $\{\lambda \in C(R^+, R^+) : \lambda(t)$ *is decreasing in* t *and* $\lim_{t\to\infty} \lambda(t) = 0\}$.

Definition 2.2. *A set* $B \subset R^n$ *is said to be asymptotically self* p*-invariant with respect to the differential system* (1), *if given any monotonic decreasing sequence* $\{\varepsilon_j\}$, $\varepsilon_j \to 0$ *as* $j \to \infty$, *there exists a monotonic increasing sequence* $\{t_j(\varepsilon)\}$, $t_j(\varepsilon) \to \infty$ as $j \to \infty$ *and a real number* $A > 0$, *such that if for any solution* $x(t, t_0, x_0)$ *of* (1), $x_0 \in B$, $t_0 \geq t_j(\varepsilon)$ *implies* $x(t, t_0, x_0) \in S(B, \varepsilon_j)$,

jand
$$x^{(r)}(t, t_0, x_0) \in S(B, A\varepsilon_j), \ \textit{for } t \geq t_0,$$

$=1, 2, 3, \ldots$ *and* $r = 1, 2, \ldots p$, *where* $x^{(r)}(t, t_0, x_0)$ *is the* r^{th} *derivative of* $x(t, t_0, x_0)$ *with respect to* t *and*

$$x^{(0)}(t, t_0, x_0) \equiv x(t, t_0, x_0).$$

Remark 1. Definition 2.2 clearly contains the well-known definition of an asymptotically self invariant set. In fact, an asymptotically self o-invariant set is asymptotically self invariant. Thus with $p \neq 0$, our definition generalizes the well-known definition of an asymptotically self invariant set.

Let M_{n-k} be an $(n-k)$ dimensional manifold of R_n containing the set B and assume that the set B is asymptotically self p-invariant with respect to the system (1). We define some new stability properties for the set B.

Definition 2.3. *The asymptotically self* p*-invariant set* B *with respect to the system* (1) *is said to be* : qS_1 : q*-conditionally equistable if for each* $\varepsilon > 0$ *there exist* $t_1(\varepsilon)$, $t_1(\varepsilon) \to \infty$ *as* $\varepsilon \to 0$, $\delta = \delta(t_0, \varepsilon)$ $t_0 \geq t_1(\varepsilon)$ *which is continuous in* t_0 *for each* ε, *a positive integer* q, *and a positive number* A *such that for any solution* $x(t, t_0, x_0)$ *of the system* (1) *with* $x_0 \in \overline{S(B, \delta)} \cap M_{n-k}$,

$$x(t, t_0, x_0) \in S(B, \varepsilon) \ \textit{and} \ x^{(r)}(t, t_0, x_0) \in S(B, A\varepsilon) \ \textit{for } r = 1, 2, \ldots q$$

and for $t \geq t_0 \geq t_1(\varepsilon)$ qS_2 : q*-conditionally uniform stable if* δ *in* qS_1 *is independent of* t_0.

The other definitions for $qS_3 - qS_8$ can be easily formulated along the same lines as qS_1 and qS_2 and parallel to the stability definitions $QI_3 - QI_8$ of [1]. To spare space we do not state them. If $k=0$, then $qS_1 - qS_8$ reduce to the q-stability definitions of an asymptotically self p-invariant set.

Remark 2. If B is self p-invariant and $k=0$, in which case $M_{n-k} = R^n$, then definitions of qS_1 to qS_8 coincide with QI_1 to QI_8 of [1] and remark 2 of [1] applies without change. Let $0<k<n$ and $p>0$ while $q=0$, then qS_1 to qS_8 reduce to the conditional stability definitions of an asymptotically self p-invariant set which are new. Let $q>0$, $p=0$ and $0<k<n$, then $qS_1 - qS_8$ reduce to the q-conditional stability definitions of an asymptotically self invariant set which are also new. If $p=0$, $q=0$ and $0<k<n$, then $qS_1 - qS_8$ reduce to the usual definitions of conditional stability of an asymptotically self invariant set [cf, 4, 5]. Moreover, if $B = \{0\}$ and a self invariant set with

$q>0$, $p=0$, $\| \Phi(x) \| = \| x \|$ and $0<k<n$, then $qS_1 - qS_8$ reduce to q-conditional stability definitions of the trivial solution of the system (1). Finally if $0<k<n$, $q=0$, $p=0$, $B=\{0\}$ is self invariant and $\| \Phi(x) \| = \| x \|$, then $qS_1 - qS_8$ reduce to the conditional stability definitions of the trivial solution of (1) [4, def 4.4.1]. It becomes obvious that our definitions include as special cases a wide range of other well-known and new stability concepts.

3. Main Results. In this section we investigate sufficient conditions for the conditional stability properties of the invariant set introduced in section 2. We begin by stating sufficient conditions for the set B to be asymptotically self p-invariant.

We shall be concerned with the function $V \in C(R^+ \times S(B, \rho), R^N)$ and the auxiliary differential system

$$\frac{du}{dt} = g(t, u), \ u(t_0) = u_0. \tag{2}$$

We shall assume that the equation (2) and the functions V and f satisfy the following hypothesis :

(H_1) $g \in C(R^+ \times R^N, R^N)$ and $g(t, u)$ is quasimonotone nondecreasing in u for each $t \in R^+$,

(H_2) $V \in C(R^+ \times S(B, \rho), R^N)$, $V(t, x)$ is locally Lipschitzian in x and for $(t, x) \in R^+ \times S(B, \rho)$, $b(\| \Phi(x) \|) \leq \sum_{i=1}^{N} V_i(t, x)$ where $b \in \mathcal{K}$ and $\sum_{i=1}^{N} V_i(t, x) = \sigma(t)$, for $\sigma \in \mathcal{L}$ if $x \in B$,

(H_3) $V_i(t, x) \equiv 0$ $(i = 1, 2, \ldots k)$ $k<n$, if $x \in M_{n-k}$ where M_{n-k} is an $(n-k)$ dimensional manifold containing the set B,

(H_4) $f \in C(R^+ \times S(B, \rho), R^n)$ and for $(t, x) \in R^+ \times S(B, \rho)$, $D^+V(t, x) \leq g(t, V(t, x))$,

(H_5) For $(t, x^{(r)}) \in R^+ \times S(B, \rho)$, $r = 1, 2, \ldots, p$, $t \geq t_0$ and $\alpha \in \mathcal{K}$,

$$\max_{1 \leq r \leq p} \alpha(\| \Phi(x^{(r)}(t, t_0, x_0)) \|) \leq \sum_{i=1}^{N} V_i(t, x)$$

(H_6) For any function $\beta(t, u)$ which is continuous for $t \geq 0$, $u \geq 0$, decreasing in t for each fixed u, increasing in u for each fixed t, $\lim_{\substack{t\to\infty \\ r\to 0}} \beta(t, u) = 0$ implies

$$\sum_{i=1}^{N} u_i(t, t_0, u_0) \leq \beta\left(t_0, \sum_{i=1}^{N} x_{i0}\right), \ t \geq t_0$$

provided $u_{i0} = 0$, $i = 1, 2, \ldots k$ where $u(t, t_0, u_0)$ is any solution of (2) through (t_0, u_0),

(H_7) : Every solution $x(t, t_0, x_0)$ of (1) is $S = \max\{p, q\}$ times differentiable for $t \geq t_0$ where p, q are non-negative integers and there exists $\alpha \in \mathcal{K}$ such that if $(t, x^{(r)}) \in R^+ \times S(B, \rho)$ $r = 1, 2, \ldots, S$, then

$$\max_{1 \leq r \leq S} \alpha(\| \Phi(x^{(r)}(t) \|) \leq \sum_{i=1}^{N} V_i(t, x),$$

(H_8) For any function $\alpha(t, r)$ defined and continuous on $R^+ \times R^+$, monotone increasing in r for each fixed t and monotone decreasing in t for each fixed t, $\lim_{\substack{t\to\infty \\ r\to 0}} \alpha(t, r)=0$ implies for $(t, x) \in R^+ \times S(B, \rho)$, $\sum_{i=1}^{N} V_i(t, x) \leqslant \alpha(t, \|\Phi(x)\|)$.

Theorem 3.1. *Let hypothesis* (H_1), (H_2), (H_3), (H_4), (H_5) *and* (H_6) *be satisfied. Then the set* $B=\{x \in R^n : \|\Phi(x)\|=0\}$ *is asymptotically self* p*-invariant.*

Proof. Let $x_0 \in B$ and $\{\varepsilon_j\}$ be a decreasing sequence such that $\varepsilon_j \to 0$ as $j\to\infty$. Define $\eta(t)=\beta(t, \sigma(t))$, then using arguments similar to those of Theorem 4.7.1 of [4] we can show that $\{b(\varepsilon_j)\}$ is a decreasing sequence and that $\exists$ an increasing sequence $t_j(\varepsilon)$, $t_j(\varepsilon)\to\infty$ as $j\to\infty$ such that $\eta(t_0)<b(\varepsilon_j)$, $j=1, 2, 3, \ldots$ for $t_0 \geqslant t_j(\varepsilon)$. Consequently $x(t, t_0, x_0) \in S(B, \varepsilon_j)$, $j=1, 2, \ldots$, for $t \geqslant t_0 \geqslant t_j(\varepsilon)$, where $x(t, t_0, x_0)$ is any solution of (1) through (t_0, x_0).

Now choose the real number A such that $b(\varepsilon_j) \leqslant \alpha(A\varepsilon_j)$ with $A\varepsilon_j \in (0, \rho)$, then we claim that $x^{(r)}(t, t_0, x_0) \in S(B, A\varepsilon_j)$ for $t \geqslant t_0 \geqslant t(\varepsilon_j)$ and $r=1, 2, \ldots p$. Suppose not, then $\exists$ a solution $x(t, t_0, x_0)$, an integer l, $0<l\leqslant p$ such that if $x_0 \in B$, $t_1 \geqslant t_0 \geqslant t_j(\varepsilon)$ for certain j, then

$$\|\Phi(x^{(l)}(t_1, t_0, x_0))\|=A\varepsilon_j \text{ for } t_1>t_0\geqslant t_j(\varepsilon),$$

and
$$\|\Phi(x^{(l)}(t, t_0, x_0))\| \leqslant A\varepsilon_j<\rho \text{ for } t\in[t_0, t_1].$$

By Theorem 4.4.1 of [4], $\sum_{i=1}^{N} V_i(t, x(t, t_0, x_0)) \leqslant \sum_{i=1}^{N} r_i(t, t_0, u_0)$ for $t\in[t_0, t_1]$ where the vector function $r(t, t_0, u_0)$ is the maximal solution of (2). Hence at $t=t_1$, we obtain

$$\alpha(A\varepsilon_j) \leqslant \sum_{i=1}^{N} V_i(t_1, x(t_1)) \leqslant \sum_{i=1}^{N} r_i(t_1, t_0, u_0) \leqslant \beta(t_0, \sigma(t_0)) = \eta(t_0).$$

Thus $\alpha(A\varepsilon_j) \leqslant \eta(t_0) < b(\varepsilon_j) \leqslant \alpha(A\varepsilon_j)$ which is a contradiction. Hence B is asymptotically self p-invariant.

Theorem 3.2. *Assume that the hypothesis* $(H_1)-(H_4)$, *and* $(H_5)-(H_8)$ *hold, and that the set* $u=0$ *of the differential system* (2) *is asymptotically self invariant (ASI). If the ASI set* $u=0$ *of the equation* (2) *is conditionally equistable, then the set* B *is asymptotically self* p*-invariant and* q*-conditionally equistable with respect to the system* (1).

Proof. That B is asymptotically self p-invariant is obvious from the last result. Let $0<\varepsilon<\rho$, $t_0 \in R^+$ be given and assume that the ASI set $u=0$ of (2) is conditionally equistable, then [4] given $b(\varepsilon)>0 \exists t_1(\varepsilon)$, $t_1(\varepsilon)\to\infty$ as $\varepsilon\to 0$ and $\delta(t_0, \varepsilon)>0$, $t_0\geqslant t_1(\varepsilon)$ such that $\sum_{i=1}^{N} u_i(t, t_0, u_0) < b(\varepsilon)$, for $t \geqslant t_0 \geqslant t_1(\varepsilon)$ provided $\sum_{i=1}^{N} u_{i0} \leqslant \delta$, $u_{i0}=0$, $i=1, 2, \ldots, k$. Choose $u_{i0}=V_i(t_0, x_0)$, $i=1, 2, \ldots k$ and $x_0 \in M_{n-k}$, so that $u_{i0}=0$, $i=1, 2 \ldots k$ by hypothesis (H_3).

Let $\sum_{i=1}^{N} u_{i0}=\alpha(t_0, \|\Phi(x_0)\|)$, then $\exists$ $t_2(\varepsilon) \geqslant 0$ and $\delta_1=\delta_1(t_0, \varepsilon)$, $t_0 \geqslant t_2(\varepsilon)$ such that

$$\alpha(t_0, \|\Phi(x_0)\|) \leqslant \delta \text{ and } \|\Phi(x_0)\| \leqslant \delta_1 \text{ hold together.}$$

Choose $t_3(\varepsilon)=\max\{t_1(\varepsilon), t_2(\varepsilon)\}$ and let $t \geqslant t_0 \geqslant t_3(\varepsilon)$. Suppose $\exists$ a solution $x(t, t_0, x_0)$ of (1) $\ni$ for $x_0 \in \overline{S(B, \delta_1)} \cap M_{n-k}$ and $t_1 \geqslant t_0 \geqslant t_3(\varepsilon)$

$$\|\Phi(x(t_1))\|=\varepsilon \text{ and } \|\Phi(x(t))\| \leqslant \varepsilon \text{ for } t\in[t_0, t_1],$$

then by the choice of u_{i0}, $\alpha(t, \|\Phi(x_0)\|) \leqslant \delta$ and $\|\Phi(x_0)\| \leqslant \delta_1$ imply that whenever $x_0 \in \overline{S(B, \delta_1)} \cap M_{n-k}$, $\sum_{i=1}^{N} u_{i0} \leqslant \delta$ where $u_{i0}=0$, $i=1, 2, \ldots k$. Hence at $t=t_1$, (H_1), (H_2), (H_4) and Theorem 4.4.1 of [4], imply

$$b(\varepsilon) \leqslant \sum_{i=1}^{N} V_i(t_1, x(t_1)) \leqslant \sum_{i=1}^{N} r_i(t_1, t_0 x_0) < b(\varepsilon),$$

which is impossible. Hence $x(t, t_0, x_0) \in S(B, \varepsilon)$ for

$$t \geqslant t_0 \geqslant t_3(\varepsilon) \text{ and } x_0 \in \overline{S(B, \delta_1)} \cap M_{n-k}.$$

Now choose a real number $A>0$ such that $b(\varepsilon) \leqslant \alpha(A\varepsilon)$ with $A\varepsilon \in (0, \rho)$, then with δ_1 and $t_3(\varepsilon)$, $x^{(r)}(t, t_0, x_0) \in S(B, A\varepsilon)$ for $r=1, 2, \ldots, q$ provided $x_0 \in \overline{S(B, \delta_1)} \cap M_{n-k}$ and $t \geqslant t_0 \geqslant t_3(\varepsilon)$. Otherwise, $\exists$ an integer l such that $1 \leqslant l \leqslant q$, and a solution $x(t, t_0, x_0)$ such that $x_0 \in S(B, \delta_1) \cap M_{n-k}$, and $t_1 > t_0 \geqslant t(\varepsilon)$ with

$$\|\Phi(x^{(l)}(t_1, t_0, x_0))\| = A\varepsilon$$

and

$$\|\Phi(x^{(l)}(t, t_0, x_0))\| \leqslant A\varepsilon \text{ for } t\in[t_0, t_1].$$

Hence we have,

$$\alpha(A\varepsilon) \leqslant \alpha(\|\Phi(x^{(l)}(t_1, t_0, x_0))\|) \leqslant \max_{1\leqslant r\leqslant q} \alpha(\|\Phi(x^{(l)}(t_1))\|),$$

and so

$$\alpha(A\varepsilon) \leqslant \sum_{i=1}^{N} V_i(t_1, x(t_1)) \leqslant \sum_{i=1}^{N} r_i(t_1, t_0, x_0) < b(\varepsilon) \leqslant \alpha(A\varepsilon),$$

which is a contradiction, and so the proof is complete.

Assume that V and f satisfy the following hypotheses:

(H_9) $V\in C(R^+ \times S(B, \rho), R^N)$, $V(t, x)$ is locally Lipschitzian in x and

$$\max_{1\leqslant r\leqslant S} \alpha(\|\Phi(x^{(r)}(t))\|) \leqslant \sum_{i=1}^{N} V_i(t, x) \leqslant \min_{1\leqslant r\leqslant S} \alpha(\|\Phi(x^{(r)}(t))\|) \text{ for } 0<r<$$

$\|\Phi(x)\|<\rho$ and $t>\theta(r)$ where a, $\alpha\in\mathcal{K}$ and $\theta(r)\geqslant 0$ is monotone decreasing. Also $\exists$ b, $\beta\in\mathcal{K}$ such that $b(\|\Phi(x)\|) \leqslant \sum_{i=1}^{N} V_i(t, x) \leqslant \beta(\|\Phi(x)\|)$.

(H_{10}) $f \in C(R^+ \times S(B, \rho), R^n)$ and for $(t, x) \in R^+ \times S(B, \rho)$ $t \geq 0(r)$ with $0 < r < \| \Phi(x) \| < \rho$, $D^+V(t, x) \leq g(t, V(t, x))$.

We now give sufficient conditions for the q-conditional uniform stability of an asymptotically self p-invariant set.

Theorem 3.3. *Assume the hypotheses (H_1), (H_3), (H_6), (H_9) and (H_{10}) hold. Then the conditional uniform stability of the ASI set $u=0$ implies the set B is asymptotically self p-invariant and q-conditionally uniform stable with respect to* (1).

Proof. By hypothesis, we can let δ_1 of the last theorem be independent of t_0 and if we set $\sum_{i=1}^{N} u_{i0} = \beta(\|\Phi(x_0)\|)$, then δ is independent of t_0. That B is q-conditionally uniform stable follows easily.

Theorem 3.4. *Let all the hypotheses of Theorem 3.2 hold. Then the conditional equi-asymptotic stability of the ASI set $u=0$ of system* (2) *implies that the asymptotically self p-invariant set B is q-conditionally equi-asymptotically stable with respect to* (1).

Proof. It suffices to show that $\exists$ $t(\varepsilon) \geq 0$, $T_1 = T_1(t_0, \varepsilon) > 0$ $\delta = \delta(t_0, \varepsilon) > 0$ and $A > 0$ such that if $x_0 \in \overline{S(B, \delta)} \cap M_{n-k}$ and $t_0 \geq t(\varepsilon)$, then

(3) $$x(t, t_0, x_0) \in S(B, \varepsilon) \text{ and } x^{(r)}(t, t_0, x_0 \in S(B, A\varepsilon)$$

for $t \geq t_0 + T_1$ and $r = 1, 2, \ldots, q$. By hypothesis, given $b(\varepsilon) > 0$ and $t_0 \in R^+$, $\exists \delta = \delta(t_0)$, $t_1(\varepsilon) > 0$ and $T_1 = T_1(t_0, \varepsilon) > 0 \Rightarrow$

$$\sum_{i=1}^{N} u_i(t, t_0, u_0) < b(\varepsilon)$$

for $t_0 \geq t_1(\varepsilon)$ and $t \geq t_0 + T_1$, provided $\sum_{i=1}^{N} u_{i0} \leq \delta$ with $u_{i0} \equiv 0$ $(i = 1, 2, \ldots k)$.

Set $\sum_{i=1}^{N} u_{i0} = \alpha(t_0, \| \Phi(x_0) \|)$, then $\exists t_2(\varepsilon)$, $t_2(\varepsilon) \to \infty$ as $\varepsilon \to 0$ and $\delta_1 = \delta_1(t_0, \varepsilon) > 0 \Rightarrow$ $\alpha(t_0, \| \Phi(x_0) \|) \leq \delta$ and $\| \Phi(x_0) \| \leq \delta_1$ hold simultaneously, for $t_0 \geq t_2(\varepsilon)$. Let $\delta_0 = \delta_0(t_0, \rho)$, set $\delta_2 = \min \{\delta_0, \delta_1\}$ and $t(\varepsilon) = \max \{t_1(\varepsilon), t_2(\varepsilon)\}$. Choose a real number $A > 0$ such that $b(\varepsilon) \leq \alpha(A\varepsilon)$ with $A\varepsilon \in (0, \rho)$. Suppose (3) were false, then $\exists$ an integer l, $1 \leq l \leq q$, a solution $x(t, t_0, x)$ of the system (1) and a sequence $\{t_j\}$, $t_j \geq t_0 + T_1$ with $t_j \to \infty$ as $j \to \infty$ such that for $x_0 \in \overline{S(B, \delta_2)} \cap M_{n-k}$ and $t_0 \geq t(\varepsilon)$,

$$\| \Phi(x(t_j)) \| \leq \varepsilon \text{ and } \| \Phi(x^{(l)}(t_j, t_0, x_0)) \| \geq A\varepsilon.$$

Now $x_0 \in \overline{S(B, \delta_2)} \cap M_{n-k}$ and the choice of $\sum_{i=1}^{n} u_{i0}$ imply $\sum_{i=1}^{N} u_{i0} \leq \delta$ where $u_{i0} = 0$, $i = 1, 2, \ldots k$. Hence for the sequence $\{t_j\}$, and Theorem 4.4.1 of [4],

$$b(\varepsilon) \leq b(\|\Phi(x(t_j))\|) \leq \sum_{i=1}^{N} V_i(t_j, x(t_j)) \leq \sum_{i=1}^{N} r_i(t_j, t_0, x_0) < b(\varepsilon)$$

which is a contradiction. That is, $x(t, t_0, x_0) \in S(B, \varepsilon)$ for $t \geq t_0 + T_1$, $t_0 \geq t(\varepsilon)$, and $x_0 \in S(B, \delta_2) \cap M_{n-k}$. Also

$$a(A\varepsilon) \leq a(\| \Phi(x^{(l)}(t_j, t_0, x_0)) \|) \leq \max_{1 \leq r \leq q} a(\| \Phi(x^{(r)}(t_j)) \|) \leq \sum_{i=1}^{N} V_i(t_j x(t_j)) \leq$$

$$\leq \sum_{i=1}^{N} r_i(t_j, t_0, x_0) < b(\varepsilon) \leq a(A\varepsilon),$$

which again is a contradiction and so $x^{(r)}(t, t_0, x_0) \in S(B, A\varepsilon)$ for $t \geq t_0 + T_1$, $t_0 \geq t(\varepsilon)$ and $x_0 \in \overline{S(B, \delta_2)} \cap M_{n-k}$. The proof is complete.

Theorem 3.5. *Assume all the hypothesis of Theorem 3.3 hold. Then the conditional uniform asymptotic stability for the ASI set $u=0$ of the system* (2) *implies the q-conditional uniform asymptotic stability of the asymptotically self p-invariant set B.*

Proof. Following from the last theorem the proof is straightforward.

Remark 3. In view of remarks 1 and 2, our results in this section extend well-known results [1, 4] and include many new results as special cases.

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Department of Mathematics
University of Ibadan
Ibadan, Nigeria