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THE CONSTRUCTION OF A CONTROL OPERATOR

BY

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Introduction. Here, we consider the one-dimensional optimal control problem:

Problem 1. Minimize $\int_0^T [ax^2(t)+bu^2(t)]dt$ subject to the dynamic constraint: $\dot{x}(t)=cx(t)+du(t), 0\leq t\leq T$ where $a, b>0$ and c and d are constants which are not necessarily positive.

Generally speaking, to estimate the convergence rate of the conjugate gradient method in solving problem (1), we require an account of the greatest lower and least upper bounds of the spectrum of operator A associated with this problem where, A is

$$(1)\quad <z, Az>_K=\int_0^T [ax^2(t)+bu^2(t)]dt+\mu\int_0^T [(\dot{x}(t)-cx(t)-du(t))^2]dt\,;\quad \mu>0.$$

where

$$z(t)=(x(t), u(t))\,;\ K=H_1[0, T]\times L_2[0, T]$$

and the norm of K is defined as

$$\| z(t)\|_K^2=\| x(t)\|_{L_2}^2+\|\dot{x}(t)\|_{L_2}^2+\| u(t)\|_{L_2}^2.$$

The bounds on A can be obtained computationally without constructing this operator [1]. But, from the functional analysis view-point, some useful contributions to knowledge can be gained by actually constructing such an operator. This desire motivated the presented study.

In developing our major results we proceed as follows. Instead of constructing operator A that satisfies (1) directly, we find that it is more convenient to construct A such that it meets the following requirements:

$$(2)\quad \begin{aligned} <z_1(t), Az_2(t)>_K=\int_0^T &[ax_1(t)x_2(t)+bu_1(t)u_2(t)+\mu\dot{x}_1(t)\dot{x}_2(t)+\\ &+\mu c^2x_1(t)x_2(t)+\mu d^2u_1(t)u_2(t)-\mu c\dot{x}_1(t)x_2(t)-\mu c\dot{x}_2(t)x_1(t)+\\ &+\mu cdx_1(t)u_2(t)+\mu cdx_2(t)u_1(t)-\mu d\dot{x}_1(t)u_2(t)-\mu d\dot{x}_2(t)u_1(t)]\,dt.\end{aligned}$$

where

$$z_1(t) = (x_1(t),\ u_1(t))\ ;\ \ 0 \leqslant t \leqslant T,$$
$$z_2(t) = (x_2(t),\ u_2(t))\ ;\ \ 0 \leqslant t \leqslant T\ ;$$

and equation (2) presents the bilinear form which is associated with the control problem (1). For our subsequent developments, we shall henceforth use (2) as the equivalent form of (1).

It is not difficult to see that equivalence is attained in (2) when

$$x_1(t) \equiv x_2(t),$$

and

$$u_1(t) \equiv u_2(t).$$

The underlying idea for the construction of operator A satisfying (2) is contained in the remark that now follows.

Remark. The Riesz representation theorem guarantees [4] the existence of operator A satisfying (2) since the right hand side of (2) contains a bilinear, self-adjoint and continuous form.

2. Presentation of the main result. Let

$$(3)\qquad Az_2 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_2 \end{bmatrix} = \begin{bmatrix} (A_{11}x_2)(t) + (A_{12}u_2)(t) \\ (A_{21}x_2)(t) + (A_{22}u_2)(t) \end{bmatrix}$$

such that

$$(4)\qquad < z_1,\ Az_2 > = (x_1(t),\ u_1(t)) \begin{bmatrix} (A_{11}x_2)(t) + (A_{12}u_2)(t) \\ (A_{21}x_2)(t) + (A_{22}u_2)(t) \end{bmatrix} = R(z_1, z_2, \mu).$$

Let us consider the situation in (2) when $u_2 = 0$ and let us denote the resulting functional corresponding to this situation by $< z_1,\ \omega_{z_1} >_K$; thus

$$(5)\qquad \begin{aligned} < z_1,\ \omega_{z_1} >_K &= \int_0^T [x_1(t)\{ax_2(t) + \mu c^2 x_2(t) - \mu c\dot{x}_2(t)\} + \\ &+ \dot{x}_1(t)\{\mu\dot{x}_2(t) - \mu c x_2(t)\} + u_1(t)\{\mu c d x_2(t) - \mu d\dot{x}_2(t)\}]dt = \\ &= \int_0^T [x_1(t)\hat{x}_2(t) + \dot{x}_1(t)\hat{\dot{x}}_2(t) + u_1(t)\hat{u}_2(t)]dt, \end{aligned}$$

where

$$< z_1(t),\ Az_2(t) >_K \big|_{u_2=0} = 0 = \begin{bmatrix} (A_{11}x_2)(t) \\ (A_{21}x_2)(t) \end{bmatrix} = \begin{bmatrix} \hat{x}_2(t) \\ \hat{u}_2(t) \end{bmatrix}.$$

Therefore, to construct A_{11} and A_{21}, in effect, implies finding uniquely $\hat{x}_2(t)$ and $\hat{u}_2(t)$. To this end, we assume that

$$\alpha(t) = ax_2(t) + \mu c^2 x_2(t) - \mu c\dot{x}_2(t), \qquad \beta(t) = \mu\dot{x}_2(t) - \mu c x_2(t),$$

where $\alpha(t) - \hat{x}_2(t)$, $\beta(t) - \hat{\dot{x}}_2(t)$ are continuous functions on $[0, T]$ and consider that $x(.) \in D_1[0, T]$ with $x_1(0) = x_1(T) = 0$. Here, $x(.)$ denotes a function with the dot standing for undesignated variable and $D_1[0, T]$ designates the space of all continuous functions $y(t)$ with continuous first derivatives $y^{(1)}(t)$ and equipped with the norm

$$\|y\|_{D_1[0,\,T]} = \max_{0 \leqslant t \leqslant T} |\,y(t)\,| + \max_{0 \leqslant t \leqslant T} |\,y^{(1)}(t)\,|.$$

On considering the above assumptions, it follows from [3] that

$$(6)\qquad \int_0^T [x_1(t)(\alpha(t) - \hat{x}_2(t)) - \dot{x}_1(t)(\beta(t) - \hat{\dot{x}}_2(t))]dt = 0.$$

It is easy to see that the problem of finding $\hat{x}_2(t)$ and $\hat{u}_2(t)$ is equivalent to that of solving the second order differential equation in $\hat{x}_2(t)$:

$$(7)\qquad \ddot{\hat{x}}_2(t) - \hat{x}_2(t) = \mu\ddot{x}_2(t) - (a + \mu c^2)x_2(t).$$

To find the solution of (7) we make the change of variables $y = \hat{x}_2 - \mu x_2$ and on using appropriate boundary conditions generated by (6), we obtain the desired results stated below:

$$\begin{aligned} (A_{11}x_2)(t) = &- (\mu\dot{x}_2(0) - \mu c x_2(0))\ \mathrm{Sinh}\ t + \int_0^t (\mu\dot{x}_2 - \mu c x_2)(s)\ \ \mathrm{Cosh}\ (t - s)ds - \\ &- \int_0^t ((a + \mu c^2)x_2 - \mu c\dot{x}_2)(s)\ \mathrm{Sinh}\ (t - s)ds + \{-(\mu\dot{x}_2(0) - \mu c x_2(0))\ \mathrm{Sinh}\ T + \\ &+ \int_0^T (\mu\dot{x}_2 - \mu c x_2)(s). \\ &\mathrm{Cosh}\ (T - s)ds - \int_0^T ((a + \mu c^2)x_2 - \mu c\dot{x}_2)(s)\ \mathrm{Sinh}\ (T - s)ds\}e^T, \end{aligned}$$

$$(A_{12}u_2)(t) = bu_2(t) + \mu d^2 u_2(t), \qquad (A_{21}x_2)(t) = \mu c d x_2(t) - \mu d\dot{x}_2(t),$$

$$\begin{aligned} (A_{22}u_2)(t) = &\ \mu d u_2(0)\ \mathrm{Sinh}\ t - \int_0^t \mu d u_2(s)\ \mathrm{Cosh}\ (t - s)ds + \\ &+ \int_0^T \mu c d u_2(s)\ \mathrm{Sinh}\ (t - s)ds + \{\mu d u_2(0)\ \mathrm{Sinh}\ T - \\ &- \int_0^T \mu d u_2(s)\ \mathrm{Cosh}\ (T - s)ds + \int_0^T \mu c d u_2(s)\ \mathrm{Sinh}\ (T - s)ds\}e^T \end{aligned}$$

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CONNECTIONS MAKING PARALLEL A METRIC ($J^4 = 1$)-STRUCTURE

BY

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1. Introduction. Let M be a C^∞ differentiable manifold, J a (1,1) tensor field on M such that $J^4 = 1$, and g a (perhaps pseudo-Riemannian) metric on M such that $g(JX, Y) + g(X, JY) = 0$, X, Y being vector fields on M. This situation generalizes in a sense that of a manifold of dimension 4 with an (1,1) electromagnetic tensor field of the first class (see Hlavaty [2], Mishra [3]), as is proved by following A. Montesinos ideas [4] in Hernando [1]. We want to determine all linear connections ∇ on such a manifold (M, J, g) with $\nabla J = 0$ and $\nabla g = 0$. We call such a structure a metric $(J^4 = 1)$-structure.

2. Connections parallelizing J. Firstly, we shall obtain the family of all linear connections ∇ such that $\nabla J = 0$. For this, let ∇ be an arbitrary linear connection. We essay whether $\widetilde{\nabla}$, defined by

$$\widetilde{\nabla}_X Y : \nabla_X Y + aJ(\nabla_X J^3)\, Y + bJ^2 (\nabla_X J^2)\, Y + cJ^2 (\nabla_X J)\, Y$$

with a, b, c, functions and M satisfies $\widetilde{\nabla} J = 0$; but we have

$$(\widetilde{\nabla}_X J)\, Y = \widetilde{\nabla}_X JY - J\widetilde{\nabla}_X Y \quad \nabla_X JY + aJ(\nabla_X J^3)\, JY + bJ^2(\nabla_X J^2)\, JY +$$
$$+ cJ^3(\nabla_X J) JY - J\nabla_X Y - aJ^2(\nabla_X J^3)\, Y - bJ^3(\nabla_X J^2)\, Y - c(\nabla_X J)\, Y = (1 -$$
$$- c)\, (\nabla_X J)\, Y - a(\nabla_X J)\, Y + bJ^2(\nabla_X J^3)\, Y - b(\nabla_X J)\, Y + cJ^3(\nabla_X J^2) Y -$$
$$- c(\nabla_X J)\, Y - aJ^2(\nabla_X J^3) Y - bJ^3(\nabla_X J^2)\, Y \quad = (1 - c - a - b - c)\, (\nabla_X J)\, Y +$$
$$(b - a)\, J^2 (\nabla_X J^3)\, Y + (c - b)\, J^3 (\nabla_X J^2)\, Y = 0.$$

We obtain the equations $1 = a + b + 2c$, $b = a$, $c = b$, with unique solution $a = b = c = 1/4$. Thus, for any linear connection ∇, the new connection $\widetilde{\nabla}$ defined by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{1}{4}\{J(\nabla_X J^3)\, Y + J^2(\nabla_X J^2)\, Y + J^3(\nabla_X J)\, Y\} \tag{1}$$

satisfies $\widetilde{\nabla} J = 0$.

Now, let ∇ and $\widetilde{\nabla}$ two linear connections such that $\nabla J = \widetilde{\nabla} J = 0$, and put $A(X, Y) = \widetilde{\nabla}_X Y - \nabla_X Y$. Then, we must have $JA(X, Y) = A(X, JY)$. We construct as before, the linear combination