Acknowledgements. This work forms part of the Thesis submitted by the author in fulfilment of the requirement for the degree of Doctor of Philosophy (Ph. D) to the University of Leeds, England. The author very thankfully acknowledges the guidance, advice and keen interest of Dr. J. E. Rubio, Reader in Applied Mathematics and the encouragements of Professor D. Orighton, both, from the School of Mathematics, The University of Leeds, England.

REFERENCES

1. Ibiejugba, M. A. — *Computing Method in Optimal Control*, Ph. D. Thesis, University of Leeds, England, May 1980
2. Liusternik, L. A. and Sobolev, I. — *Elements of Functional Analysis*, Frederick Unger, English Translation, New York, 1961
3. Gelfand, I. M. and Fomin, S. — *Calculus of Variations*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1963
4. Luenberger, D. G. — *Optimization by Vector Space Methods*, John Wiley and Sons New York, 1969

Received 1.VII.1981

Department of Mathematics
University of Ife
Ile-Ife, Nigeria

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Supliment la tomul XXVIII, s. I a, 1982

CONNECTIONS MAKING PARALLEL A METRIC ($J^4 = 1$)-STRUCTURE

BY

E. REYES, A. MONTESINOS and P. M. GADEA

1. Introduction. Let M be a C^∞ differentiable manifold, J a (1,1) tensor field on M such that $J^4 = 1$, and g a (perhaps pseudo-Riemannian) metric on M such that $g(JX, Y)+g(X, JY)=0$, X, Y being vector fields on M. This situation generalizes in a sense that of a manifold of dimension 4 with an (1,1) electromagnetic tensor field of the first class (see Hlavaty [2], Mishra [3]), as is proved by following A. Montesinos ideas [4] in Hernando [1]. We want to determine all linear connections ∇ on such a manifold (M, J, g) with $\nabla J=0$ and $\nabla g=0$. We call such a structure a metric ($J^4=1$)-structure.

2. Connections parallelizing J. Firstly, we shall obtain the family of all linear connections ∇ such that $\nabla J=0$. For this, let ∇ be an arbitrary linear connection. We essay whether $\widetilde{\nabla}$, defined by

$$\widetilde{\nabla}_X Y : \nabla_X Y + aJ(\nabla_X J^3)\, Y + bJ^2(\nabla_X J^2)\, Y + cJ^2(\nabla_X J)\, Y$$

with a, b, c, functions and M satisfies $\widetilde{\nabla} J=0$; but we have

$$(\widetilde{\nabla}_X J)\, Y = \widetilde{\nabla}_X JY - J\widetilde{\nabla}_X Y \quad \nabla_X JY + aJ(\nabla_X J^3)\, JY + bJ^2(\nabla_X J^2)\, JY +$$
$$+ cJ^3(\nabla_X J)JY - J\nabla_X Y - aJ^2(\nabla_X J^3)\, Y - bJ^3(\nabla_X J^2)\, Y - c(\nabla_X J)\, Y = (1-$$
$$-c)\,(\nabla_X J)\, Y - a(\nabla_X J)\, Y + bJ^2(\nabla_X J^3)\, Y - b(\nabla_X J)\, Y + cJ^3(\nabla_X J^2) Y -$$
$$-c(\nabla_X J)\, Y - aJ^2(\nabla_X J^3) Y - bJ^3(\nabla_X J^2)\, Y = (1-c-a-b-c)\,(\nabla_X J)\, Y +$$
$$(b-a)\, J^2\,(\nabla_X J^3)\, Y + (c-b)\, J^3\,(\nabla_X J^2)\, Y = 0.$$

We obtain the equations $1=a+b+2c$, $b=a$, $c=b$, with unique solution $a=b=c=1/4$. Thus, for any linear connection ∇, the new connection $\widetilde{\nabla}$ defined by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{1}{4}\{J(\nabla_X J^3)\, Y + J^2(\nabla_X J^2)\, Y + J^3(\nabla_X J)\, Y\} \tag{1}$$

satisfies $\widetilde{\nabla} J=0$.

Now, let ∇ and $\widetilde{\nabla}$ two linear connections such that $\nabla J=\widetilde{\nabla} J=0$, and put $A(X, Y)=\widetilde{\nabla}_X Y - \nabla_X Y$. Then, we must have $JA(X, Y)=A(X, JY)$. We construct as before, the linear combination

$$\frac{1}{4}\{A(X,Y)+JA(X,J^3Y)+J^2A(X,J^2Y)+J^3A(X,JY)\}=$$

$$=\frac{1}{4}\{A(X,Y)+J^4A(X,Y)+J^4A(X,Y)+J^4A(X,Y)\}=A(X,Y)\,.$$

Hence,

$$\widetilde{\nabla}_XY=\nabla_XY+\frac{1}{4}\{A(X,Y)+JA(X,J^3Y)+J^2A(X,J^2Y)+J^3A(X,JY)\}\,.$$

Conversely, let A be an arbitrary (1, 2) tensor field on M and let ∇ be a linear connection such that $\nabla J=0$. Then, we shall prove that $\widetilde{\nabla}$ defined by (1) satisfies $\widetilde{\nabla}J=0$. For this, it is necessary and sufficient that

$$0=J\frac{1}{4}\{A(X,Y)+JA(X,J^3Y)+J^2A(X,J^2Y)+J^3A(X,JY)\}-$$

$$-\frac{1}{4}\{A(X,JY)+JA(X,J^4Y)+J^2A(X,J^3Y)+J^3A(X,J^2Y)\}.$$

But this relation, from $J^4=1$, is an identity. Hence, we can state the following result.

Theorem 2.1. *Let M be a manifold and J a (1,1) tensor field on M such that $J^4=1$. Let ∇ be an arbitrary (fixed) linear connection on M, and A an arbitrary (1, 2) tensor field on M. Then the following formula gives us all (and only) the linear connections $\widetilde{\nabla}$ such that $\widetilde{\nabla}J=0$:*

$$\widetilde{\nabla}_XY=\nabla_XY+\frac{1}{4}\{J(\nabla_XJ^3)\,Y+J^2(\nabla_XJ^2)\,Y+J^3(\nabla_XJ)\,Y\}+$$

$$+\frac{1}{4}\{A(X,Y)+JA(X,J^3Y)+J^2A(X,J^2Y)+J^3A(X,JY)\}\,.$$

Also, as it is well known, we have:

Theorem 2.2. (see O b a t a [5]) *Let (M,g) be a Riemannian manifold, ∇ an arbitrary (fixed) linear connection, and A an arbitrary (1, 2) tensor field on M. Then the following formula gives all (and only) the linear connections $\widetilde{\nabla}$ such that $\widetilde{\nabla}_g=0$:*

$$g(\widetilde{\nabla}_XY,Z)=g(\nabla_XY,Z)+\frac{1}{2}(\nabla_Xg)\,(Y,Z)+$$

$$+\frac{1}{2}g(A(X,Y),\,Z)-\frac{1}{2}g\,(Y,A(X,Z)).$$

These facts motivate the following technique to determine all the linear connections making parallel a metric $(J^4=1)$-structure.

3. The operators Φ_1, Φ_2, Ψ_1, Ψ_2. We now consider the following 4 operators: Φ_1, Φ_2, Ψ_1, Ψ_2, which act on linear connections ∇ on M in

the case of index 1 (that is, Φ_1 and Ψ_1), and on (1, 2) tensor fields A on M in the case of index 2 (Φ_2 and Ψ_2) in the following way (where X, Y, Z are vector fields on M):

$$(\Phi_1\nabla)_XY=\nabla_XY+\frac{1}{4}\{J(\nabla_XJ^3)\,Y+J^2(\nabla_XJ^2)\,Y+J^3(\nabla_XJ)\,Y\}\,;$$

$$(\Phi_2A)\,(X,Y)=\frac{1}{4}\{A(X,Y)+JA(X,J^3Y)+J^2A\,(X,J^2Y)+J^3A(X,JY)\}\,;$$

$$g((\Psi_1\nabla)_X\,Y,Z)=g(\nabla_XY,Z)+\frac{1}{2}(\nabla_Xg)\,(Y,Z)\,;$$

$$g((\Psi_2A)\,(X,Y),Z)\,=\frac{1}{2}g\,(A(X,Y),Z)-\frac{1}{2}g\,(Y,A(X,Z)).$$

We next prove 3 lemmas concerning these operators, using the usual notations. For example, ∇, $\widetilde{\nabla}$ denote linear connections on M.

Lemma 3.1. *If $\widetilde{\nabla}_XY=\nabla_XY+A(X,Y)$, then $\Phi_1\widetilde{\nabla}=\Phi_1\nabla+\Phi_2A$.*

Proof. From $J^4=1$, we deduce

(1) $J(\widetilde{\nabla}_XJ^3)\,Y=J\widetilde{\nabla}_XJ^3Y-\widetilde{\nabla}_XY,\qquad J(\nabla_XJ^3)\,Y=J\nabla_XJ^3Y-\nabla_XY,$

(2) $J^2(\widetilde{\nabla}_XJ^2)\,Y=J^2\widetilde{\nabla}_XJ^2Y-\widetilde{\nabla}_XY,\qquad J^2(\nabla_XJ^2)\,Y=J^2\nabla_XJ^2Y-\nabla_XY,$

(3) $J^3(\widetilde{\nabla}_X)\,Y=J^2\widetilde{\nabla}_XJY-\widetilde{\nabla}_XY,\qquad J^3(\nabla_XJ)\,Y=J^3\nabla_XJY-\nabla_XY.$

Then, substracting each of both identities in (1), (2), (3) we obtain

$$J(\widetilde{\nabla}_XJ^3)\,Y=J(\nabla_XJ^3)\,Y+JA(X,J^3Y)-A(X,Y),$$

$$J^2(\widetilde{\nabla}_XJ^2)\,Y=J^2(\nabla_XJ)\,Y+J^2A(X,J^2Y)-A(X,Y),$$

$$J^3(\widetilde{\nabla}_XJ)\,Y=J^3(\nabla_XJ)\,Y+J^3A(X,JY)-A(X,Y).$$

Thus,

$$(\Phi_1\widetilde{\nabla})_XY=\widetilde{\nabla}_XY+\frac{1}{4}\{J(\widetilde{\nabla}_XJ^3)\,Y+J^2(\widetilde{\nabla}_XJ^2)\,Y+J^3(\widetilde{\nabla}_XJ)\,Y\}=$$

$$=\nabla_XY+A(X,Y)+\frac{1}{4}\{J(\nabla_XJ^3)\,Y+J^2(\nabla_XJ^2)\,Y+J^3(\nabla_XJ)\,Y+$$

$$+JA(X,J^3Y)+J^2A(X,\,J^2Y)+J^3A(X,JY)-3A(X,Y)\}=$$

$$=(\Phi_1\nabla)_XY+(\Phi_2A)\,(X,Y).$$

Lemma 3.2. *If $\widetilde{\nabla}_XY=\nabla_XY+A(X,Y)$, then $\Psi_1\widetilde{\nabla}=\Psi_1\nabla+\Psi_2A$.*

Proof. We have

$$(\widetilde{\nabla}_Xg)\,(Y,Z)=Xg(Y,Z)-g(\nabla_XY,Z)-g\,(A(X,Y),Z)-g(Y,\nabla_XZ)-$$

$$-g(Y,A(X,Z))=(\nabla_Xg)\,(Y,Z)-g(A(X,Y),Z)-g(Y,A(X,Z)).$$

Thus,

$$g((\Psi_1,\widetilde{\nabla})_X Y,Z)=g(\widetilde{\nabla}_X Y,Z)+\frac{1}{2}(\widetilde{\nabla}_X g)(Y,Z)=g(\nabla_X Y,Z)+g(A(X,Y),Z)+$$
$$+\frac{1}{2}(\nabla_X g)(Y,Z)-\frac{1}{2}g(A(X,Y),Z)-\frac{1}{2}g(Y,A(X,Z))=g(\nabla_X Y,Z)+$$
$$+\frac{1}{2}(\nabla_X g)(Y,Z)+\frac{1}{2}g(A(X,Y),Z)-\frac{1}{2}g(Y,A(X,Z))=$$
$$=g((\Psi_1\nabla)_X Y,Z)+g((\Psi_2 A)(X,Y),Z).$$

That is, $\Psi_1\widetilde{\nabla}=\Psi_1\nabla+\Psi_2 A$.

Lemma 3.3. $\Phi_1\Psi_1=\Psi_1\Phi_1$.

Proof. With regard to $\Psi_1\Phi_1$, we have

$$g((\Psi_1\Phi_1\nabla)_X Y,Z)=g((\Phi_1\nabla)_X Y,Z)+\frac{1}{2}((\Phi_1\nabla)_X g)(Y,Z)=g(\nabla_X Y,Z)+$$
$$+\frac{1}{4}\sum_{k=1}^{3}g(J^k(\nabla_X J^{4-k})Y,Z)+\frac{1}{2}Xg(Y,Z)-\frac{1}{2}g((\Phi_1,\nabla)_X Y,Z)-$$
$$-\frac{1}{2}g(Y,(\Phi_1\nabla)_X Z)=g(\nabla_X Y,Z)+\frac{1}{2}(\nabla_X g)(Y,Z)+\frac{1}{8}\sum_{k=1}^{3}g(J^k(\nabla_X J^{4-k})Y,Z)-$$
$$-\frac{1}{8}\sum_{k=1}^{3}g(Y,J^k(\nabla_X J^{4-k})Z).$$

On the other hand,

$$g((\Phi_1\Psi_1\nabla)_X Y,Z)=g((\Psi_1\nabla)_X Y,Z)+\frac{1}{4}\sum_{k=1}^{3}g(J^k((\Psi_1\nabla)_X J^{4-k})Y,Z). \tag{1}$$

Having in mind that $g(J^k X,Y)=(-1)^k g(X,J^k Y)$ and $J^4=1$, a straigthforward computation leads to:

$$g(J^k((\Psi_1\nabla)_X J^{4-k})Y,Z)=\frac{1}{2}g(J^k(\nabla_X J^{4-k})Y,Z)-\frac{1}{2}g(Y,J^k(\nabla_X J^{4-k})Z).$$

Substitution in (1) gives the result. q.e.d.

4. Connections making J and q parallel

Theorem 4.1. *Let (M,J,g) be in the given conditions. A linear connection $\widetilde{\nabla}$ on M satisfies $\widetilde{\nabla}J=0$ and $\widetilde{\nabla}g=0$ if and only if there exists a linear connection ∇ on M such that*

$$\widetilde{\nabla}=\Phi_1\Psi_1\nabla.$$

Proof. From the definitions of Φ_1 and Ψ_1, the images of these operators are, respectively, the connections which make J and g parallel. Thus, $(\Phi_1\Psi_1,\nabla)J=0$ and $(\Phi_1\Psi_1\nabla)g=(\Psi_1\Phi_1\nabla)g=0$.

Conversely, if $\widetilde{\nabla}J=0$ and $\widetilde{\nabla}g=0$, then $\Psi_1\widetilde{\nabla}=\widetilde{\nabla}$ from the formula of Ψ_1; and $\Phi_1\widetilde{\nabla}=\widetilde{\nabla}$. Hence, $\Phi_1\Psi_1\widetilde{\nabla}=\Psi_1\Phi_1\widetilde{\nabla}=\widetilde{\nabla}$, and the condition is necessary.

Theorem 4.2. *Let ∇ be a fixed arbitrary linear connection on (M,J,g). Then, the following formula gives, for any $(1,2)$ tensor field A on M all (and only) the connections $\widetilde{\nabla}$ such that $\widetilde{\nabla}J=0$ and $\widetilde{\nabla}g=0$:*

$$\widetilde{\nabla}=\Psi_1\Phi_1(\nabla+A)=\Psi_1,(\Phi_1\nabla+\Phi_2 A)=\Psi_1\Phi_1\nabla+\Psi_2\Phi_2 A$$

Proof. Let ∇ be fixed. We consider the following map from $Z^1_2(M)$ to the set of linear connections on M,

$$A\to\Psi_1\Phi_1(\nabla+A).$$

Clearly, this map has its image in the subset of linear connections on M which make parallel J and g. The map is also onto: For if $\widetilde{\nabla}$ satisfies $\widetilde{\nabla}J=0$ and $\widetilde{\nabla}g=0$, then $\widetilde{\nabla}=\Psi_1\Phi_1\widetilde{\nabla}$, and thus we have, for $A=\nabla-\widetilde{\nabla}$, the relation $\Psi_1,\Phi_1(\nabla+A)=\Psi_1\Phi_1\widetilde{\nabla}=\widetilde{\nabla}$. q.e.d.

5. **Expressions with indices.** Finally, we give some expressions with indices, which can be useful in the calculations, the indices varying in the natural sets from the dimension of the manifold. If we then denote Γ^i_{jk} the coefficients of the connection ∇, and A^i_{jk} the components of the $(1,2)$ tensor field A, we have the following expressions for $(\Phi_1\nabla)^i_{jk}$, $(\Phi_2 A)^i_{jk}$ $(\Psi_1\nabla)^i_{jk}$, $(\Psi_2 A)^i_{jk}$, and $(\Phi_1\Psi_1\nabla)^i_{jk}$ $(\Psi_1\Phi_1\nabla)^i_{jk}$, respectively:

a) $\Phi_1\Gamma^i_{jk}=\Gamma^i_{jk}+\frac{1}{4}\{J^i_l(\nabla_j J^3)^l_k+(J^2)^i_k(\nabla_j\ J^2)^l_k+(J^3)^i_l(\nabla_j\ J)^l_k\}$;

b) $\Phi_2 A^i_{jk}=\frac{1}{4}\{A^i_{jk}+J^m_k A^l_{jm}(J^3)^i_l+(J^2)^m_k A^l_{jm}(J^2)^i_l+(J^3)^m_k A^l_{jm}J^i_l\}$;

c) $\Psi_1\Gamma^i_{jk}=\Gamma^i_{jk}+\frac{1}{2}g^{li}\ \nabla_j\ g_{kl})$ (see O b a t a [5]);

d) $\Psi_2 A^i_{jk}=\frac{1}{2}\ A^i_{jk}-g^{il}\,g_{km}A^m_{jl})$;

e) $\Phi_1\Psi_1\Gamma^i_{jk}=\Psi_1\Phi_1\Gamma^i_{jk}=\Gamma^i_{jk}+\frac{1}{8}(\nabla_j\ g_{kl})g^{li}+\frac{1}{8}\{J^i_l(\nabla_j\ J^3)^l_k+$

$+(J^2)^i_l(\nabla_j\ J^2)^l_k+(J^3)^i_l(\nabla_j\ J)^l_k\}+\frac{1}{8}\{\nabla_j\ \varkappa^3_{kl}))\,\varkappa^{li}+(\nabla_j\ \varkappa^2_{kl})\,\varkappa^{li}_2+(\nabla_j\ \varkappa_{kl})\,\varkappa^{li}_3\}$,

where

$$\varkappa_{ij}=J^h_i g_{kj},\qquad \varkappa^2_{ij}=(J^2)^k_i g_{kj},\qquad \varkappa^3_i=(J^3)^k_i g_{kj},$$
$$\varkappa^{ij}=J^j_h g^{ik}\qquad \varkappa^{ij}_2=(J^2)^j_k g^{ik},\qquad \varkappa^{ij}_3=(J^3)^j_k g^{ik},$$

as it is easily proved.

Similar formulae and results can be stated for the case corresponding to the electromagnetic field of second class (R e y e s [6]).

REFERENCES

1. Hernando, J. M. — *Estructuras geométricas asociadas al campo electromagnétic de tipo (1,1) de 1 y 2ª clase*, (Thesis), Dept. de Geometria, Valladolid (1981)
2. Hlavaty, V. — *Geometry of Einstein's unified field theory*, P. Noordhoff, (1958)
3. Mishra, R. S. — *Structures in electromagnetic tensor fields*, Tensor N.S. 30 (1976), 145—156
4. Montesinos, A. — *A generalisation of certain class of G-structures*, (to appear)
5. Obata, M. — *Affine connections on manifolds with almost complex, quaternion or hermitian structure*, Japanese J. of Math. 26 (1956), 43—77
6. Reyes, E. — *Conexiones asociadas al campo electromagnético de 2 clase*, (to appear)

Received 18.VI.1981

Facultad de Ciencias, Valladolid
Facultad de Matemáticas, Valencia and Instituto Jorge Juan C.S.I.C., Madrid (Spain)

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Supliment la tomul XXVIII, s. I a, 1982

p-DISTRIBUTIONS ON DIFFERENTIABLE MANIFOLDS

BY

SORIN DRAGOMIR

We generalize the concept of differentiable distribution to the induced bundle $p^{-1}TM$ of a given differentiable manifold M. A new class of submanifolds of a Finsler manifold is pointed out.

1. **Introduction.** Let M be a n-dimensional real differentiable manifold. Let $T(M)$ and $T^*(M)$ be the tangent and cotangent bundles over M. Let $p : \widetilde{M} \to M$ be the subbundle of non-nul tangent vectors on M. Let $p^{-1}TM \to \widetilde{M}$, $p^{-1}T^*M \to \widetilde{M}$ be the bundles naturally induced by $T(M)$ and p, respectively by $T^*(M)$ and p. The fibres over $\tilde{x} \in \widetilde{M}$ are denoted respectively by $p_{\tilde{x}}^{-1}TM$ and $p_{\tilde{x}}^{-1}T^*M$. A cross-section $\widetilde{X} : \widetilde{M} \to p^{-1}TM$ is called a p-vector field on M.

2. **p-Forms on differentiable manifolds.** Let $\tilde{x}$ be a non-nul tangent vector on M; we have an obvious isomorphism between $(p_{\tilde{x}}^{-1}TM)^*$ and $p_{\tilde{x}}^{-1}T^*M$. A field $\overline{K}$ of R-linear maps $\overline{K}_{\tilde{x}} : p_{\tilde{x}}^{-1}TM \to R$ defined on $\widetilde{M}$ such that $\tilde{x} \to \overline{K}_{\tilde{x}}$ is differentiable of class C^∞, is called a differentiable p-form of degree 1 on M. The immediate conclusion of the isomorphism above is that a differentiable p-form of degree 1 can be also regarded as a cross-section: $\overline{K} : \widetilde{M} \to p^{-1}T^*M$. Every differentiable 1-form K on M admits a natural lift $\overline{K} : \widetilde{M} \to p^{-1}T^*M$, $\overline{K}(\tilde{x}) = (\tilde{x}, K(p\tilde{x}))$, $\tilde{x} \in \widetilde{M}$.

Let (U, y^i) be a local coordinate system on M. The natural lifts of the local 1-forms $\{dy^i\}$ are denoted by $\{\overline{dy^i}\}$ and give a local basis of the fibre $p_{\tilde{x}}^{-1}T^*M$, at any $\tilde{x} \in p^{-1}(U)$. Let ∇ be a connection in $p^{-1}TM$; we recall, [4], that a tangent vector field $\widetilde{X}$ on $\widetilde{M}$ is said to be vertical if: $(dp)\widetilde{X} = 0$, and horizontal if: $\nabla_{\widetilde{X}}\bar{v} = 0$, where: $\bar{v} : \widetilde{M} \to p^{-1}TM$, $\bar{v}(\tilde{x}) = (\tilde{x}, \tilde{x})$, is the intrinsic p-vector field on M.

Let $\widetilde{K} \in \Lambda^1(\widetilde{M})$ be a 1-form on $\widetilde{M}$. Then $\widetilde{K}$ is said to be vertical (horizontal) if $\widetilde{K}(\widetilde{X}) = 0$ for any horizontal (vertical) tangent vector field $\widetilde{X}$ on $\widetilde{M}$.

Let $T\widetilde{M}_v$ and $T\widetilde{M}_h$ be the bundles of vertical, respectively horizontal tangent vectors on $\widetilde{M}$. For any regular connection ∇ in $p^{-1}TM$, i.e. for which we have the direct sum decomposition $p^{-1}TM = T\widetilde{M}_v + T\widetilde{M}_h$, the restric-