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Facultad de Ciencias, Valladolid
Facultad de Matemáticas,
Valencia and Instituto Jorge Juan
C.S.I.C., Madrid (Spain)

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p-DISTRIBUTIONS ON DIFFERENTIABLE MANIFOLDS

BY

SORIN DRAGOMIR

We generalize the concept of differentiable distribution to the induced bundle $p^{-1}TM$ of a given differentiable manifold M. A new class of submanifolds of a Finsler manifold is pointed out.

1. **Introduction.** Let M be a n-dimensional real differentiable manifold. Let $T(M)$ and $T^*(M)$ be the tangent and cotangent bundles over M. Let $p : \widetilde{M} \to M$ be the subbundle of non-nul tangent vectors on M. Let $p^{-1}TM \to \widetilde{M}$, $p^{-1}T^*M \to \widetilde{M}$ be the bundles naturally induced by $T(M)$ and p, respectively by $T^*(M)$ and p. The fibres over $\tilde{x} \in \widetilde{M}$ are denoted respectively by $p^{-1}_{\tilde{x}}TM$ and $p^{-1}_{\tilde{x}}T^*M$. A cross-section $\widetilde{X} : \widetilde{M} \to p^{-1}TM$ is called a p-vector field on M.

2. **p-Forms on differentiable manifolds.** Let $\tilde{x}$ be a non-nul tangent vector on M; we have an obvious isomorphism between $(p^{-1}_{\tilde{x}}TM)^*$ and $p^{-1}_{\tilde{x}}T^*M$. A field $\overline{K}$ of R-linear maps $\overline{K}_{\tilde{x}} : p^{-1}_{\tilde{x}}TM \to R$ defined on $\widetilde{M}$ such that $\tilde{x} \to \overline{K}_{\tilde{x}}$ is differentiable of class C^∞, is called a differentiable p-form of degree 1 on M. The immediate conclusion of the isomorphism above is that a differentiable p-form of degree 1 can be also regarded as a cross-section: $\overline{K} : \widetilde{M} \to p^{-1}T^*M$. Every differentiable 1-form K on M admits a natural lift $\overline{K} : \widetilde{M} \to p^{-1}T^*M$, $\overline{K}(\tilde{x}) = (\tilde{x}, K(p\tilde{x}))$, $\tilde{x} \in \widetilde{M}$.

Let (U, y^i) be a local coordinate system on M. The natural lifts of the local 1-forms $\{dy^i\}$ are denoted by $\{\overline{dy}^i\}$ and give a local basis of the fibre $p^{-1}_{\tilde{x}}T^*M$, at any $\tilde{x} \in p^{-1}(U)$. Let ∇ be a connection in $p^{-1}_{\tilde{x}}TM$; we recall, [4], that a tangent vector field $\widetilde{X}$ on $\widetilde{M}$ is said to be vertical if: $(dp)\widetilde{X} = 0$, and horizontal if: $\nabla_{\widetilde{X}}\bar{v} = 0$, where: $\bar{v} : \widetilde{M} \to p^{-1}TM$, $\bar{v}(\tilde{x}) = (\tilde{x}, \tilde{x})$, is the intrinsic p-vector field on M.

Let $\widetilde{K} \in \Lambda^1(\widetilde{M})$ be a 1-form on $\widetilde{M}$. Then $\widetilde{K}$ is said to be vertical (horizontal) if $\widetilde{K}(\widetilde{X}) = 0$ for any horizontal (vertical) tangent vector field $\widetilde{X}$ on $\widetilde{M}$.

Let $T\widetilde{M}_v$ and $T\widetilde{M}_h$ be the bundles of vertical, respectively horizontal tangent vectors on $\widetilde{M}$. For any regular connection ∇ in $p^{-1}TM$, i.e. for which we have the direct sum decomposition $p^{-1}TM = T\widetilde{M}_v + T\widetilde{M}_h$, the restriction $L|T\widetilde{M}_h$ of the bundle epimorphism $L: T(\widetilde{M}) \to p^{-1}TM$, $L(\widetilde{X}) = (\tilde{x}, (dp)\widetilde{X})$, $\widetilde{X} \in T_{\tilde{x}}(\widetilde{M})$, is an isomorphism whose inverse is denoted by B. We denote by $\Lambda^1(\widetilde{M})_v$, $\Lambda^1(\widetilde{M})_h$ respectively the spaces of vertical and horizontal 1-forms on M. Obviously the morphism: $L^*: \Lambda^1(\widetilde{M}) \to p^{-1}T^*M$, $L^*(\widetilde{K}_{\tilde{x}}) = \widetilde{K}_{\tilde{x}} \circ B_{\tilde{x}}$, $\widetilde{K}_{\tilde{x}} \in T^*_{\tilde{x}}(\widetilde{M})$, is an epimorphism and $\mathrm{Ker}(L^*) = \Lambda^1(\widetilde{M})_v$. It is now easy to check that for any regular connection ∇ in $p^{-1}TM$ we have the direct sum decomposition $\Lambda^1(\widetilde{M}) = \Lambda^1(\widetilde{M})_v + \Lambda^1(\widetilde{M})_h$; thus $L^*: \Lambda^1(\widetilde{M})_h \to p^{-1}T^*M$ is an isomorphism.

3. **p-Distributions on differentiable manifolds.** Let U be an open set in M and $\widetilde{U} = p^{-1}(U)$. An assignment $\widetilde{D}: \tilde{x} \in \widetilde{U} \to \widetilde{D}_{\tilde{x}} \subseteq p^{-1}_{\tilde{x}}TM$, such that every $\widetilde{D}_{\tilde{x}}$ is a real m-dimensional vector subspace of $p^{-1}_{\tilde{x}}TM$ is called a m-dimensional p-distribution on M. For example, any p-vector field $\widetilde{X}$ locally defined on $\widetilde{U}$ defines a 1-dimensional p-distribution on M, i.e. $\widetilde{D}_{\tilde{x}} = \{k\overline{X}(\tilde{x}) / k \in R\}$, for any $\tilde{x} \in \bar{U}$.

Any m-dimensional distribution D on M admits a natural lift $\widetilde{D}$, given by: $\widetilde{D}_{\tilde{x}} = \{\tilde{x}\} \times D_{p(\tilde{x})}$, $\tilde{x} \in \widetilde{M}$. Let $\widetilde{D}$ be a m-dimensional p-distribution on M. Let $h_a: \widetilde{U}_a \to R^{2n}$ be a local chart on $\widetilde{M}$ with $\widetilde{U}_a \cap \widetilde{U} \neq \emptyset$. Let $\{\overline{X}_i\}$, $i = 1, 2, \ldots, m$, be independent p-vector fields defined on $\widetilde{U}_a \cap \widetilde{U}$, so that for any $\tilde{x} \in \widetilde{U}_a \cap \widetilde{U}$, the set $\{\overline{X}_i(\tilde{x})\}_{i=1,m}$ gives a basis of $\widetilde{D}_{\tilde{x}}$. Then $\{\overline{X}_i\}$ is said to be a local basis, defined on $\widetilde{U}_a \cap \widetilde{U}$, of the p-distribution $\widetilde{D}$. If for $(\widetilde{U}_a, h_a)$ there is a local basis on $\widetilde{U}_a \cap \widetilde{U}$ of $\widetilde{D}$, then $(\widetilde{U}_a, h_a)$ is said to be an admissible local chart of the p-distribution $\widetilde{D}$. Then $\widetilde{D}$ is said to be differentiable (of class C^∞) if there is an atlas of admissible local charts on $\widetilde{M}$: $\{(\widetilde{U}_a, h_a)\}_{a \in I}$ so that: i) $\widetilde{U}$ is contained in the union of $\widetilde{U}_a$, $a \in I$, ii) For any $a \in I$, there is a local basis of $\widetilde{D}$, defined on $\widetilde{U}_a \cap \widetilde{U}$, consisting of differentiable (of class C^∞) p-vector fields. Let $\widetilde{D}$ be a m-dimensional p-distribution on $\widetilde{M}$, defined on $\widetilde{U}$, and $(\widetilde{U}_a, h_a)$ an admissible local chart on $\widetilde{M}$. Let $\{\overline{X}_i\}_{i=1,m}$ be a local basis of $\widetilde{D}$ defined on $\widetilde{U}_a \cap \widetilde{U}$. The natural lifts $\{\bar{\partial}_i\}$ of $\dfrac{\partial}{\partial y^i}$ give a basis of $p^{-1}_{\tilde{x}}TM$ at each $\tilde{x} \in \widetilde{M}$. Hence $\overline{X}_j(\tilde{x}) = X^i_j(\tilde{x})\bar{\partial}_i(\tilde{x})$, $j = 1, 2, \ldots, m$, and we obtain $m.n$ functions with rang $\|X^i_j\| = m$ on $\widetilde{U}_a \cap \widetilde{U}$. Hence the p-distribution $\widetilde{D}$ is differentiable if and only if the functions X^i_j are differentiable. If $\overline{X}$ is a p-vector field defined on $\widetilde{U}$, with $\overline{X}(\tilde{x}) \in \widetilde{D}_{\tilde{x}}$, for any $\tilde{x} \in \widetilde{U}$, we write $\overline{X} \in \widetilde{D}$. With these notations $\overline{X} \in \widetilde{D}$ if and only if there are m differentiable functions $f^j: \widetilde{U}_a \cap \widetilde{U} \to R$, so that $X^i(\tilde{x}) = f^j(\tilde{x})X^i_j(\tilde{x})$, $i = 1, 2, \ldots, n$, for any $\tilde{x} \in \widetilde{U}_a \cap \widetilde{U}$. Also, for a given m-dimensional differentiable p-distribution, $\widetilde{D}$ on M there exist $(n-m)$ differentiable p-forms of degree 1 on M: $\{\overline{K}^i\}_{i=1,n-m}$ so that $\overline{X} \in \widetilde{D}$ if and only if $\overline{K}^i(\overline{X}) = 0$, for all i.

4. **Integral manifolds of a p-distribution.** Let V be a n-dimensional-differentiable manifold and M an immersed submanifold of V, of dimension m. Then M is called an integral manifold of the differentiable p-distribution $\widetilde{D}$ on V if for any non-nul vector $\tilde{x}$ tangent to M we have: $\widetilde{D}_{\tilde{x}} = p^{-1}_{\tilde{x}}TM$. It is easy to prove that any 1-dimensional p-distribution admits locally a unique integral manifold, under certain initial conditions. For a given regular connection ∇ in $p^{-1}TV$, a m-dimensional p-distribution $\widetilde{D}$ on V is said to be involutive if for any p-vector fields $\overline{X}$, $\overline{Y}$ on V, the condition $\overline{X}, \overline{Y} \in \widetilde{D}$ yields $L[B\overline{X}, B\overline{Y}] \in \widetilde{D}$. A local coordinate neighbourhood (U, h) of a point $x \in V$ is said to be cubic if $h(U) \subseteq R^n$ is an dimensional cube $h(U) = [-r, r]^n$, $r > 0$, centred at $h(x) = (0, \ldots, 0)$. Using the classical theorem of Frobenius, [2], for a given distribution D on V we obtain the following:

Theorem. *If the natural lift $\widetilde{D}$ of the distribution D on V, defined on $\widetilde{U} = p^{-1}(U)$, $U \subseteq V$, is involutive, for a given regular connection in $p^{-1}TV$, then there is a cubic local chart around any $x \in U$, with $r > 0$, such that the submanifold: $M: x^{m+i} = c^{m+i}$, $|c^{m+i}| < r$, $i = 1, 2, \ldots, n-m$ is an integral manifold of the p-distribution $\widetilde{D}$.*

Let $\widetilde{D}$ be a p-distribution on V. We call $\widetilde{D}$ integrable if for any non-nul tangent vector $\tilde{x} \in \widetilde{U}$, there is a submanifold M of V, so that: i) M is an integral manifold of $\widetilde{D}$, ii) $\tilde{x}$ is tangent to M.

Corollary. *Let V be endowed with a regular connection in the induced bundle $p^{-1}TV$. Hence any C^∞-differentiable involutive p-distribution on V obtained by natural lifting from a distribution on V is integrable.*

5. **C.R.- submanifolds of Finsler manifolds.** Let V be an even dimensional Finsler manifold with the Finsler energy $E: T(V) \to R_+$ and the associated Finsler metric tensor $g_{ij} = \dfrac{\partial^2 E}{\partial v^i \partial v^j}$. A morphism $J: p^{-1}TV \to p^{-1}TV$ with $J^2 = -I$, is said to be an p-almost complex structure on V. Hence J is exactly a Finsler almost complex structure on V, that is $J^i_j(x, v)J^j_k(x, v) = -\delta^i_k$. Then g is called a p-almost hermitian Finsler metric if $g(J\overline{X}, J\overline{Y}) = g(\overline{X}, \overline{Y})$, for any p-vector fields $\overline{X}, \overline{Y}$ on V. Then (V, E, g, J) is an almost hermitian Finsler manifold. Let M be an immersed submanifold of V. We denote by $\eta \to \widetilde{M}$ the p-normal bundle of the given immersion, i.e. we have the direct sum decomposition: $p^{-1}TV|_{\widetilde{M}} = p^{-1}TM + \eta$ (see [4]). Assume M is endowed with a p-distribution $\widetilde{D}$ which is invariant under the p-almost complex structure, i.e. $J_{\tilde{x}}(\widetilde{D}_{\tilde{x}}) = \widetilde{D}_{\tilde{x}}$, $\tilde{x} \in \widetilde{M}$, and so that the orthogonal (with respect to $g_{\tilde{x}}$) complement $\widetilde{D}^\perp$ of $\widetilde{D}$ is an anti-invariant p-distribution on M, i.e. $J_{\tilde{x}}(\widetilde{D}^\perp_{\tilde{x}}) \subseteq \eta_{\tilde{x}}$, $\tilde{x} \in \widetilde{M}$. Then $(M, \widetilde{D}, \widetilde{D}^\perp)$ is called a C.R.-submanifold of the almost hermitian Finsler manifold V.

Let P, Q be the natural projectors on $\widetilde{D}$ and $\widetilde{D}^\perp$. We consider the endomorphism $F: p^{-1}TM \to p^{-1}TM$, $F\overline{X} = JP\overline{X}$, and a bundle valued p-form of degree 1, $\bar{u}\,\overline{X} = JQ\overline{X}$, $\overline{X} \in p^{-1}TM$. It is now easy to check:

Proposition.

i) $F^2 = -I + Q$,

ii) $\bar{u} \circ F = 0$

iii) $F\bar{X} = 0$ if and only if $\bar{X} \in \tilde{D}^{\perp}$

iv) $\bar{u}\,\bar{Y} = 0$ if and only if $\bar{Y} \in \tilde{D}$.

Then $(F, \bar{u})$ is called the natural Finsler $C.R.$-structure on M. The study of $C.R.$-submanifolds of almost hermitian Finsler manifolds is considerably more difficult in comparisson with the classical case, [1], because of the non-uniqueness of way of choosing an induced connection.

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Faculty of Mathematics
University of Bucharest
R. S. România

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ALMOST PRODUCT FINSLER STRUCTURES

BY

FRANCISC C. KLEPP

The almost complex Finsler structures was studied in [1], [2], [3], [4]. The set of almost complex Finsler connections was determined in [1], and in [2] was prsented modifications in the writting of the general form of this set, using the method of R. Miron and M. Hashiguchi [5]. The group of the transformations of almost complex Finsler connections and their invariants are established in [4], the integrability conditions of the almost complex Finsler structures being determined in the same paper.

In the present paper the notion of almost product Finsler structure and the notion of almost product Finsler connection are introduced and are studied with the method used by R. Miron [4].

The set of all almost product Finsler connections is determined (Theorem 2.5 and Theorem 2.6) and the groups G_{ap} and $G_{ap}(N)$ of transformations of the almost product Finsler connections, respectively of the transformations of almost product Finsler connections having the same non-linear connection N, are presented. Furthermore the invariants of the group $G_{ap}(N)$ are established (Theorem 3.2), and the integrability conditions of the almost product Finsler structures are determined (Theorem 5.4), by means of these invariants.

The notations and terminology of M. Matsumoto [6] are used with slight modifications from [5].

§1. Preliminaries. Let M be an n-dimensional differential manifold, $T(M)$ the tangent bundle of M and N a non-linear connection on $T(M)$. Let (x^i, y^i) be the canonic coordinates on $T(M)$. Then $\dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} - N^k_i \dfrac{\partial}{\partial y^k}$; $i = 1, 2, \ldots, n$ is a local basis for the distribution defining the non-linear connection N, and $\dfrac{\partial}{\partial y^i}$; $i = 1, 2, \ldots, n$ is a local basis for the vertical distribution $T(M)^v$. The dual basis is given by $(dx^i, \delta y^i)$, where $\delta y^i = dy^i + N^i_k dx^k$ satisfies:

$$dx^i\left(\frac{\delta}{\delta x^j}\right) = \delta^i_j\,;\ dx^i\left(\frac{\partial}{\partial y^j}\right) = 0\,;\ \delta y^i\left(\frac{\delta}{\delta x^j}\right) = 0\,;\ dy^i\left(\frac{\partial}{\partial y^j}\right) = \delta^i_j. \tag{1.1}$$