Proposition.

i) $F^2 = -I+Q$,

ii) $\bar{u} \circ F = 0$

iii) $F\overline{X} = 0$ if and only if $\overline{X} \in \tilde{D}^{\perp}$

iv) $\bar{u}\,\overline{Y} = 0$ if and only if $\overline{Y} \in \tilde{D}$.

Then $(F, \bar{u})$ is called the natural Finsler $C.R.$-structure on M. The study of $C.R.$-submanifolds of almost hermitian Finsler manifolds is considerably more difficult in comparisson with the classical case, [1], because of the non-uniqueness of way of choosing an induced connection.

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ALMOST PRODUCT FINSLER STRUCTURES

BY

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The almost complex Finsler structures was studied in [1], [2], [3], [4]. The set of almost complex Finsler connections was determined in [1], and in [2] was prsented modifications in the writting of the general form of this set, using the method of R. Miron and M. Hashiguchi [5]. The group of the transformations of almost complex Finsler connections and their invariants are established in [4], the integrability conditions of the almost complex Finsler structures being determined in the same paper.

In the present paper the notion of almost product Finsler structure and the notion of almost product Finsler connection are introduced and are studied with the method used by R. Miron [4].

The set of all almost product Finsler connections is determined (Theorem 2.5 and Theorem 2.6) and the groups G_{ap} and $G_{ap}(N)$ of transformations of the almost product Finsler connections, respectively of the transformations of almost product Finsler connections having the same non-linear connection N, are presented. Furthermore the invariants of the group $G_{ap}(N)$ are established (Theorem 3.2), and the integrability conditions of the almost product Finsler structures are determined (Theorem 5.4), by means of these invariants.

The notations and terminology of M. Matsumoto [6] are used with slight modifications from [5].

§1. Preliminaries. Let M be an n-dimensional differential manifold, $T(M)$ the tangent bundle of M and N a non-linear connection on $T(M)$. Let (x^i, y^i) be the canonic coordinates on $T(M)$. Then $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^k_i \frac{\partial}{\partial y^k}$; $i=1, 2, \ldots, n$ is a local basis for the distribution defining the non-linear connection N, and $\frac{\partial}{\partial y^i}$; $i=1, 2, \ldots, n$ is a local basis for the vertical distribution $T(M)^v$. The dual basis is given by $(dx^i, \delta y^i)$, where $\delta y^i = dy^i + N^i_k dx^k$ satisfies:

$$dx^i\left(\frac{\delta}{\delta x^j}\right) = \delta^i_j\,;\; dx^i\left(\frac{\partial}{\partial y^j}\right) = 0\,;\; \delta y^i\left(\frac{\delta}{\delta x^j}\right) = 0\,;\; dy^i\left(\frac{\partial}{\partial y^j}\right) = \delta^i_j. \tag{1.1}$$

A Finsler connection on M is a triade $F\Gamma=(N, F, C)$, where N the non-linear connection on $T(M)$, and F respectively C are the h-respectively v-connection coefficients [6] given by :

$$(1.2)\quad \begin{cases} \nabla_{\frac{\delta}{\delta x^k}} \dfrac{\delta}{\delta x^j} = F^i_{jk} \dfrac{\delta}{\delta x^i}; \quad \nabla_{\frac{\delta}{\delta y^j}} \dfrac{\partial}{\partial y^j} = F^i_{jk} \dfrac{\partial}{\partial y^i}; \\ \dot{\nabla}_{\frac{\partial}{\partial y^k}} \dfrac{\delta}{\delta x^j} = C^i_{jk} \dfrac{\delta}{\delta x^i}; \quad \dot{\nabla}_{\frac{\partial}{\partial y^k}} \dfrac{\partial}{\partial y^j} = C^i_{jk} \dfrac{\partial}{\partial y^i}. \end{cases}$$

To calculate the curvatures and the torsions of $F=(N, F, C)$, we remark that :

$$(1.3)\quad \begin{cases} \left[\dfrac{\delta}{\delta x^j}, \dfrac{\delta}{\delta x^k}\right] = R^i_{jk} \dfrac{\partial}{\partial y^i}; \quad \left[\dfrac{\partial}{\partial y^j}, \dfrac{\delta}{\delta x^k}\right] = -\dfrac{\partial N^i_k}{\partial y^j} \dfrac{\partial}{\partial y^i}; \\ \left[\dfrac{\delta}{\delta x^j}, \dfrac{\partial}{\partial y^k}\right] = \dfrac{\partial N^i_j}{\partial y^k} \dfrac{\partial}{\partial y^i}; \quad \left[\dfrac{\partial}{\partial y^j}, \dfrac{\partial}{\partial y^k}\right] = 0; \end{cases}$$

where :

$$(1.4)\quad R^i_{jk} = \frac{\delta N^i_j}{\partial x^k} - \frac{\delta N^i_k}{\delta x^j}.$$

For a Finsler tensor field K of the type (1.1) with the components K^i_j the h-and v-covariant derivatives are given by :

$$(1.5)\quad K^i_{j|k} = \frac{\delta K^i_j}{\delta x^k} + F^i_{mk} K^m_j - F^m_{jk} K^i_m; \quad K^i_j|_k = \frac{\partial K^i_j}{\partial y^k} + C^i_{mk} K^m_j - C^m_{jk} K^i_m.$$

The commutation formulae for the second order derivatives are :

$$(1.6)\quad \begin{cases} K^i_{j|k|h} - K^i_{j|h|k} = -K^i_m R^m_{jkh} + K^m_j R^i_{mkh} - K^m_j| R^m_{kh} - K^i_{j|m} T^m_{kh}, \\ K^i_{j|k}|_h - K^i_j|_{h|k} = -K^i_m P^m_{jkh} + K^m_j P^i_{mkh} - K^i_j|_m P^m_{kh} - K^i_{j|m} C^m_{kh}, \\ K^i_j|_k|_h - K^i_j|_h|_k = -K^i_m S^m_{jkh} + K^m_j S^i_{mkh} - K^i_j|_m S^m_{kh}, \end{cases}$$

where five torsion fields R^1, P^1, S^1, T, C and three curvature tensor fields R^2, P^2, S^2 appear and their components are given by :

$$(1.7)\quad \begin{cases} R^1: \ R^i_{jk} = \dfrac{\delta N^i_j}{\delta x^k} - \dfrac{\delta N^i_k}{\delta x^j}; \quad P^1: \ P^i_{jk} = \dfrac{\partial N^i_j}{\partial y^k} - F^i_{kj}; \\ S^1: \ S^i_{jk} = C^i_{jk} - C^i_{kj}; \quad T: \ T^i_{jk} = F^i_{jk} - F^i_{kj}; \quad C: \ C^i_{jk}; \\ R^2: \ R^i_{jkh} = \dfrac{\delta F^i_{jk}}{\delta x^h} - \dfrac{\delta F^i_{jh}}{\delta x^k} + F^a_{jk} F^i_{ah} - F^a_{jh} F^i_{ak} + C^i_{ja} R^a_{kh}; \\ P^2: \ P^i_{jkh} = \dfrac{\partial F^i_{jk}}{\partial y^h} - C^i_{jh|k} + C^i_{ja} P^a_{kh}; \\ S^2: \ S^i_{jkh} = \dfrac{\partial C^i_{jk}}{\partial y^h} - \dfrac{\partial C^i_{jh}}{\partial y^k} + C^a_{jk} C^i_{ah} - C^a_{jh} C^i_{ak}. \end{cases}$$

Let C be a curve on the differential manifold M and $\tilde{C}$ a curve on the tangent bundle $T(M)$ of M, such that $\pi(\tilde{C})=C$. A Finsler vector field X is said to be parallel on C relative to any curve $\tilde{C}[\pi(\tilde{C})=C]$ and relative to the Finsler connection $F\Gamma=(N, F, C)$, if and only if $X^i{}_{|k}=0$ and $X^i|_k=0$ [7].

If $F\Gamma=(N, F, C)$ and $F\overline{\Gamma}=(\overline{N}, \overline{F}, \overline{C})$ are two Finsler connections on M, then a unique triade of Finsler tensor fileds (A, B, D) is determined such that :

$$(1.8)\quad \overline{N}^i_j = N^i_j + A^i_j; \quad \overline{F}^i_{jk} = F^i_{jk} - C^i_{jh} A^h_k + B^i_{jk}; \quad \overline{C}^i_{jk} = C^i_{jk} + D^i_{jk}.$$

Conversely: given the Finsler connection $F\Gamma=(N, F, C)$ and the triade (A, B, D) of Finsler tensor fields, the connection $F\overline{\Gamma}=(\overline{N}, \overline{F}, \overline{C})$ given by (1.8) is a Finsler connection. The map $F\Gamma \leftarrow F\overline{\Gamma}$ defined by (1.8) is called transformation of the Finsler connections.

§2. The notion of almost product Finsler connection. Definition 2.1. *An almost product Finsler structure on M is a Finsler tensor field P of type* (1, 1) *with components $P^i_j(x, y)$ satisfying :*

$$(2.1)\quad P^k_j P^i_k = \delta^i_j.$$

It follows that the matrix (P^i_j) is non-singular.

To such a Finsler tensor field P, we can associate the O b a t a operators :

$$(2.2)\quad O^{ij}_{hk} = \frac{1}{2}(\delta^i_h\delta^j_k - P^i_h P^j_k); \quad O'^{ij}_{hk} = \frac{1}{2}(\delta^i_h\delta^j_k + P^i_h P^j_k),$$

which are projectors with the following properties :

$$(2.3)\quad \begin{cases} O^{ij}_{hk} + O'^{ij}_{hk} = \delta^i_h\delta^j_k; \quad O'^{ij}_{hk} O'^{hk}_{rs} = O'^{ij}_{rs}, \\ O^{ij}_{hk} O'^{hk}_{rs} = O'^{ij}_{hk} O^{hk}_{rs} = 0; \quad O^{ij}_{hk} O^{hk}_{rs} = O^{ij}_{rs}. \end{cases}$$

Definition 2.2. *A Finsler connection $F\Gamma=(N, F, C)$ is called almost product relative to the almost product Finsler structure P, if an arbitrary Finsler tensor field $X \in \mathscr{X}_F(M)$ parallel on any curve C on M relative to $\tilde{C}$ on $T(M)$ $[\pi(\tilde{C})=C]$ and relative to $F\Gamma$, determines a Finsler vector field $Y=PX$ parallel on C relative to $\tilde{C}$ and relative to $F\Gamma$.*

Here $\mathscr{X}_F(M)$ denotes the set of Finsler vector fields on M.

It can be proved :

Theorem 2.1. *The Finsler connection $F\Gamma=(N, F, C)$ is an almost product Finsler connection relative to P, if and only if :*

$$(2.4)\quad P^i_{j|k} = 0; \quad P^i_j|_k = 0.$$

We have immediately :

Theorem 2.2. *If $F\dot{\Gamma}=(\dot{N}, \dot{F}, \dot{C})$ is an arbitrary but fixed Finsler connection on M, then the Finsler connection :*

$$(2.5)\quad \overset{k}{N}{}^i_j = \dot{N}^i_j; \quad \overset{k}{F}{}^i_{jk} = \dot{F}^i_{jk} - \frac{1}{2} P^h_j \dot{P}^i_{h|k}; \quad \overset{k}{C}{}^i_{jk} = \dot{C}^i_{jk} - \frac{1}{2} P^h_j \dot{P}^i_h|_k$$

is an almost product Finsler connection, where $\overset{\bullet}{|}$ *and* $\overset{\circ}{|}$ *are the h- and v-covariant derivatives relative to* $F\overset{\circ}{\Gamma}$.

A straightforward calculus shows that the Finsler connection $K\overset{\circ}{\Gamma} = (\overset{k}{N}, \overset{k}{F}, \overset{k}{C})$ given by (2.5) satisfies the equations (2.4).

Definition 2.3. *The Finsler connection* $K\overset{\circ}{\Gamma} = (\overset{k}{N}, \overset{k}{F}, \overset{k}{C})$ *given by* (2.5) *is called the almost product Kawaguchi connection derived from* $F\overset{\circ}{\Gamma}$.

Using the properties (2.3) one can prove:

Lemma 2.3. *A tensorial equation of the form:*

$$O^{ir}_{sj} X^{s}_{rk} = U^{i}_{jk} \quad \text{respectively} \quad O'^{ir}_{sj} X^{s}_{rk} = U^{i}_{jk}$$

with the Finsler tensor field X^{s}_{rk} *as unknown, has a solution, if and only if:*

$$O^{*ir}_{sj} U^{s}_{rk} = 0, \quad \text{respectively} \quad O^{ir}_{sj} U^{s}_{rk} = 0.$$

In this case the general solution of the equation is given by:

$$X^{i}_{jk} = O'^{ir}_{sj} Y^{s}_{rk} + U^{i}_{jk} \ \text{resp.} \ X^{i}_{jk} = O^{ir}_{sj} Y^{s}_{rk} + U^{i}_{jk}, \tag{2.6}$$

where $Y^{i}_{jk}(x, y)$ is an arbitrary Finsler tensor field, of the same type as X^{i}_{jk}.

From (2.2) and (2.4) it follows:

Theorem 2.4. *The h-and v-convariant derivatives of the Obata operators of the almost product Finsler structure vanish.*

$$O^{ij}_{hk|p} = 0\ ;\ O^{ij}_{hk}|_{p} = 0\ ;\ O'^{ij}_{hk|p} = 0\ ;\ O'^{ij}_{hk}|_{p} = 0. \tag{2.7}$$

The set of all almost product Finsler connections can be determined by a known method based on Lemma 2.3 [5]. Let $F\overset{\circ}{\Gamma} = (\overset{\circ}{N}, \overset{\circ}{F}, \overset{\circ}{C})$ be an arbitrary but fixed Finsler connection; then according to (1.8), the triad $F\Gamma = (N, F, C)$ given by:

$$N^{i}_{j} = \overset{\circ}{N}{}^{i}_{j} - X^{i}_{j}\ ;\ F^{i}_{jk} = \overset{\circ}{F}{}^{i}_{jk} + \overset{\circ}{C}{}^{i}_{j}{}^{h} + B^{i}_{jk}\ ;\ C^{i}_{jk} = \overset{\circ}{C}{}^{i}_{jk} + D^{i}_{jk}, \tag{2.8}$$

where X, B and D are arbitrary Finsler tensor fields, is a Finsler connection. Requiring $F\Gamma$ to be an almost product Finsler connection, we obtain for the Finsler tensor fields X, B and D the following expressions:

$$B^{i}_{jk} = \frac{1}{2} P^{i}_{m}(P^{m}_{j}{}_{\overset{\bullet}{|}k} + P^{m}_{j}\overset{\circ}{|}_{h} X^{h}_{k}) + O^{mi}_{jh} X^{h}_{mk},$$

$$D^{i}_{jk} = \frac{1}{2} P^{i}_{m} P^{m}_{j}\overset{\circ}{|}_{k} + O^{mi}_{jh} Y^{h}_{mk},$$

where X and Y are arbitrary Finsler tensor fields.

Thus we can obtain:

Theorem 2.5. *The set of all almost product Finsler connections relative to the almost product Finsler structure* P *is given by:*

$$\begin{aligned} &N^{i}_{j} = \overset{\circ}{N}{}^{i}_{j} - X^{i}_{j}\ ;\ C^{i}_{jk} = \overset{\circ}{C}{}^{i}_{jk} - \frac{1}{2} P^{m}_{j} P^{i}_{m}\overset{\bullet}{|}_{k} + O^{mi}_{jh} Z^{h}_{mk},\\ &F^{i}_{jk} = \overset{\circ}{F}{}^{i}_{jk} + \overset{\circ}{C}{}^{i}_{jm} X^{m}_{k} - \frac{1}{2} P^{m}_{j}(P^{i}_{m}{}_{\overset{\bullet}{|}k} + P^{i}_{m}\overset{\circ}{|}_{h} X^{h}_{k}) + O^{mi}_{jh} Y^{h}_{mk}, \end{aligned} \tag{2.9}$$

where $F\overset{\circ}{\Gamma} = (\overset{\circ}{N}, \overset{\circ}{F}, \overset{\circ}{C})$ *is a fixed Finsler connection on* M, $\overset{\circ}{|}$ *and* $\overset{\circ}{|}$ *are the h- and v-covariant derivatives relative to* $F\overset{\circ}{\Gamma}$ *and* X, Y, Z *are arbitrary Finsler tensor fields.*

Remark 2.1. The almost product Kawaguchi connection $K\overset{\circ}{\Gamma} = (\overset{k}{N}, \overset{k}{F}, \overset{k}{C})$ given by (2.5) can be obtained from (2.9) by taking $X = Y = Z = 0$.

Corollary 2.1. *If* $F\overset{\circ}{\Gamma} = (\overset{\circ}{N}, \overset{\circ}{F}, \overset{\circ}{C})$ *is a fixed almost product Finsler connection, then the set of all almost product Finsler connections* $F\Gamma = (N, F, C)$ *is given by:*

$$N^{i}_{j} = \overset{\circ}{N}{}^{i}_{j} - X^{i}_{j}\ ;\ F^{i}_{jk} = \overset{\circ}{F}{}^{i}_{jk} + \overset{\circ}{C}{}^{i}_{jh} X^{h}_{k} + O^{hi}_{jm} Y^{m}_{hk}\ ;\ C^{i}_{jk} = \overset{\circ}{C}{}^{i}_{jk} + O^{ih}_{jm} Z^{m}_{hk}, \tag{2.10}$$

where X, Y, Z *are arbitrary Finsler tensor fields.*

If we denote by $F\Gamma(\overset{\circ}{N})$ the Finsler connections having the same non-linear connection $\overset{\circ}{N}$, then one can state:

Theorem 2.6. *The set of all almost product Finsler connections* $F\Gamma(\overset{\circ}{N})$ *having the same non-linear connection* N° *is given by:*

$$N^{i}_{j} = \overset{\circ}{N}{}^{i}_{j}\ ;\ F^{i}_{jk} = \overset{\circ}{F}{}^{i}_{jk} + O^{hi}_{jm} X^{m}_{hk}\ ;\ C^{i}_{jk} = \overset{\circ}{C}{}^{i}_{jk} + O^{hi}_{jm} Y^{m}_{hk}, \tag{2.11}$$

where X *and* Y *are arbitrary Finsler tensor fields.*

Starting with of the identities of Ricci (1.6) we can prove:

Proposition 2.7. *For every almost product connection* $F\Gamma$ *relativ to the almost product Finsler structure* P, *the following tensor fields:* $O^{*ri}_{js} P^{s}_{rkh}$; $O^{*ri}_{js} R^{s}_{rkh}$; $O^{*ri}_{js} S^{s}_{rkh}$ *as well as its covariant h- and v-derivatives of any order vanish.*

§3. The group of the transformations of almost product Finsler connections. Let $F\Gamma$ and $F\overline{\Gamma}$ be two almost product Finsler connections by Corollary 2.1 we have:

$$\overline{N}{}^{i}_{j} = N^{i}_{j} - X^{i}_{j}\ ;\ \overline{F}{}^{i}_{jk} = F^{i}_{jk} + C^{i}_{jh} X^{h}_{k} + O^{ih}_{jm} Y^{m}_{hk}\ ;\ \overline{C}{}^{i}_{jk} = C^{i}_{jk} + O^{ih}_{jm} Z^{m}_{hk}, \tag{3.1}$$

where X, Y, Z are arbitrary Finsler tensor fields.

Conversely: given an almost product Finsler connection $F\Gamma = (N, F, C)$ and a triade of Finsler tensor fields X, Y, Z with the components X^{i}_{j}, Y^{i}_{jk} and Z^{i}_{jk} respectively, the Finsler connection given by (3.1) is an almost product Finsler connection.

Theorem 3.1. *The set* G_{ap} *of the transformations of almost product Finsler connections, given by* (3.1) *form an abelian group under the product of the transformations.*

The structure of the group G_{ap} can be studied in the same way as the group of the metric Finsler connections [5]. Denoting by $G_{ap}(N)$ the set of the transformations of almost product Finsler connections having the same non-linear connection, it is clear that $G_{ap}(N)$ is the group of transformations given by:

$$\overline{N}{}^{i}_{j} = N^{i}_{j}\ ;\ \overline{F}{}^{i}_{jk} = F^{i}_{jk} + O^{hi}_{jm} X^{m}_{hk}\ ;\ \overline{C}{}^{i}_{jk} = C^{i}_{jk} + O^{hi}_{jm} Y^{m}_{hk}, \tag{3.2}$$

where X and Y are arbitrary Finsler tensor fields.

Determinating the invariants of the group $G_{ap}(N)$ we have:

Theorem 3.2. *The following Finsler tensor fields are invariant by the action of the group $G_{ap}(N)$:*

$$\tilde{N}^i_{jk} = T^i_{jk} - P^h_j P^m_k T^i_{hm} + P^i_h(P^m_j T^h_{mk} + P^m_k T^h_{jm}),$$

$$\tilde{R}^i_{jk} = -R^i_{jk} - P^h_j P^m_k R^i_{hm} + P^i_h(P^m_j R^h_{mk} + P^m_k R^h_{jm}),$$

$$\tilde{C}^i_{jk} = C^i_{jk} - P^h_j P^m_k C^i_{hm} + P^i_h(P^m_j C^h_{mk} + P^m_k C^h_{jm}),$$

$$P^i_{jk} = P^i_{jk} - P^h_j P^m_k P^i_{hm} + P^i_h(P^m_j P^h_{mk} + P^m_k P^h_{jm}),$$

$$\dot{R}^i_{jk} = R^i_{jk} + P^h_j P^m_k R^i_{hm} + P^i_h(P^m_j R^h_{mk} + P^m_k R^h_{jm}),$$

$$\dot{N}^i_{jk} = S^i_{jk} - P^h_j P^m_k S^i_{hm} + P^i_h(P^m_j S^h_{mk} + P^m_k S^h_{jm}),$$

$$\dot{C}^i_{jk} = C^i_{jk} + P^h_j P^m_k C^i_{hm} + P^i_h(P^m_j C^h_{mk} + P^m_k C^h_{jm}),$$

$$\dot{P}^i_{jk} = P^i_{jk} + P^h_j P^m_k P^i_{hm} - P^i_h(P^m_j P^h_{mk} - P^m_k P^h_{jm}),$$

$$\ddot{T}^i_{jk} = -P^i_h T^h_{jk} + P^h_k P^i_{jh} - P^h_j P^i_{kh},$$

$$\ddot{S}^i_{jk} = -R^i_{jk} - P^h_j P^m_k S^i_{hm} + P^i_h(P^m_k C^h_{jm} - P^m_j C^h_{km}).$$

Proof. Taking into account (3.2) it results:

$$\overline{T}^i_{jk} = T^i_{jk} + \frac{1}{2}(X^i_{jk} - X^i_{jk}) - \frac{1}{2} P^i_m(P^h_j X^m_{hk} - P^h_k X^m_{hj}),$$

$$\overline{S}^i_{jk} = S^i_{jk} + \frac{1}{2}(Y^i_{jk} - Y^i_{kj}) - \frac{1}{2} P^i_m(P^h_j Y^m_{hk} - P^h_k Y^m_{hj})\ ;\ \overline{R}^i_{jk} = R^i_{jk},$$

$$\overline{C}^i_{jk} = C^i_{jk} + \frac{1}{2}(Y^i_{jk} - P^h_j P^i_m Y^m_{hk})\ ;\ \overline{P}^i_{jk} = P^i_{jk} - \frac{1}{2}(X^i_{kj} - P^h_k P^i_m X^m_{hj}),$$

and by a straightforward calculus we can prove the invariancy of the above Finsler tensor fields.

§4. Almost product structures on the tangent bundle. An almost product structure on the tangent bundle $T(M)$ is given by a tensor field $\tilde{P}$ of type (1, 1) with the property $\tilde{P}.\tilde{P} = \tilde{1}$. In the local base $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ a tensor field $\tilde{P}$ of type (1, 1) is given by:

$$\tilde{P} = \overset{1}{P}{}^i_j \frac{\delta}{\delta x^i} \otimes dx^j + \overset{2}{P}{}^i_j \frac{\delta}{\delta x^i} \otimes \delta y^j + \overset{3}{P}{}^i_j \frac{\partial}{\partial y^i} \otimes dx^j + \overset{4}{P}{}^i_j \frac{\partial}{\partial y^i} \otimes \delta y^j. \tag{4.1}$$

Then we have:

$$\tilde{P}\left(\frac{\delta}{\delta x^j}\right) = \overset{1}{P}{}^i_j \frac{\delta}{\delta x^i} + \overset{3}{P}{}^i_j \frac{\partial}{\partial y^i}\ ;\ \tilde{P}\left(\frac{\partial}{\partial y^j}\right) = \overset{2}{P}{}^i_j \frac{\delta}{\delta x^i} + \overset{4}{P}{}^i_j \frac{\partial}{\partial y^i}, \tag{4.2}$$

If $X \in \mathcal{X}(TM)$ then $X = \tilde{X}^i \frac{\delta}{\delta x^i} + \dot{X}^i \frac{\partial}{\partial y^i}$ and by (4.2) it results:

$$\tilde{P}(X) = (\overset{1}{P}{}^i_j \tilde{X}^j + \overset{2}{P}{}^i_j \dot{X}^j) \frac{\delta}{\delta x^i} + (\overset{3}{P}{}^i_j \tilde{X}^j + \overset{4}{P}{}^i_j \dot{X}^j) \frac{\partial}{\partial y^i}.$$

So the condition $\tilde{P}[\tilde{P}(X)] = X$; $\forall X \in \mathcal{X}(TM)$ leads to:

$$\begin{cases} \overset{1}{P}{}^i_k \overset{1}{P}{}^j_i + \overset{3}{P}{}^i_k \overset{2}{P}{}^j_i = \delta^j_k\ ; & \overset{2}{P}{}^i_k \overset{1}{P}{}^j_i + \overset{4}{P}{}^i_k \overset{2}{P}{}^j_i = 0, \\ \overset{1}{P}{}^i_k \overset{3}{P}{}^j_i + \overset{3}{P}{}^i_k \overset{4}{P}{}^j_i = 0\ ; & \overset{2}{P}{}^i_k \overset{3}{P}{}^j_i + \overset{4}{P}{}^i_k \overset{4}{P}{}^j_i = \delta^j_k. \end{cases} \tag{4.3}$$

The Nijenhuis tensor of $\tilde{P}$ is given by:

$$N(X, Y) = -[X, Y] + \tilde{P}[\tilde{P}X, Y] + \tilde{P}[X, \tilde{P}Y] - [\tilde{P}X, \tilde{P}Y]. \tag{4.4}$$

The integrability condition of the almost product structure $\tilde{P}$ is $N(X, Y) = 0$; $\forall X, Y \in \mathcal{X}(TM)$. It is sufficient to calculate

$$N\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right);\ N\left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\right) \text{ and } N\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right)$$

and we can determine $N(X, Y)$. The calculus leads to:

Theorem 4.1. *The Nijenhuis tensor of the almost product structure $\tilde{P}$ has the following properties:*

$$N\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right) = -R^i_{jk} \frac{\partial}{\partial y^i} + \left(\frac{\delta \overset{1}{P}{}^m_k}{\delta x^j} - \frac{\delta \overset{1}{P}{}^m_j}{\delta x^k}\right) \tilde{P}\left(\frac{\delta}{\delta x^m}\right) + \left\{\overset{1}{P}{}^r_j R^m_{rk} - \overset{1}{P}{}^r_k R^m_{rj} + \right.$$

$$\left. + \left(\overset{3}{P}{}^r_k \frac{\partial N^m_j}{\partial y^r} - \overset{3}{P}{}^r_j \frac{\partial N^m_k}{\partial y^r}\right) + \left(\frac{\delta \overset{3}{P}{}^m_k}{\delta x^j} - \frac{\delta \overset{3}{P}{}^m_j}{\delta x^k}\right)\right\} \tilde{P}\left(\frac{\partial}{\partial y^m}\right) -$$

$$- \left[\overset{1}{P}{}^i_j \frac{\delta}{\delta x^i} + \overset{3}{P}{}^i_j \frac{\partial}{\partial y^i}, \overset{1}{P}{}^l_k \frac{\delta}{\delta x^l} + \overset{3}{P}{}^l_k \frac{\partial}{\partial y^l}\right],$$

$$N\left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\right) = -\frac{\partial N^i_j}{\partial y^k} \frac{\partial}{\partial y^i} + \left(\frac{\delta \overset{2}{P}{}^m_k}{\delta x^j} - \frac{\partial \overset{1}{P}{}^m_j}{\partial y^k}\right) \tilde{P}\left(\frac{\delta}{\delta x^m}\right) +$$

$$+ \left\{\overset{1}{P}{}^r_j \frac{\partial N^m_r}{\partial y^k} - \frac{\partial \overset{3}{P}{}^m_j}{\partial y^k} + \overset{4}{P}{}^r_k \frac{\partial N^m_j}{\partial y^r} + \frac{\delta \overset{4}{P}{}^m_k}{\delta x^j} - \overset{2}{P}{}^r_k R^m_{rj}\right\} \tilde{P}\left(\frac{\partial}{\partial y^m}\right) -$$

$$- \left[\overset{1}{P}{}^i_j \frac{\delta}{\delta x^i} + \overset{3}{P}{}^i_j \frac{\partial}{\partial y^i}, \overset{2}{P}{}^m_k \frac{\delta}{\delta x^m} + \overset{4}{P}{}^m_k \frac{\partial}{\partial y^m}\right],$$

$$N\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right) = \left(\frac{\partial \overset{2}{P}{}^m_k}{\partial y^j} - \frac{\partial \overset{2}{P}{}^m_j}{\partial y^k}\right) \tilde{P}\left(\frac{\delta}{\delta x^m}\right) + \left\{\overset{2}{P}{}^p_j \frac{\partial N^m_p}{\partial y^k} - \overset{2}{P}{}^p_k \frac{\partial N^m_p}{\partial y^j} + \right.$$

$$\left. + \frac{\partial \overset{4}{P}{}^m_k}{\partial y^j} - \frac{\partial P^m_j}{\partial y^k}\right\} \tilde{P}\left(\frac{\partial}{\partial y^m}\right) - \left[\overset{2}{P}{}^i_j \frac{\delta}{\delta x^i} + \overset{4}{P}{}^i_j \frac{\partial}{\partial y^i}, \overset{2}{P}{}^l_k \frac{\delta}{\delta x^l} + \overset{4}{P}{}^l_k \frac{\partial}{\partial y^l}\right].$$

§5. The integrability of the almost product Finsler structures. Let N be a non-linear connection on $T(M)$ and P an almost product Finsler structure on M. Then P can be lifted to $T(M)$ by the aid of the distributions N and $T(M)^v$. The most important liftings are:

$$P^I = P^i_j \frac{\delta}{\delta x^i} \otimes dx^j + P^i_j \frac{\partial}{\partial y^i} \otimes \delta y^j,$$

$$P^{II} = P^i_j \frac{\delta}{\delta x^i} \otimes dx^j - P^i_j \frac{\partial}{\partial y^i} \otimes \delta y^j, \tag{5.1}$$

$$P^{III} = P^i_j \frac{\delta}{\delta x^i} \otimes \delta y^j + P^i_j \frac{\partial}{\partial y^i} \otimes dx^j.$$

These are almost product structures $\tilde{P}$ on $T(M)$ having the expression (4.1) with the values of the coefficients given by the following list:

$\tilde{P}$	$\overset{1}{P}{}^i_j$	$\overset{2}{P}{}^i_j$	$\overset{3}{P}{}^i_j$	$\overset{4}{P}{}^i_j$
P^I	P^i_j	0	0	P^i_j
P^{II}	P^i_j	0	0	$-P^i_j$
P^{III}	0	P^i_j	P^i_j	0

We remark the following relations:

$$\begin{cases} P^I\left(\frac{\delta}{\delta x^j}\right) = P^i_j \frac{\delta}{\delta x^i}; \; P^{II}\left(\frac{\delta}{\delta x^j}\right) = P^i_j \frac{\delta}{\delta x^i}; \; P^{III}\left(\frac{\delta}{\delta x^j}\right) = P^i_j \frac{\partial}{\partial y^i}; \\ P^I\left(\frac{\partial}{\partial y^j}\right) = P^i_j \frac{\partial}{\partial y^i}; \; P^{II}\left(\frac{\partial}{\partial y^j}\right) = -P^i_j \frac{\partial}{\partial y^i}; \; P^{III}\left(\frac{\partial}{\partial y^j}\right) = P^i_j \frac{\delta}{\delta x^i}. \end{cases} \tag{5.2}$$

Definition 5.1. *The almost product Finsler structure P with the components $P^i_j(x, y)$ is integrable of the type I, II and respectively III, if the almost products structure P^I, P^{II} and respectively P^{III} are integrable.*

Applying to P^I the formulae of Theorem 4.1 and using the properties of the almost product structures it results:

Theorem 5.1. *The almost product Finsler structure P is integrable of the type I, if and only if the following invariant Finsler tensor fields vanish:*

$$\tilde{N}^i_{jk} = 0; \; \tilde{R}^i_{jk} = 0; \; \tilde{C}^i_{jk} = 0; \; \tilde{P}^i_{jk} = 0; \; \dot{N}^i_{jk} = 0. \tag{5.3}$$

Applying to P^{II} the formulae of Theorem 4.1 an analogous calculus leads to:

Theorem 5.2. *The almost product Finsler structure P is integrable of the type II, if and only if the following invariant Finsler tensor fields vanish*

$$\tilde{N}^i_{jk} = 0; \; \dot{R}^i_{jk} = 0; \; \dot{C}^i_{jk} = 0; \; \dot{P}^i_{jk} = 0; \; \dot{N}^i_{jk} = 0. \tag{5.4}$$

Applying to P^{III} the formulae of Theorem 4.1 an analogous calculus lead to:

Theorem 5.3. *The almost product Finsler structure P is integrable of the type III, if and only if the following invariant Finsler tensor fields vanish:*

$$\ddot{S}^i_{jk} = 0; \; \ddot{T}^i_{jk} = 0. \tag{5.5}$$

Concluding, by Theorems 5.1, 5.2 and 5.3 we obtain:

Theorem 5.4. *The almost product Finsler structure P is integrable of the type I, II respectively III, if and only if the following invariants of the group $G_{ap}(N)$, listed below, vanish:*

Type of integrability	Characterization by invariants
I	$\tilde{N}^i_{jk}=0$; $\tilde{R}^i_{jk}=0$; $\tilde{C}^i_{jk}=0$; $\tilde{P}^i_{jk}=0$; $\dot{N}^i_{jk}=0$
II	$\tilde{N}^i_{jk}=0$; $\dot{R}^i_{jk}=0$; $\dot{C}^i_{jk}=0$; $\dot{P}^i_{jk}=0$; $\dot{N}^i_{jk}=0$
III	$\ddot{S}^i_{jk}=0$; $\ddot{T}^i_{jk}=0$

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