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JACOBIAN CRITERION OF NAGATA-GROTHENDIECK AND APPLICATIONS

BY

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0. The rings of this paper are commutative and unitary. A noetherian local ring A of maximal ideal M is called *regular* if $\dim_{A/M} M/M^2 = \dim A$, i.e. M is generated by a system of parameters (or equivalently : the graded ring $A/M \oplus M/M^2 \oplus \cdots$ is isomorphic to a polynomial ring over A/M). It is known that the localizations of a regular local ring are still regular (J. P. Serre). A noetherian ring is called regular if all its localizations are regular. The regular (i.e. non-singular) points are the geometric interpretation of the regular local rings. In order to characterize the regular points in algebraic geometry and to prove that the singular locus is closed, O. Zariski proved in 1942 the following jacobian criterion of regularity (JCZ for short).

0.1. **Theorem 1** ([Z]). *Let K be a perfect field, $R = K[X_1, \ldots, X_n]$ with X_i indeterminates, $I \subseteq P$ be ideals of R with P prime and $A = R/I$. Then the following assertions are equivalent :*

a) *A_P is a regular ring;*

b) *there exist $f_1, \ldots, f_m \in I$ s.t. $I_P = (f_1, \ldots, f_m) R_P$ and the derivations $D_1, \ldots, D_m \in \mathrm{Der}_K(R, R)$ s.t. $\det(D_i f_j) \notin P$ (and $D_1, \ldots, D_m$ can be choosen between the partial derivatives $\partial/\partial X_1, \ldots, \partial/\partial X_n$).*

In order to prove that b) implies a), observe that the images of $f_1, \ldots, f_m$ in PR_P/P^2R_P are linearly independent over $K(P)$, hence $f_1, \ldots, f_m$ can be extended to a regular system of parameters ; then $A_P = R_P/IR_P$ is a regular ring.

Remark that in this implication we used only the regularity of R_P! Under which more general conditions as in th.1 does the implication a)$\Rightarrow$b) remain true? Nagata proved an analogous criterion for rings of formal (even convergent) power series ([N1]) ; it was generalized by Grothendieck, replacing the regularity by formal smoothness (see th 2').

1.1. **Theorem 2**(Jacobian Criterion of Nagata, JCN for short). *Let k be a perfect field, K a field extension of k, $R = K[[X_1, .., X_n]]$ with X_i indeterminates, $I \subseteq P$ be ideals of R with P prime and $A = R/I$. Then the following assertions are equivalent :*

a) *A_P is a regular ring ;*

b) *there exist $f_1, \ldots, f_m \in I$ s.t. $I_P = (f_1, \ldots, f_m) R_P$ and $D_1, \ldots, D_m \in \mathrm{Der}_k(R, R)$ s.t. $\det(D_i f_j) \notin P$;*

c) *there exist* $f_1, \ldots, f_m \in I$ *s.t.* $I_P = (f_1, \ldots, f_m) R_P$, *there exist a cofinite subfield* k' *of* K (i.e. $[K : k'] < \infty$) *and* $D_1, \ldots, D_m \in \mathrm{Der}_{k'}(R, R)$ *s.t.* $\det(D_i f_j) \notin P$.

1.2. **Consequence 1.** Theorem ([*N*1]). *If A is a complete noetherian local ring, then the set* Reg A *(of prime ideals P of A s.t.* A_P *is regular) is open in* Spec A.

1.3. **Consequence 2.** *A simple proof of the theorem* (*EGA*, 0_{IV}, 22, 3.3) ; *if A is a complete noetherian local ring and Q a prime ideal of A, then the completion morphism* $A_Q \to (A_Q)^\wedge$ *is regular* (see [*BR* 3]).

1.4. **Consequence 3.** *A simple, non-homological proof of the localization theorem of formal smoothness* ([*S*2], see th. 3) ; *this proof uses essentially condition c) of th. 2.*

Following *JCN* and *JCZ* we proved :

1.5. **Theorem** ([BR3]). *Let* k, K, *and* R *be as in th. 2, let* $S = R[Z_1, \ldots, Z_t]$ *with* Z_i *indeterminates,* $I \subseteq P$ *be ideals of S with P prime and* $P + XS \neq S$ *and* $A = S/I$. *Then the conditions* a), b), c) *from th. 2, where we replace R by S, are equivalent.*

This theorem is true also for $k = \mathbf{Z}_{pZ}$, with p a prime number, and K a Cohen p-ring, if $p \notin P$([BR 3]).

Let A be a notherian ring and B an A-algebra of finite type. Using the preceding theorem we obtained a simultaneous and simple proof of the theorems.

1.6. **Consequence 4.** ([N2]). *If A is a Nagata ring, then B is also a Nagata ring.*

1.7. **Consequence 5.** (*EGA*, 0_{IV}). *If A is a reg-ring, then B is also a reg-ring.*

2.0. The formal smoothness is a generalization and, in the same time, a relativization of regularity and separability too. We consider only adic topologies. A morphism of topological rings $u : A \to B$ is called *formally smooth* if for any discrete topological A-algebra C and any nilpotent ideal J of C, every morphism of topological A-algebras $f : B \to C/J$ can be lifted to a morphism of A-algebras $f' : B \to C$, i.e. $f = pf'$, where $p : C \to C/J$ is the canonical surjection. It is proved that the lifting holds also when C is a complete A-algebra and J is topologically nilpotent. The Jacobian criterion of smoothness (see e.g. EGA, IV, or [R], chap. 12) generalizes *JCZ* and leads to the homology of commutative algebras ([*A* 1]).

In particular cases, the formal smoothness is equivalent with well-known properties. Recall that a notherian algebra B over a field k is *geometrically reduced* (resp. *geometrically regular*) if for any field k', finite extension of k, the ring $B \otimes k^{k'}$ is reduced (resp. regular). Obviously a reduced noetherian ring of dimension 0 is regular.

2.1. **Theorem** (C o h e n (1960). EGA, 0_{IV}, §19). *Let* $k \subseteq K$ *be a field extension of characteristic exponent p. Then the following assertions are equivalent :*

a) $k \to K$ *is formally smooth ;*

b) $k \to K$ *is geometrically reduced* (i.e. *the extension is separable*) ;

c) $K \otimes_k k^{1/p}$ *is a field ...*

This property is generalized in :

2.2. **Theorem** (A. G r o t h e n d i e c k, EGA, 0_{IV}, 22.5.8 and N. R a d u, [R1]). *Let k be a field of characteristic exponent p and A a noetherian local k-algebra. Then the following assertions are equivalent :*

a) $k \to A$ *is formally smooth in the radicial topology ;*

b) $k \to A$ *is geometrically regular ;*

c) *the ring* $A \otimes_k k^{1/p}$ *is noetherian and regular.*

The conditions b) and c) related the formal smoothness and the regularity. If k is perfect, condition c) shows that A is a formally smooth k-algebra iff A is a regular ring.

The equivalence between a) and b) was proved by A. Grothendieck in EGA ; the implication b)$\Rightarrow$a) is difficult and technical. G. F a l t i n g s [F] gave a simpler proof of it using the p-bases. In 1977 N. R a d u proved that a) and b) are equivalent with c), obtaining also a simpler and natural proof of b)$\Rightarrow$a). In assertion c) it is interesting the condition that the ring $A \otimes_k k^{1/p}$ is noetherian. The equivalence between a) and b) shows that if A is a formally smooth k-algebra and P is a prime ideal of A, then A_P is a formally smooth k-algebra (in the P-adic topology).

It is almost obvious that it could exist criteria for formal smoothness, analogous to *JCZ* and *JCN*. The most important is :

2.3. **Theorem** 2′ (Jacobian Criterion of Nagata-Grothendieck, EGA, 0_{IV}, 22.7.3). *Let k be a field of characteristic exponent p, K a field, separable extension of k, R a formally smooth complete noetherian local K-algebra,* $I \subseteq P$ *be ideals of R with P prime and* $A = R/I$. *Suppose that* $[k : k^p] < \infty$ *and the residue field L of R is finite over K. Then the following assertions are equivalent :*

a) A_P *is a formally smooth k-algebra in the P-adic topology ;*

b) *there exist* $f_1, \ldots, f_m \in I$ *s.t.* $I_P = (f_1, \ldots, f_m) R_P$ *and* $D_1, \ldots, D_m \in \mathrm{Der}_k(R, R)$ *s.t.* $\det(D_i f_j) \notin P$;

c) *there exist* $f_1, \ldots, f_m \in I$ *s.t.* $I_P = (f_1, \ldots, f_m) R_P$, *there exist a cofinite subfield* k' *of K and* $D_1, \ldots, D_m \in \mathrm{Der}_{k'}(R, R)$ *s.t.* $\det(D_i f_j) \notin P$.

The proof in EGA is difficult and rather technical. Using JCN and the equivalence between a) and c) in 2.2, N. R a d u proved the equivalence between a) and b) in th.2, without the hypothesis $[L : K] < \infty$([R 2]). Using 1.5 instead of *JCN*, we obtain

2.4. **Theorem** ([BR 3]). *Let k be a field of characteristic exponent p with* $[k : k^p] < \infty$, *R a formally smooth complete noetherian k-algebra,* $S = R[Z_1, \ldots, Z_t]$ *with* Z_i *indeterminates,* $I \subseteq P$ *be ideals of S with P prime,* $P + MS \neq S$, *where M is the maximal ideal of R and* $A = S/I$. *Then the following assertions are equivalent :*

a) A_P *is a formally smooth k-algebra ;*

b) *there exist* $f_1, \ldots, f_m \in I$ *s.t.* $I_P = (f_1, \ldots, f_m) S_P$ *and* $D_1, \ldots, D_m \in \mathrm{Der}_k(S, S)$ *s.t.* $\det(D_i f_j) \notin P$.

It seems that a) and b) from 2.4 are also equivalent with a condition c) as in 2.3.

3.0 Let $u : A \to B$ be a local morphism of noetherian local rings. Then u is formally smooth iff u is flat and $u \otimes_A k$ is formally smooth (i.e. the special fibre of u is geometrically regular), where k is the residue field of A (EGA, 0_{IV}, 19.7.1). The morphism u is called *regular* if it is flat and all of its fibres are geometrically regular (i.e. for any $Q \in \operatorname{Spec} B$ and $P = Q \cap A$, the morphism $A_P \to B_Q$ is formally smooth). If u is regular, obviously it is formally smooth. The converse is true for instance if A is a field (see above). In EGA, IV, 7.5.6, A. Grothendieck conjectured that the formal smoothness localizes; this became „one of the most brilliant theorems from local algebra" ([Ni]):

3.1. **Theorem 3** (André-Radu-Seydi). *If a local morphism $u : A \to B$ of noetherian local rings is formally smooth and the completion morphism $A \to \hat{A}$ is regular, then u is regular.*

A. Grothendieck proved 3.1 when the residue field K of B is finite over the residue field k of A. In [BR 1], 3.1 was proved when $[k : k^p] < \infty$, hence also when char $k = 0$; also the general case was reduced to the case when A and B are formal power series rings. Using this reduction and the (co) homology of algebras from [A1], M. André proved 3.1 in general ([A2]). In 1976 N. Radu gave another proof, using the good separability of the module of differentials ([BR 2]). In [S2], see also [S1], M. Seydi proved 3.1 using only JCN. The sketch of Seydi's proof is the following: reduce 3.1 to the case when A and B are complete, A is a domain and it suffices to show that for any $Q \in \operatorname{Spec} B$ with $Q \cap A = 0$, the ring B_Q is regular. By the structure theorem of Cohen and EGA, 0_{IV}, §19 there exist a formally smooth local morphism $v : R \to S$ of complete regular local rings and $P \in \operatorname{Spec} R$ s.t. $A = R/P$ and $u = v \otimes_R A$. The field A_P is a regular ring; the condition c) from *JCN* gives $f_1, \ldots, f_m \in P$, and $D_1, \ldots, D_m \in \operatorname{Der}_{k'}(R, R)$ s.t. $\det(D_i f_j) \notin P$. Fortunately D_i can be extended to $D'_i \in \operatorname{Der}_{k'}(S, S)$. Using these derivations $D'_1, \ldots, D'_m$ and *JCN*, it follows that the fibre of v in P, which is the fibre of u in zero, is regular. (For details, see e.g. [BR], 9.19).

This idea of Seydi (announced in 1973, but explained in 1980) increased the interest for *JCN* and 2.3, by the possibility to apply it to solve another important problems in local algebra (see e.g. consequences 2—5)

3.2. The localization theorem for formal smoothness (i.e. of geometric regularity) has as consequences localization theorems for other properties: reduced, normal, (R_n) and (S_{n+1}), (R_n) etc. (see [Ni], [Ma], [BI], [I] etc.).

4.0. Let A be a noetherian ring and I an ideal of A. The „lifting theorem" asserts that if A/I is a **P**-ring, and A is I-adically complete then A itself is a **P**-ring (EGA, IV, 7, 4, 8). The answer for the „lifting theorem", even when **P** = regular, was open for a long time.

4.1. An important application of th. 3 is the lifting theorem for reg-rings in semilocal case ([Ro 1]).

4.2. The theorem from 4.1 was used to show that if A is a noetherian local ring and $A[[X]]$, with X an indeterminate, has the approximation property of Artin, then A is an excellent ring ([Ro 3]).

4.3. In the non-semilocal case, the „lifting theorem" is not true for reg-rings, as a counterexample of Nishimura shows ([Ni]). Under natural hypotheses on the regular locus and catenarity, the lifting theorem

holds for noetherian reg-rings of finite dimension which contain a field of characteristic zero (Ro 2); the proof uses 4.1 and the resolution of singularity of Hironaka.

Remark that a counterexample to „lifting theorem" for excellent rings would give a counterexample to resolution of singularities.

The lifting theorem holds also for noetherian nor-ring of finite dimension under natural hypotheses on normal locus and catenarity ([BRo]).

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