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THE METHOD OF THE CANONICAL ELEMENT

BY

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1. **Introduction.** This paper is strongly connected to the informative lecture given by the author under the title of „Homological methods in Commutative Algebra" during the National Conference of Algebra, Iaşi, 1984. The lecture was divided into 3 distinct parts, namely :

A. Introduction (including the precise enounces of the homological conjectures, their history and their logical connections, as known at this moment).

B. The algebraic direction (the reduction to positive characteristic, presented after the remarkable paper of L. van den Dries: „Reducing to prime characteristic", Comm. of the Math. Institute, Rijksuniversiteit Utrecht, No. 16 (1983)).

C. The homological direction (Hochster's method of the canonical element — see the bibliography).

Since the parts A and B were generously treated during the lecture, we restrict ourselves here to part C.

2. **The method of the canonical element.** Let $(R, \underline{m}, k)$ be a local (noetherian) ring and P an R-module. If $L_* : \ldots \to L_m \to L_{m-1} \ldots \to L_1 \to L_0 \to \to P \to 0$ is a projective ($\Leftrightarrow$free) resolution of P, for each $m \geqslant 0$ one defines: $\mathrm{syz}^m(L_*) = \mathrm{coker}\,(L_{m+1} \to L_m) = \ker\,(L_{m-1} \to L_{m-2})$ $(L_{-1} = P,\ L_2 = 0$ etc. etc.), such that there is an exact sequence : $L_* |_m : 0 \to \mathrm{syz}^m(L_*) \to L_{m-1} \to L_{m-2} \ldots \to \to L_0 \to P \to 0$.

If I is an ideal of R and $P = R/I$, the truncated sequence $L^*/_m$ determines an element $\varepsilon(L_* |_m) \in \mathrm{Ext}^m_R(R/I, \mathrm{syz}^m(L_*))$ and therefore an element $\eta(L_* |_m) \in H^m_I(\mathrm{syz}^m(L_*))$.

We reffer to $\varepsilon(L_* |_m)$ and $\eta(L_* |_m)$ as *"the canonical elements"* of order m, associated to the resolution L^* of R/I. This name is justified by the following

1. **Proposition.** *Let L_*, L'_* be two projective resolutions of R/I, $m \geqslant 0$ an integer. Then any map of complexes $f : L_* \to L'_*$, lifting* id: $R/I \to R/I$, *induces a map f:* $\mathrm{syz}^m(L_*) \to \mathrm{syz}^m(L'_*)$, *and hence maps* $f_1 : \mathrm{Ext}^m_R(R/I, \mathrm{syz}^m(L_*)) \to \to \mathrm{Ext}^m_R(R/I, \mathrm{syz}^m(L'_*))$, $f_2 : H^m_I(\mathrm{syz}^m(L_*)) \to H^m_I(\mathrm{syz}^m(L'_*))$.

With these notations : $f_1(\varepsilon(L_* |_m)) = \varepsilon(L'_* |_m)$ and $f_2(\eta(L_* |_m)) = \eta(L'_* |_m)$.

From here one sees that the canonical elements do not depend on the resolutions L_* of R/I and on the liftings to maps of complexes of id: $R/I \to R/I$.

Therefore, it is correct to denote these canonical elements by ε^m_I, resp. η^m_I (provided $I \neq R$). When $m = \mathrm{ht}\, I$, the upper script is ommited. For $I = \underline{m}$ (the maximal ideal of R), one uses the notation : ε_R, η_R respectively.

The advantage of the canonical elements is that they behave functorially in a characteristic-free manner.

2. **Proposition.** *Let $g : R \to S$ be a homomorphism of noetherian rings, $I \subset R$ and $J \subset S$ ideals such that $g(I) \subset J$, $\sqrt{J} = \sqrt{(IS)}$.*

Let f : $\mathrm{syz}^m(R/I) \to \mathrm{syz}^m\ (S/J)$ $(m \geqslant 0)$ *be any lifting of $R/I \to S/J$ and $\tilde{f} : H^m_I(\mathrm{syz}^m(R/I)) \to H^m_I(\mathrm{syz}^m(S/J))$ the induced map. Then : $\tilde{f}(\eta^m_I) = \eta^m_J$.*

When $I = (x_1 \ldots x_n)$ is generated by an R-sequence, the canonical element η_I may be computed as follows: one takes the Koszul complex $K_*(\underline{x})$ as a resolution of R/I and identifies $K_n(\underline{x}) = \mathrm{syz}^n$. $(R/\underline{x})$ to R and $\mathrm{Ext}^n_R(R/\underline{x}, R)$ to $R/\underline{x}$; then $\varepsilon_I = \varepsilon_{\underline{x}}$ is I (mod $\underline{x}) \in R/\underline{x}$, such that $\eta_{\underline{x}} = \eta_I$ is the image of I (mod $\underline{x}$) in $H^n_{\underline{x}}(R) = \varinjlim_i R/(x_1^i, x_2^i, \ldots, x_n^i)$.

The importance of the notion of *„canonical element"* is underlined by the following set of results, yielding a new homological conjecture (Hochster).

I. *Let $x = (x_1, \ldots, x_n)$ be a sequence of elements in the ring R. If there is an R-module M, such that x is M-regular, then $\eta_{\underline{x}R} \neq 0$.*

II. *Let R be regular and $R \to S$ a finite algebra. If $\eta_R \neq 0$, then R is a direct summand of S.*

Conjecture (The canonical element conjecture). *For every local ring R, $\eta_R \neq 0$.*

From I and II one sees that this new conjecture is situated between the one of the big CM-modules and the direct summand conjecture. In fact Hochster proved that it is *equivalent* to the direct summand conjecture. Thus, the canonical element conjecture, although weaker than the „big-CM conjecture", is still strong enough as to imply the rest of the homological conjectures, as well as to supply a new proof to the syzygy problem of Evans and Griffith. What Hochster proved (between other things) in connection with this new conjecture, is enounced in the following :

Theorem. (i) *Let R be a local ring of positive characteristic. Then $\eta_R \neq 0$.*

(ii) *Let R be a local ring such that no prime $p > 0$ is part of a system of parameters for R. Then $\eta_R \neq 0$.*

A very useful way of dealing with the canonical elements is to put their definition under a down-to-earth equivalent form, which is the following.

Let $(R, \underline{m}, k)$ be as above. One says that „R has the property CE" if :

(i) *for every projective resolution of the residue field $L_* : \ldots L_m \to L_{m-1} \ldots \ldots L_1 \to L_0 \to k \to 0$;* (ii) *for every system of parameters $x = (x_1, \ldots, x_n)$ of R* $(n = \dim\ R)$; (iii) *for every lifting $f_* : K_*(x) \to L_*$ to a map of complexes of* can : $R/x \to k$, *where $K_*(x)$ is the Koszul complex over x, the following holds: the n-th component $f_n : R = K_n(x) \to L_n$ of f_* is non-zero.*

Proposition. *The local ring R has the property CE iff $\eta_R \neq 0$.*

The advantage of this equivalent form to the non-vanishing of the canonical element is that it supplies entirely new homological proofs to some of the old homological conjectures (such as, for instance, to the intersection conjecture of Peskine-Szpiro), starting from the new conjecture formulated above and not directly from the „big CM-modules" conjecture. It also makes clear the independence of the system of parameters of most of the results connected to the direct summand conjecture.

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ON THE STRUCTURE OF FORMALLY SMOOTH MORPHISMS

BY

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For proving results about local formally smooth morphisms $u : A \to B$, the case dim A = dim B is sometimes easier. If dim $A <$ dim B, it arises the problem to find a noetherian local A-algebra $\tilde{A}$ and a local A-morphism $\tilde{u} : \tilde{A} \to B$ such that : 1) dim $\tilde{A}$ = dim B ; 2) $\tilde{u}$ is formally smooth ; 3) $\tilde{A}$ has a „good" structure, for example is a localization of a polynomial A-algebra.

The main result of this paper is the following structure theorem (for proofs see [NP]) :

Theorem 1. *Let $u : A \to B$ be a local formally smooth morphism of noetherian local rings and $k \subset K$ the residue field extension induced by u. Suppose that $p : =$ char $k > 0$ and that one of the following conditions holds : i) K has a separate p-basis over k (i.e. a p-basis $x = (x_i)_{i \in I}$ such that K is separable over $k(x)$) ; ii) B is a complete ring.*

Then there exist a noetherian local A-algebra $\tilde{A}$ and a local A-morphism $\tilde{u} : \tilde{A} \to B$, such that : 1) dim $\tilde{A}$ = dim B ; 2) *$\tilde{u}$ is formally smooth ;* 3) *$\tilde{A}$ is a filtered inductive limit of localizations of some polynomial A-algebras (each in a finite number of indeterminates).*

Moreover in the case i), we can choose $\tilde{A}$ to be simply a localisation of a polynomial A-algebra.

If the residue field extension $k \subset K$ is separable, the construction of $\tilde{A}$ and $\tilde{u}$ is well known and not difficult. For example, if $A = k$ is a field and B is a local formally smooth one dimensional k-algebra, i.e. a discrete valuation ring, then if $\sigma : k[T] \to B$ is given by $T \to t$, t being a local parameter of B, we can take $\tilde{A} : = k[T]_{\sigma^{-1}(tB)}$.

As an application of the above result, we get an easier proof of Th.(7.3) from [P], concerning general Néron desingularization.

The proof of the Theorem 1 needs some structure theorems about field extensions of positive characteristic.

First note that, in the hypotheses of Th. 1, rank $\Gamma_{K/k} < \infty$, where $\Gamma_{K/k}$ denotes the conormal module of the extension $k \subset K$ (see [M]).

Let $k \subset K$ be a field extension of positive characteristic p. A p-base $x = (x_i)_{i \in I}$ of K over k is called *separate* if K is separable over $k(x)$. We shall say that $k \subset K$ is a *separate extension* if it has a separate p-basis.

Theorem 2. *In the above hypotheses, suppose additionally that rank $\Gamma_{K/k} < \infty$. Then the following statements are equivalent :*

i) *$k \subset K$ is a separate extension,*

i) *every p-base of K over k is separate,*