The advantage of this equivalent form to the non-vanishing of the canonical element is that it supplies entirely new homological proofs to some of the old homological conjectures (such as, for instance, to the intersection conjecture of Peskine-Szpiro), starting from the new conjecture formulated above and not directly from the „big $CM$-modules" conjecture. It also makes clear the independence of the system of parameters of most of the results connected to the direct summand conjecture.

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*Institute of Mathematics*
*University of Bucharest*

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# ON THE STRUCTURE OF FORMALLY SMOOTH MORPHISMS

BY

VASILE NICA and DORIN POPESCU

For proving results about local formally smooth morphisms $u : A \to B$, the case dim $A$ = dim $B$ is sometimes easier. If dim $A <$ dim $B$, it arises the problem to find a noetherian local $A$-algebra $\tilde{A}$ and a local $A$-morphism $\tilde{u} : \tilde{A} \to B$ such that : 1) dim $\tilde{A}$ = dim $B$ ; 2) $\tilde{u}$ is formally smooth ; 3) $\tilde{A}$ has a „good" structure, for example is a localization of a polynomial $A$-algebra.

The main result of this paper is the following structure theorem (for proofs see [NP]) :

**Theorem 1.** *Let $u : A \to B$ be a local formally smooth morphism of noetherian local rings and $k \subset K$ the residue field extension induced by $u$. Suppose that $p := \text{char}\ k > 0$ and that one of the following conditions holds :* i) *$K$ has a separate $p$-basis over $k$ (i.e. a $p$-basis $x = (x_i)_{i \in I}$ such that $K$ is separable over $k(x)$)* ; ii) *$B$ is a complete ring.*

*Then there exist a noetherian local $A$-algebra $\tilde{A}$ and a local $A$-morphism $\tilde{u} : \tilde{A} \to B$, such that :* 1) dim $\tilde{A}$ = dim $B$ ; 2) *$\tilde{u}$ is formally smooth* ; 3) *$\tilde{A}$ is a filtered inductive limit of localizations of some polynomial $A$-algebras (each in a finite number of indeterminates).*

Moreover in the case i), we can choose $\tilde{A}$ to be simply a localisation of a polynomial $A$-algebra.

If the residue field extension $k \subset K$ is separable, the construction of $\tilde{A}$ and $\tilde{u}$ is well known and not difficult. For example, if $A = k$ is a field and $B$ is a local formally smooth one dimensional $k$-algebra, i.e. a discrete valuation ring, then if $\sigma : k[T] \to B$ is given by $T \to t$, $t$ being a local parameter of $B$, we can take $\tilde{A} := k[T]_{\sigma^{-1}(m_B)}$.

As an application of the above result, we get an easier proof of Th.(7.3) from [P], concerning general Néron desingularization.

The proof of the Theorem 1 needs some structure theorems about field extensions of positive characteristic.

First note that, in the hypotheses of Th. 1, rank $\Gamma_{K/k} < \infty$, where $\Gamma_{K/k}$ denotes the conormal module of the extension $k \subset K$ (see [M]).

Let $k \subset K$ be a field extension of positive characteristic $p$. A $p$-base $x = (x_i)_{i \in I}$ of $K$ over $k$ is called *separate* if $K$ is separable over $k(x)$. We shall say that $k \subset K$ is a *separate extension* if it has a separate $p$-basis.

**Theorem 2.** *In the above hypotheses, suppose additionally that rank $\Gamma_{K/k} < \infty$. Then the following statements are equivalent :*

i) *$k \subset K$ is a separate extension,*

i) *every $p$-base of $K$ over $k$ is separate,*

iii) *there exists a finite type field extension $F$ of $k$ contained in $K$ such that $K$ is separable over $F$,*

iv) *there exists a finite field extension of $k$, $E \subset k^{p^{-\infty}}$ such that $E \subset E(K)$ is separable.*

There exist field extensions without separate $p$-bases. For example, let $k_0$ be a perfect field, $T$ an indeterminate, $k = k_0(T)$ and $K = k^{p^{-\infty}}$.

**Theorem 3.** ([*P*], *Th.* (6.3)). *Let $k \subset K$ be a field exension of characteristic $p > 0$ with $t := \text{rank}\, \Gamma_{K/k}$. Suppose that for every $n \in \mathrm{N}$ rank $\Gamma_{K_n/k} = t$ where $K_n = k(K^{p^n})$. Then there exists an ascending chain of subfields of $K$, all containing $k$, $F_1 \subset F_2 \subset \ldots \subset F_n \subset \ldots$, such that for every positive integer $n$: 1) $F_n$ is a finite type field extension of $k$ and rank $\Gamma_{F_n/k} = t$; 2) every $p$-basis of $F_n$ over $k$ generates $F_n$ over $k$; 3) $F_n \subset k(F^p_{n+1})$.*

*Moreover if $F = \bigcup_{n \in N} F_n$ then: 4) $F = k(F^p)$ and rank $\Gamma_{F/k} = t$; 5) $K$ is a separable extension of $F$.*

Note that the conditions rank $\Gamma_{K_n/k}$ = constant $n \geqslant 0$ holds when $K = k(K^p)$.

Using the proof of the Theorem 1, we get as a corollary the following:

**Theorem** ([*EGA*], *Th.* 22.2.6). *Let $A$ be a noetherian local ring, $k$ its residue field and $K/k$ a field extension such that $t :=$ rank $\Gamma_{K/k}$. Then there exists a formally smooth noetherian complete local $A$-algebra $B$ such that: 1) the residue field of $B$ is $K$; 2)* dim $B$ = dim $A + t$.

This application was suggested to us by prof. N. Radu to whom we owe thanks.

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# ALGEBRAICALLY INDEPENDENT HAHN SERIES

BY
DORU ŞTEFĂNESCU

Let $k$ be a commutative field of characteristic $p \neq 0$. We considered in [4] the field $k(T^{Q,p}))$ of the restricted power series over $k$. This field is a subfield of the field of the general power series considered by H. Hahn ([1]) and, if $k$ is algebraically closed, it contains the algebraic closure of the field $k(T))$ of the meromorphic power series. We proved in [4] that the extension $k(T)) \subset k(T^{Q,p}))$ is not algebraic. Recently we proved that the transcendence degree of the former extension is infinite. (We recall that $k((T^{Q,p})) = \{f = \sum_{i \in S(f)} a_i T^i;\ S(f)$ is well ordered and there is $m = m(f) \in \mathbf{N}^*$ such that for every $i \in S(f)$ there are $s_i, n_i \in \mathbf{Z}$ with $i = s_i/(mp^{n_i})\}$.

1. **Proposition.** *Let $k$ be a commutative field of positive characteristic $p$. Let $v_i : \mathbf{N}^* \to \mathbf{N}^*$ be a family of functions such that*: a) $sv_i(s) < v_i(s+1)$ *for every* $s \in \mathbf{N}^*$, b) $v_i(s) < v_{i+1}(s)$ *for every* $i \in \mathbf{N}^*$, c) $v_1(m+1) > (m/n)v_n(m)$ *for every* $m, n \in \mathbf{N}^*$.

$$\text{Let } F_q = \sum_{j=1}^{\infty} a_{jq} T^{p^{\frac{-1}{v_q(j)}}} \in k((T^{Q,p})), \text{ where } a_{jq} \in k \setminus \{0\}.$$

*Then $F_1, \ldots, F_n, \ldots$ are algebraically independent over $k(T))$.*

*Proof.* Letus suppose that there is $G(Y_1, \ldots, Y_n) = \sum_{0 \leqslant i_1, \ldots, i_n \leqslant s} a_{i_1 \ldots i_n} Y_1^{i_1} \ldots Y_n^{i_n} \in k((T))[Y_1, \ldots, Y_n] \setminus \{0\}$ such that

$$G(F_1, \ldots, F_n) = 0. \tag{1}$$

We may suppose that $a_{i_1 \ldots i_n} = a_{i_1 \ldots i_n}(T) = a'_{i_1 \ldots i_n} + Tb_{i_1 \ldots i_n}$, where $a'_{i_1 \ldots i_n} \in k$ (not all $a'_{i_1 \ldots i_n}$ zero), $b_{i_1 \ldots i_n} \in k[[T]]$.

If $(i_1, \ldots, i_n)$ is such that $a'_{i_1 \ldots i_n} \in 0$ then, for $m$ sufficiently large, the coefficient of $T^{p^{\sum_{h=1}^{n} ((-i_h)/v_h(m))}}$ in the left side of (1) is not zero, a contradiction. It follows that $F_1, \ldots, F_n$ are algebraically independent over $k(T))$ for every $n$.

The proof of 1 suggested a construction of a class of algebraically independent formal power series over the field of the rational functions in one indeterminate.

Let $k$ be a commutative field of arbitrary characteristic, $k(T)$ the field of the rational functions in one indeterminate over $k$, $k[[T]]$ the ring of the formal power series in one indeterminate over $k$ and $k((T))$ its quo-