iii) *there exists a finite type field extension F of k contained in K such that K is separable over F,*

iv) *there exists a finite field extension of k, $E \subset k^{p^{-\infty}}$ such that $E \subset E(K)$ is separable.*

There exist field extensions without separate p-bases. For example, let k_0 be a perfect field, T an indeterminate, $k=k_0(T)$ and $K=k^{p^{-\infty}}$.

Theorem 3. ([*P*], *Th.* (6.3)). *Let $k \subset K$ be a field exension of characteristic $p>0$ with $t := \operatorname{rank} \Gamma_{K/k}$. Suppose that for every $n \in N$ rank $\Gamma_{K_n/k}=t$ where $K_n=k(K^{p^n})$. Then there exists an ascending chain of subfields of K, all containing k, $F_1 \subset F_2 \subset \ldots \subset F_n \subset \ldots$, such that for every positive integer n: 1) F_n is a finite type field extension of k and rank $\Gamma_{F_n/k}=t$; 2) every p-basis of F_n over k generates F_n over k; 3) $F_n \subset k(F_{n+1}^p)$.*

Moreover if $F=\bigcup_{n \in N} F_n$ then: 4) $F=k(F^p)$ and rank $\Gamma_{F/k}=t$; 5) K is a separable extension of F.

Note that the conditions rank $\Gamma_{K_n/k}$ = constant $n \geqslant 0$ holds when $K=k(K^p)$.

Using the proof of the Theorem 1, we get as a corollary the following:

Theorem ([*EGA*], *Th.* 22.2.6). *Let A be a noetherian local ring, k its residue field and K/k a field extension such that $t := \operatorname{rank} \Gamma_{K/k}$. Then there exists a formally smooth noetherian complete local A-algebra B such that:* 1) *the residue field of B is K*; 2) $\dim B = \dim A + t$.

This application was suggested to us by prof. N. Radu to whom we owe thanks.

REFERENCES

[EGA] Grothendieck, A. Dieudonné, J. — *Eléments de Géométrie Algébrique*, IV, Part. 1, Publ. Math. IHES 20 (1964).

[M] Matsumura, H. — *Commutative algebra*, Benjamin, New York, 1980.

[NP] Nica V., Popescu, D. — *A structure theorem of formally smooth morphisms in positive characteristic*, INCREST, Preprint Series in Mathematics, no. 34 (1984).

[P] Popescu, D. — *General Néron desingularization and Approximations*, II (revised form), to appear.

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXI, s. I a, 1985 (supliment)

ALGEBRAICALLY INDEPENDENT HAHN SERIES

BY

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Let k be a commutative field of characteristic $p \neq 0$. We considered in [4] the field $k(T^{Q,p}))$ of the restricted power series over k. This field is a subfield of the field of the general power series considered by H. Hahn ([1]) and, if k is algebraically closed, it contains the algebraic closure of the field $k(T))$ of the meromorphic power series. We proved in [4] that the extension $k(T)) \subset k(T^{Q,p}))$ is not algebraic. Recently we proved that the transcendence degree of the former extension is infinite. (We recall that $k((T^{Q,p})) = \{f = \sum_{i \in S(f)} a_i T^i$; $S(f)$ is well ordered and there is $m=m(f) \in \mathbf{N}^*$ such that for every $i \in S(f)$ there are $s_i, n_i \in \mathbf{Z}$ with $i=s_i/(mp^{n_i})\}$.

1. **Proposition.** *Let k be a commutative field of positive characteristic p. Let $v_i : \mathbf{N}^* \to \mathbf{N}^*$ be a family of functions such that:* a) $sv_i(s)<v_i(s+1)$ *for every $s \in \mathbf{N}^*$,* b) $v_i(s)<v_{i+1}(s)$ *for every $i \in \mathbf{N}^*$,* c) $v_1(m+1)>(m/n)v_n(m)$ *for every $m, n \in \mathbf{N}^*$.*

Let $F_q = \sum_{j=1}^{\infty} a_{jq} T^{p^{\frac{-1}{v_q(j)}}} \in k((T^{Q,p}))$, *where* $a_{jq} \in k \setminus \{0\}$.

Then $F_1, \ldots, F_n, \ldots$ are algebraically independent over $k(T))$.

Proof. Letus suppose that there is $G(Y_1, \ldots, Y_n) = \sum_{0 \leqslant i_1, \ldots, i_n \leqslant s} a_{i_1 \ldots i_n} Y_1^{i_1} \ldots Y_n^{i_n} \in k((T))[Y_1, \ldots, Y_n] \setminus \{0\}$ such that

$$G(F_1, \ldots, F_n)=0. \tag{1}$$

We may suppose that $a_{i_1 \ldots i_n}=a_{i_1 \ldots i_n}(T)=a'_{i_1 \ldots i_n}+Tb_{i_1 \ldots i_n}$, where $a'_{i_1 \ldots i_n} \in k$ (not all $a'_{i_1 \ldots i_n}$ zero), $b_{i_1 \ldots i_n} \in k[[T]]$.

If $(i_1, \ldots, i_n)$ is such that $a'_{i_1 \ldots i_n} \neq 0$ then, for m sufficiently large, the coefficient of $T^{\sum_{k=1}^{n} ((-i_k)/p^{v_k(m)})}$ in the left side of (1) is not zero, a contradiction. It follows that $F_1, \ldots, F_n$ are algebraically independent over $k(T))$ for every n.

The proof of 1 suggested a construction of a class of algebraically independent formal power series over the field of the rational functions in one indeterminate.

Let k be a commutative field of arbitrary characteristic, $k(T)$ the field of the rational functions in one indeterminate over k, $k[[T]]$ the ring of the formal power series in one indeterminate over k and $k((T))$ its quotient field. It is well known that the transcendence degree of the extension $k(T)\subset k((T))$ is infinite, but there are known few examples of series algebraically independent over $k(T)$. We mention the results of Rothaus ([2]) obtained in the case k is a field of positive characteristic of infinite transcendence degree over its prime field. We describe below a general construction.

2. **Proposition.** *Let $v_i : \mathbf{N}^* \to \mathbf{N}^*$ be a family of functions such that:*

a) $v_i(s) < v_i(s+1)$ *for every* $s \in \mathbf{N}^*$,

b) $v_i(s) < v_{i+1}(s)$ *for every* $i \in \mathbf{N}^*$,

c) *If* $m \in \mathbf{N}^*$, *then for every m sufficiently large we have* $v_1(m+1) > (m/n)v_n(m)$.

d) *If* $n, t \in \mathbf{N}^*$, *then for every m sufficiently large there is a unique n-uple* $(j_1, \ldots, j_n)$ *such that* $j_1v_1(m) + \ldots + j_nv_n(m) = \max\{i_1v_1(m) + \ldots + i_nv_n(m) : 0 \leqslant i_1, \ldots, i_n \leqslant t\}$.

Let k be a commutative field of arbitrary characteristic and let $f_q = \sum_{i=1}^{\infty} a_{iq}T^{v_q(i)} \in k[[T]]$, *where* $a_{iq} \in k \setminus \{0\}$. *Then* $f_1, f_2, \ldots, f_n, \ldots$ *are algebraically independent over* $k(T)$.

Remark. There are functions $v_i : \mathbf{N}^* \to \mathbf{N}^*$ that satisfy the conditions a, b, c, d of 2. For instance: $v_1(1) = a \in \mathbf{N}^*$, $v_1(m+1) = 1 + m^m v_1(m)$, $v_n(m) = nm^n v_1(m)$ for every $n \geqslant 2$.

Idea of the proof of 2. We suppose that there is $F(Y_1, \ldots, Y_n) \in k(T)[Y_1, \ldots, Y_n] \setminus \{0\}$ such that $F(f_1, \ldots, f_n) = 0$.

Let $m \in \mathbf{N}$ (m sufficiently large). Let $g_q = a_{1q}T^{v_q(1)} + \ldots + a_{mq}T^{v_q(m)}$, $h_q = f_q - g_q$. We remark that $F(f_1, \ldots, f_n) = F(g_1, \ldots, g_n) + S = 0$, where $S \in k[[T]]$. We may choose m such that $F(g_1, \ldots, g_n) \neq 0$ and $\deg_T F(g_1, \ldots, g_n) < \operatorname{ord}_T S$.

3. **Corollary.** (*K.F.* Schmidt [3]). *Let k be a commutative field and* $f = a_{m_1}T^{m_1} + \ldots + a_{m_n}T^{m_n} + \ldots \in k[[T]]$. *If* $m_1 < m_2 < \ldots < m_n < \ldots$ *and* $\lim\sup (m_{n+1}/m_n) = \infty$, *then f is transcendental over* $k(T)$.

Let k be a commutative field of arbitrary characteristic. We considered in [5] a criterion of analytical independence relative to some formal power series rings over k. This criterion can be obtained as a corollary of the following result.

4. **Proposition.** *Let k be a commutative field,* $X_1, \ldots, X_{t+m}$ *indeterminates over k,* $u_1, \ldots, u_n \in k[[X_1, \ldots, X_{t-1}]]$. *Let us consider* $f_1 = X_tu_1, \ldots, f_n = X_tu_n$; $f_{n+1} = X_{t+1}u_1, \ldots, f_{2n} = X_{t+1}u_n$; $\ldots$; $f_{(m-1)n+1} = X_{t+m}u_1, \ldots, f_{mn} = X_{t+m}u_n \in (X_1, \ldots, X_{t+m})k[[X_1, \ldots, X_{t+m}]]$.

If $u_1, \ldots, u_n$ *are algebraically independent over* $k(X_1, \ldots, X_{t-1})$, *then* $f_1, \ldots, f_{mn}$ *are analytically independent over* $S = k[[X_1X_t, \ldots, X_{t-1}X_t, X_t, X_1X_{t+1}, \ldots, X_{t-1}X_{t+1}, X_{t+1}, \ldots, X_1X_{t+m}, \ldots, X_{t-1}X_{t+m}, X_{t+m}]]$.

For proving 4, we must consider the map $s : S[[Y_1, \ldots, Y_{mn}]] \to k[[X_1, \ldots, X_{t+m}]]$, defined by $s(G) = G(f_1, \ldots, f_{mn})$. The analytical independence of $f_1, \ldots, f_{mn}$ over S is equivalent to the fact that the morphism s is injective. Let $F \in \ker s$. Then $F(f_1, \ldots, f_{mn}) = 0$.

We write F in a suitable way as a sum of formal power series with coefficients homogeneous polynomials. If $F \neq 0$ we arrive to a nontrivial algebraic relation between $u_1, \ldots, u_n$ over $k(X_1, \ldots, X_{t-1})$. Therefore $F = 0$ and the analytical independence of $f_1, \ldots, f_{mn}$ over S is proved.

REFERENCES

1. Hahn H. — *Über die nichtarchimedischen Grossensysteme.* S.—B.—Kön. Akad. Wiss. Berlin Kl. Math. *116* (IIa), s. 601—655 (1907).
2. Rotthaus C. — *Nichtausgezeichmete, universell japanische Ringe.* Math. Z. *152*, s. 107—125 (1977).
3. Schmidt F. K. — *Mehrfach perfekte Körper.* Math. Ann. *108*, s. 1—25 (1933).
4. Ştefănescu D. — *A method to obtain algebraic elements over $k((T))$ in positive characteristic.* Bull. Math. Soc. Sc. Math. R. S. Roumanie *26*(74), p. 77—91 (1982).
5. Ştefănescu D. — *Analytically independent elements.* Abstr. Amer. Math. Soc. *4*, p. 198 (1983).

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