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ON THE STANLEY-REISNER RING OF A SUSPENSION

BY

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In this note we consider the Stanley-Reisner ring of the suspension of a given simplicial complex.

1. Let Δ be a finite (abstract) simplicial complex on the vertex set $V=\{x_1, \ldots, x_n\}$. The Stanley-Reisner ring of Δ over a field k is $k[\Delta]= =k[x_1, \ldots, x_n]/I_\Delta$, where $I_\Delta=(x_{i_1}, \ldots, x_{i_n} / i_1 < \ldots < i_n, \{x_{i_1}, \ldots, x_{i_n}\} \overline{\in} \Delta)$.

The ring $k[\Delta]$ was studied especially for triangulations Δ of spheres related to so called Upper Bound Conjecture (UBC) [1, p. 68—69].

Since the Krull dimension of $k[\Delta]$ is the maximal cardinality of an algebraic independent set of vertices of Δ, it results the relation :

$$\dim k[\Delta]=\dim \Delta+1. \tag{1}$$

Consider $k[x_1, \ldots, x_n]$ as a N^n-graded k-algebra and let $k[\Delta]$ with the fine grading defined by $\deg x_i=(0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0) \in Z^n$. Denote by $H^i(k[\Delta])$ the local cohomology modules of $k[\Delta]$ defined with respect to the irrelevant ideal of $k[x_1, \ldots, x_n]$. Hochster's Theorem [1, p. 70] states :

$$F(H^i(k[\Delta]), \lambda)= \sum_{F \in \Delta} \dim_k \tilde{H}_{i-|F|-1}(\mathrm{Lk}_\Delta F ; k) \prod_{x_i \in F} \lambda_i^{-1}/(1-\lambda_i^{-1}). \tag{2}$$

The left part in (2) is the Hilbert series of the Z^n-graded $k[x_1, \ldots, x_n]$-module $H^i(k[\Delta])$. More precisely, for $\alpha \in Z^n$ let be $\mathrm{supp}\, \alpha=\{x_i \in V / \alpha_i \neq 0\}$. Then, $H^i(k[\Delta])_\alpha=0$ if $\alpha>0$ or if $F=\mathrm{supp}\, \alpha \overline{\in} \Delta$ and $H^i(k[\Delta])_\alpha \cong \tilde{H}_{i-|F|-1}(\mathrm{Lk}_\Delta F ; k)$ if $\alpha \leqslant 0$ and $F=\mathrm{supp}\, \alpha \in \Delta$, where $\mathrm{Lk}_\Delta F$ is the link of F, i.e. the subcomplex of Δ, $\{G \in \Delta / G \cup F \in \Delta, G \cap F=\varnothing\}$ and $\tilde{H}*(. ; k)$ denotes the reduced simplicial homology with coefficients in k. We obtain :

Lemma 1. *$H^i(k[\Delta]) \neq 0$ iff there exists $F \in \Delta$ $\tilde{H}_{i-|F|-1}(\mathrm{Lk}_\Delta F ; k) \neq 0$.*

Now we recall from [1], [2] that Δ is Cohen-Macaulay (CM) over k if $k[\Delta]$ is Cohen-Macaulay as a graded $k[x_1, \ldots, x_n]$-module with respect to the irrelevant ideal. Δ is a double-Cohen-Macaulay simplicial complex (2—CM) over k if Δ is CM over k, and for every vertex x of Δ the subcomplex $\Delta \setminus x=\{F \in \Delta / x \overline{\in} F\}$ is CM over k of the same dimension as Δ. A simplicial complex Δ is almost Cohen-Macauley (ACM) over k if for every $F \in \Delta$, $F \neq \varnothing$, $\mathrm{Lk}_\Delta F$ is CM. Δ is Gorenstein simplicial complex over k if $k[\Delta]$ is a Gorenstein ring. Finally, Δ is Buchsbaum over k if $k[\Delta]$ is Buchsbaum [1, p. 84].

2. Let Δ be a simplicial complex and $S\Delta$ its suspension, i.e. $S\Delta$ is the join $\Delta * S^0$, Let $S^0=\{a, b\}$. If $V=\{x_1, \ldots, x_n\}$ is the vertex set of Δ, then

$S\Delta$ has the vertices $\{x_1, \ldots, x_n, a, b\}$ and the simplexes of $S\Delta$ are those of Δ or S^0 or have the form $\{x_{i_0}, \ldots, x_{i_r}, a\}$, $\{x_{i_0}, \ldots, x_{i_r}, b\}$, with $\{x_{i_0}, \ldots, x_{i_r}\} \in \Delta$.

Theorem 1.
a) $\dim k[S\Delta]=\dim k[\Delta]+1$; b) $H^i(k[S\Delta]) \neq 0 \Leftrightarrow H^{i-1}(k[\Delta]) \neq 0$; c) depth $k[S\Delta]=$ depth $k[\Delta]+1$.

Proof. a) $\dim k[S\Delta]=\dim S\Delta+1=(\dim \Delta+1)+1=\dim k[\Delta]+1$.

b) Suppose that $H^i(k[S\Delta]) \neq 0$. Then, by Lemma 1 there exists $F \in S\Delta$ such that $\tilde{H}_{i-|F|-1}(\mathrm{Lk}_{S\Delta} F ; k) \neq 0$. We distinguish the cases : $F \in \Delta$ and $F \overline{\in} \Delta$. If $F \in \Delta$, then $\mathrm{Lk}_{S\Delta} F=S(\mathrm{Lk}_\Delta F)$. In fact, if $G \in \mathrm{Lk}_{S\Delta} F$, then $F \cup G \in S\Delta$ and $F \cap G=\varnothing$. If $G \in \Delta$ it is obvious that $G \in \mathrm{Lk}_\Delta F$. If $G \overline{\in} \Delta$, then $G=(G_1, a)$ or $G=(G_1, b)$ with $G_1 \in \Delta$. It follows $G_1 \cap F=\varnothing$ and $F \cup G_1 \in \Delta$, that is $G_1 \in \mathrm{Lk}_\Delta F$ and therefore $G \in S(\mathrm{Lk}_\Delta F)$. Thus we have obtained the inclusion $\mathrm{Lk}_{S\Delta} F \subseteq S(\mathrm{Lk}_\Delta F)$. Conversely, if $G \in S(\mathrm{Lk}_\Delta F)$ and if $G \in \mathrm{Lk}_\Delta F$ then $G \in \mathrm{Lk}_{S\Delta} F$. If $G=(G_1, a)$ (or $G=(G_1, b)$), with $G_1 \in \mathrm{Lk}_\Delta F$, then $G_1 \cap F= =\varnothing$, $G_1 \cup F \in \Delta$ and therefore $G \cap F=\varnothing$ and $G \cup F=(G_1 \cup F, a) \in S\Delta$, which prove the inclusion $S(\mathrm{Lk}_\Delta F) \subseteq \mathrm{Lk}_{S\Delta} F$. By the suspension isomorphism [3, Th. 4.4, 10, p. 130], it follows $0 \neq \tilde{H}_{i-|F|-1}(\mathrm{Lk}_{S\Delta} F ; k)=\tilde{H}_{i-|F|-1}(S(\mathrm{Lk}_\Delta F) ; k) \cong$ $\cong \tilde{H}_{(i-|F|-1)}(\mathrm{Lk}_\Delta F ; k)=\tilde{H}_{(i-1)-|F|-1}(\mathrm{Lk}_\Delta F ; k)$. By Lemma 1 we obtain $H^{i-1}(k[\Delta]) \neq 0$. In the second case, when $F \overline{\in} \Delta$, we have, for example, $F= =(F_1, a)$, with $F_1 \in \Delta$. Then $\mathrm{Lk}_{S\Delta} F=\mathrm{Lk}_\Delta F_1$. In fact, if $G \in \mathrm{Lk}_{S\Delta} F$, then $G \cap F=\varnothing$ implies that $a \overline{\in} G$ and since $F \cup G \in S\Delta$ and $a \in F$, we obtain that $b \overline{\in} G$ and therefore $G \in \Delta$. By $G \cap F_1=\varnothing$ and $G \cup F_1 \in \Delta$ we obtain that $G \in$ $\in \mathrm{Lk}_\Delta F_1$, which proves the inclusion $\mathrm{Lk}_{S\Delta} F \subseteq \mathrm{Lk}_\Delta F_1$. Conversely, if $G_1 \in$ $\in \mathrm{Lk}_\Delta F_1$ then $F_1 \cup G_1 \in \Delta$ and therefore $F \cup G_1 \in S\Delta$. Then, since $G_1 \cap F_1=\varnothing$ and $a \overline{\in} G_1$ it follows $G_1 \cap F=\varnothing$. Thus we have obtained the inclusion $\mathrm{Lk}_\Delta F_1 \subseteq$ $\subseteq \mathrm{Lk}_{S\Delta} F$. In this second case we have $|F|=|F_1|+1$ and therefore $0 \neq$ $\neq \tilde{H}_{i-|F|-1}(\mathrm{Lk}_{S\Delta} \tilde{F} ; k)=\tilde{H}_{i-|F|-1}(\mathrm{Lk}_\Delta \tilde{F}_1 ; k)=\tilde{H}_{(i-1)-|F_1|-1}(\mathrm{Lk}_\Delta F_1 ; k)$. By Lemma 1 we deduce $H^{i-1}(k[\Delta]) \neq 0$. In this way we have proved the implication $H^i(k[S\Delta]) \neq 0 \Rightarrow H^{i-1}(k[\Delta]) \neq 0$. Suppose now that $H^{i-1}(k[\Delta]) \neq 0$. Then, there exists $F \in \Delta$ such that $\tilde{H}_{i-|F|-1}(\mathrm{Lk}_\Delta F ; k) \neq 0$. By the above considerations we have $\tilde{H}_{i-|F|-1}(\mathrm{Lk}_\Delta F ; k) \cong \tilde{H}_{i+1-|F|-1}(S(\mathrm{Lk}_\Delta F) ; k)=\tilde{H}_{i+1-|F|-1}(\mathrm{Lk}_{S\Delta} F ; k)$ and from the Lemma 1 we deduce $H^{i+1}(k[S\Delta]) \neq 0$.

c) Denote $d=\dim k[\Delta]$, $p=$ depth $k[\Delta]$, $p'=$ depth $k[S\Delta]$. Then $p \leqslant d$, $H^i(k[\Delta])=0$ for $i \overline{\in} [p, d]$ and $H^p(k[\Delta]) \neq 0 (H^d(k[\Delta]) \neq 0)$. From b), $H^{p+1}(k[S\Delta]) \neq 0$ and therefore $p+1 \geqslant p'$. Conversely, $H^{p'}(k[S\Delta]) \neq 0$ implies $H^{p'-1}(k[\Delta]) \neq 0$. Consequently $p'-1 \geqslant p$. Thus we obtain $p'=p+1$.

Denote by $S^n\Delta$ the nth suspension of Δ defined by $S^n\Delta=S(S^{n-1}\Delta)$, $n \geqslant 1$, $(S^0\Delta=\Delta)$. Obvious consequences of Theorem 1 are

Corollary 1. a) $\dim k[S^n\Delta]=\dim k[\Delta]+n$; b) depth $k[S^n\Delta]=$ depth $k[\Delta]+n$.

Corollary 2. *$S^n\Delta$ is CM iff Δ is CM.*

Corollary 3. *$S^n\Delta$ is 2-CM iff Δ is 2-CM.*

Proof. By induction on n, it is sufficient to prove the assertion for the case $n=1$. Suppose that Δ is 2-CM and let be $x \in S\Delta$, a vertex. If $x \in$ $\in \Delta$, then $S\Delta \setminus x=S(\Delta \setminus x)$ and $\Delta \setminus x$ being CM of the same dimension as Δ it follows by the Corollary 1 that $S\Delta \setminus x$ is CM of the same dimension as $S\Delta$. If $x=a$(or $x=b$) then $S\Delta \setminus x$ is the cone $C\Delta$ over Δ, which is contractible and therefore CM [1, Prop. 4.3 (Munkres), p. 71]. Also it is clear that $\dim C\Delta = \dim S\Delta$. Thus $S\Delta$ is 2-CM. Conversely, if we suppose that $S\Delta$ is 2-CM, then Δ is CM and if x is a vertex of Δ, then $S(\Delta \setminus x) = S\Delta \setminus x$ which implies, by the Corollary 1, that $\Delta \setminus x$ is CM. Because $\dim S(\Delta \setminus x) = \dim S\Delta \setminus x = \dim S\Delta$, it follows that $\dim \Delta \setminus x = \dim \Delta$ and thus Δ is 2-CM.

Remark 1. If $S^n\Delta$ is ACM for an integer $n \geqslant 1$, then Δ is CM.

In fact, $\mathrm{Lk}_{S\Delta} a = \Delta$ and therefore if $S\Delta$ is ACM then Δ is CM. Generally, if $S^n\Delta$ is ACM, then $S^{n-1}\Delta$ is CM and by Corollary 1, Δ is CM.

Corollary 4. *If for an integer* $n \geqslant 1$, $\mathrm{Proj}(k[S^n\Delta])$ *is* CM *then* Spec $(k([\Delta])$ *is* CM.

Theorem 2. $S^n\Delta$ *is a Gorenstein simplicial complex iff* Δ *is so.*

Proof. By induction on n, it is sufficient to prove the statement for $n=1$. Let Δ be a simplicial complex over the vertex set V. Let Γ_Δ be the core of Δ, i.e. the subcomplex of Δ generated by the vertices $x \in \Delta$ for which $\mathrm{st}_\Delta x = \{F \in \Delta / F \cup \{x\}\} \neq V$. Then, by [1, Th. 5.1., p. 75] Δ is Gorenstein iff for all $F \in \Gamma_\Delta$; $\tilde{H}_i(\mathrm{Lk}_{\Gamma_\Delta} F ; k) \cong \begin{cases} k, & i = \dim \mathrm{Lk}_{\Gamma_\Delta} F \\ 0, & i < \dim \mathrm{Lk}_{\Gamma_\Delta} F \end{cases}$

Now, the theorem follows from the relation $\Gamma_{S\Delta} = S\Gamma_\Delta$. This relation holds because if x is a vertex of Γ_Δ, then $\mathrm{st}_\Delta x \neq V$ and $\mathrm{st}_{S\Delta} x = S(\mathrm{st}_\Delta x)$, therefore $\mathrm{st}_{S\Delta} x \neq V \cup \{a, b\}$ and this proves that x is a vertex of $\Gamma_{S\Delta}$. Then, since obviously $a, b \in \Gamma_{S\Delta}$, we obtain the inclusion $S\Gamma_\Delta \subseteq \Gamma_{S\Delta}$. Conversely, if x is a vertex of $\Gamma_{S\Delta}$ then $x \in S\Gamma_\Delta$ because $x \in \{a, b\}$ or $x \in \Gamma_\Delta$, hence $\Gamma_{S\Delta} \subseteq S\Gamma_\Delta$.

Now we can apply the same reasoning as in the proof of Theorem 1.

Theorem 3. *If* $S^n\Delta$ *is a Buchsbaum simplicial complex, then* Δ *is so.*

Proof. By induction on n, it is sufficient to prove the statement for $n=1$. Hence, suppose $S\Delta$ a Buchsbaum simplicial complex over k. This means [1, Th. 8.1, p. 84] that for all $F \in S\Delta$, $F \neq \varnothing$, $\tilde{H}_i(\mathrm{Lk}_{S\Delta} F ; k) = 0$, if $i < \dim \mathrm{Lk}_{S\Delta} F$. If $F \in \Delta$, $F \neq \varnothing$, then as in the proof of the Theorem 1 we have $S(\mathrm{Lk}_\Delta F) = \mathrm{Lk}_{S\Delta} F$. By the suspension isomorphism it follows $\tilde{H}_i(\mathrm{Lk}_\Delta F ; k) \cong \tilde{H}_{i+1}(S(\mathrm{Lk}_\Delta F ; k) = \tilde{H}_{i+1}(\mathrm{Lk}_{S\Delta} F ; k) = 0$, if $i+1 < \dim \mathrm{Lk}_{S\Delta} F = \dim \mathrm{Lk}_\Delta F + 1$, which proves that Δ is Buchsbaum.

Remark 2. Generally the converse of Theorem 3 is not true. In fact, if we consider, for example, Δ a triangulation of the torus, then Δ is a Buchsbaum simplicial complex. If F is a vertex, then the space of $\mathrm{Lk}_\Delta F$ is homotopy equivalent to S^1 and therefore $\tilde{H}_i(\mathrm{Lk}_\Delta F ; k) \cong \tilde{H}_i(S^1 ; k) = 0$, for $i < 1$. If F is an 1-simplex, this is a face of just two 2-simplexes and therefore the space of the subcomplex $\mathrm{Lk}_\Delta F$ is S^0, for which $\tilde{H}_i(\mathrm{Lk}_\Delta F ; k) = \tilde{H}_i(S^0 ; k) = 0$ if $i < 0 = \dim \mathrm{Lk}_\Delta F$. Finally, if F is a 2-simplex, then $\mathrm{Lk}_\Delta F = \varnothing$ and $\tilde{H}_i(\varnothing ; k) = 0$, if $i < -1 = \dim \varnothing$. Thus Δ is a Buchsbaum simplicial complex. But $S\Delta$ not is so, since if we take $F = a$, then $\mathrm{Lk}_S F = \Delta$ and we have $H_1(\mathrm{Lk}_{S\Delta} F ; k) = H_1(\Delta ; k) = 2k \neq 0$ although $\dim \mathrm{Lk}_{S\Delta} F = \dim \Delta = 2$.

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FLATNESS CRITERIA FOR LASKERIAN RINGS

BY

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Let A be a commutative unity ring, E an A-module and $F \subseteq E$ a submodule. Denote the radical of F in E by $r_E(F) = \{a \in A / \forall x \in E,\ \exists n \in \mathbb{N},\ a^n x \in F\}$ and this is an ideal of A. The submodule $F \neq E$ is called *primary*, if for each $a \in A$ and for each $x \in E$, $ax \in F$ implies $a \in r_E(F)$ or $x \in F$. The radical of a primary submodule F is a prime ideal of A, denote it by P, and F is called P-primary. The A-module E is called a *laskerian module*, if every proper submodule of E is a finite intersection of primary submodules of E. A laskerian module is called *strong laskerian*, if for any submodule F, there exists $n \in \mathbb{N}$ such that $(r_E(F))^n \cdot E \subseteq F$. The ring A is called *laskerian* or *strong laskerian*, if it is so as an A-module.

We remind other facts which are necessary for presenting our results. An A-module E is *flat*, if for each exact sequence $0 \to F' \to F$, the sequence $0 \to F' \otimes_A E \to F \otimes_A E$ is exact. For a morphism of rings $u : A \to B$ and two modules E over A and F over B, there exists the exact sequence of B-modules : $\mathrm{Tor}_1^A(E, B) \otimes_A F \to \mathrm{Tor}_1^A(E, F) \to \mathrm{Tor}_1^A(E \otimes_A B, F)$ (see [1], IV, 3.7.1).

Given an ideal I of A, we have the graded ring $\mathrm{gr}_I(A) = \bigoplus_{i \geqslant 0} \mathrm{gr}^i(A)$ and the graded module $\mathrm{gr}_I(E)$, where $\mathrm{gr}^0(A) = A/I$ and $\mathrm{gr}_I(A)$, $\mathrm{gr}_I(E)$ are A/I-modules. There exists a surjective morphism of A/I-modules $\gamma_{A,I,E}$ from $\mathrm{gr}_I(A) \otimes_{A/I} E/IE$ to $\mathrm{gr}_I(E)$, defined by $\gamma_{A,I,E}(\beta \otimes \mu) = \nu$, with $\beta = (\beta_i)_{i \geqslant 0}$, β_i being the class of an element $b_i \in I^i$ modulo I^{i+1}, μ a class of an element $m \in E$ modulo IE and $\nu = (\nu_j)_{j \geqslant 0}$, ν_j is the class of the element $b_j m$ modulo $I^{j+1}E$.

Theorem 1. ([1], IV, 6.12). *Let* A *be a commutative ring,* I *an ideal of* A *and* E *an* A*-module. Consider the conditions* : a) E *is a flat* A*-module* ; b) E/IE *is a flat* A/I*-module and* $\mathrm{Tor}_1^A(A/I, E) = 0$; c) E/IE *is a flat* A/I*-module and* $\gamma_{A,I,E}$ *is an isomorphism* ; d). E/I^nE *is a flat* A/I^n*-module, for any* $n \geqslant 1$. *Then the following implications hold* : a) $\Rightarrow$ b) $\Rightarrow$ c) $\Rightarrow$ d). *If* A *is laskerian,* I *is a finite type ideal and* E *is ideal-separated in the* I*-adic filtration, then the conditions given above are equivalent.*

It is obvious that in the case A is laskerian Theorem 1 gives criteria for E to be flat. We obtain another criterion for flatness in :

Theorem 2. *Let* $u : A \to B$ *be a morphism of laskerian rings and let* I *be a finite type ideal such that* IB *is included in the Jacobson radical of the ring* B. *If* E *is a finite type* B*-module, then the following statements are equivalent :* a) E *is a flat* A*-module* ; b) E/IE *is a flat* A/I*-module and*