and therefore CM [1, Prop. 4.3 (M u n k r e s), p. 71]. Also it is clear that $\dim C\Delta = \dim S\Delta$. Thus $S\Delta$ is 2-CM. Conversely, if we suppose that $S\Delta$ is 2-CM, then $\Delta$ is CM and if $x$ is a vertex of $\Delta$, then $S(\Delta \setminus x) = S\Delta \setminus x$ which implies, by the Corollary 1, that $\Delta \setminus x$ is CM. Because $\dim S(\Delta \setminus x) = \dim S\Delta \setminus x = \dim S\Delta$, it follows that $\dim \Delta \setminus x = \dim \Delta$ and thus $\Delta$ is 2-CM.

*Remark* 1. If $S^n\Delta$ is ACM for an integer $n \geq 1$, then $\Delta$ is CM.

In fact, $\mathrm{Lk}_{S\Delta} a = \Delta$ and therefore if $S\Delta$ is ACM then $\Delta$ is CM. Generally, if $S^n\Delta$ is ACM, then $S^{n-1}\Delta$ is CM and by Corollary 1, $\Delta$ is CM.

**Corollary 4.** *If for an integer $n \geq 1$, $\mathrm{Proj}(k[S^n\Delta])$ is CM then $\mathrm{Spec}(k([\Delta])$ is CM.*

**Theorem 2.** *$S^n\Delta$ is a Gorenstein simplicial complex iff $\Delta$ is so.*

*Proof.* By induction on $n$, it is sufficient to prove the statement for $n=1$. Let $\Delta$ be a simplicial complex over the vertex set $V$. Let $\Gamma_\Delta$ be the core of $\Delta$, i.e. the subcomplex of $\Delta$ generated by the vertices $x \in \Delta$ for which $\mathrm{st}_\Delta x = \{F \in \Delta / F \cup \{x\}\} \neq V$. Then, by [1, Th. 5.1., p. 75] $\Delta$ is Gorenstein iff for all $F \in \Gamma_\Delta$; $\tilde{H}_i(\mathrm{Lk}_{\Gamma_\Delta} F ; k) \cong \begin{cases} k, & i = \dim \mathrm{Lk}_{\Gamma_\Delta} F \\ 0, & i < \dim \mathrm{Lk}_{\Gamma_\Delta} F \end{cases}$

Now, the theorem follows from the relation $\Gamma_{S\Delta} = S\Gamma_\Delta$. This relation holds because if $x$ is a vertex of $\Gamma_\Delta$, then $\mathrm{st}_\Delta x \neq V$ and $\mathrm{st}_{S\Delta} x = S(\mathrm{st}_\Delta x)$, therefore $\mathrm{st}_{S\Delta} x \neq V \cup \{a, b\}$ and this proves that $x$ is a vertex of $\Gamma_{S\Delta}$. Then, since obviously $a, b \in \Gamma_{S\Delta}$, we obtain the inclusion $S\Gamma_\Delta \subseteq \Gamma_{S\Delta}$. Conversely, if $x$ is a vertex of $\Gamma_{S\Delta}$ then $x \in S\Gamma_\Delta$ because $x \in \{a, b\}$ or $x \in \Gamma_\Delta$, hence $\Gamma_{S\Delta} \subseteq S\Gamma_\Delta$.

Now we can apply the same reasoning as in the proof of Theorem 1.

**Theorem 3.** *If $S^n\Delta$ is a Buchsbaum simplicial complex, then $\Delta$ is so.*

*Proof.* By induction on $n$, it is sufficient to prove the statement for $n=1$. Hence, suppose $S\Delta$ a Buchsbaum simplicial complex over $k$. This means [1, Th. 8.1, p. 84] that for all $F \in S\Delta$, $F \neq \emptyset$, $\tilde{H}_i(\mathrm{Lk}_{S\Delta} F ; k) = 0$, if $i < \dim \mathrm{Lk}_{S\Delta} F$. If $F \in \Delta$, $F \neq \emptyset$, then as in the proof of the Theorem 1 we have $S(\mathrm{Lk}_\Delta F) = \mathrm{Lk}_{S\Delta} F$. By the suspension isomorphism it follows $\tilde{H}_i(\mathrm{Lk}_\Delta F ; k) \cong \tilde{H}_{i+1}(S(\mathrm{Lk}_\Delta F ; k) = \tilde{H}_{i+1}(\mathrm{Lk}_{S\Delta} F ; k) = 0$, if $i+1 < \dim \mathrm{Lk}_{S\Delta} F = \dim \mathrm{Lk}_\Delta F + 1$, which proves that $\Delta$ is Buchsbaum.

*Remark* 2. Generally the converse of Theorem 3 is not true. In fact, if we consider, for example, $\Delta$ a triangulation of the torus, then $\Delta$ is a Buchsbaum simplicial complex. If $F$ is a vertex, then the space of $\mathrm{Lk}_\Delta F$ is homotopy equivalent to $S^1$ and therefore $\tilde{H}_i(\mathrm{Lk}_\Delta F ; k) \cong \tilde{H}_i(S^1 ; k) = 0$, for $i < 1$. If $F$ is an 1-simplex, this is a face of just two 2-simplexes and therefore the space of the subcomplex $\mathrm{Lk}_\Delta F$ is $S^0$, for which $\tilde{H}_i(\mathrm{Lk}_\Delta F ; k) = \tilde{H}_i(S^0 ; k) = 0$ if $i < 0 = \dim \mathrm{Lk}_\Delta F$. Finally, if $F$ is a 2-simplex, then $\mathrm{Lk}_\Delta F = \emptyset$ and $\tilde{H}_i(\emptyset ; k) = 0$, if $i < -1 = \dim \emptyset$. Thus $\Delta$ is a Buchsbaum simplicial complex. But $S\Delta$ not is so, since if we take $F = a$, then $\mathrm{Lk}_S F = \Delta$ and we have $H_1(\mathrm{Lk}_{S\Delta} F ; k) = H_1(\Delta ; k) = 2k \neq 0$ although $\dim \mathrm{Lk}_{S\Delta} F = \dim \Delta = 2$.

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# FLATNESS CRITERIA FOR LASKERIAN RINGS

BY

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Let $A$ be a commutative unity ring, $E$ an $A$-module and $F \subseteq E$ a submodule. Denote the radical of $F$ in $E$ by $r_E(F) = \{a \in A / \forall x \in E, \exists n \in \mathbf{N}, a^n x \in F\}$ and this is an ideal of $A$. The submodule $F \neq E$ is called *primary*, if for each $a \in A$ and for each $x \in E$, $ax \in F$ implies $a \in r_E(F)$ or $x \in F$. The radical of a primary submodule $F$ is a prime ideal of $A$, denote it by $P$, and $F$ is called $P$-primary. The $A$-module $E$ is called a *laskerian module*, if every proper submodule of $E$ is a finite intersection of primary submodules of $E$. A laskerian module is called *strong laskerian*, if for any submodule $F$, there exists $n \in \mathbf{N}$ such that $(r_E(F))^n \cdot E \subseteq F$. The ring $A$ is called *laskerian* or *strong laskerian*, if it is so as an $A$-module.

We remind other facts which are necessary for presenting our results. An $A$-module $E$ is *flat*, if for each exact sequence $0 \to F' \to F$, the sequence $0 \to F' \otimes_A E \to F \otimes_A E$ is exact. For a morphism of rings $u : A \to B$ and two modules $E$ over $A$ and $F$ over $B$, there exists the exact sequence of $B$-modules : $\mathrm{Tor}_1^A(E, B) \otimes_A F \to \mathrm{Tor}_1^A(E, F) \to \mathrm{Tor}_1^A(E \otimes_A B, F)$ (see [1], IV, 3.7.1).

Given an ideal $I$ of $A$, we have the graded ring $\mathrm{gr}_I(A) = \bigoplus_{i \geq 0} \mathrm{gr}^i(A)$ and the graded module $\mathrm{gr}_I(E)$, where $\mathrm{gr}^0(A) = A/I$ and $\mathrm{gr}_I(A)$, $\mathrm{gr}_I(E)$ are $A/I$-modules. There exists a surjective morphism of $A/I$-modules $\gamma_{A,I,E}$ from $\mathrm{gr}_I(A) \otimes_{A/I} E/IE$ to $\mathrm{gr}_I(E)$, defined by $\gamma_{A,I,E}(\beta \otimes \mu) = \nu$, with $\beta = (\beta_i)_{i \geq 0}$, $\beta_i$ being the class of an element $b_i \in I^i$ modulo $I^{i+1}$, $\mu$ a class of an element $m \in E$ modulo $IE$ and $\nu = (\nu_j)_{j \geq 0}$, $\nu_j$ is the class of the element $b_j m$ modulo $I^{j+1}E$.

**Theorem 1.** ([1], IV, 6.12). *Let $A$ be a commutative ring, $I$ an ideal of $A$ and $E$ an $A$-module. Consider the conditions : a) $E$ is a flat $A$-module ; b) $E/IE$ is a flat $A/I$-module and $\mathrm{Tor}_1^A(A/I, E) = 0$ ; c) $E/IE$ is a flat $A/I$-module and $\gamma_{A,I,E}$ is an isomorphism ; d). $E/I^nE$ is a flat $A/I^n$-module, for any $n \geq 1$. Then the following implications hold : a) $\Rightarrow$b) $\Rightarrow$c) $\Rightarrow$d). If $A$ is laskerian, $I$ is a finite type ideal and $E$ is ideal-separated in the $I$-adic filtration, then the conditions given above are equivalent.*

It is obvious that in the case $A$ is laskerian Theorem 1 gives criteria for $E$ to be flat. We obtain another criterion for flatness in :

**Theorem 2.** *Let $u : A \to B$ be a morphism of laskerian rings and let $I$ be a finite type ideal such that $IB$ is included in the Jacobson radical of the ring $B$. If $E$ is a finite type $B$-module, then the following statements are equivalent : a) $E$ is a flat $A$-module ; b) $E/IE$ is a flat $A/I$-module and*

$\mathrm{Tor}_1^A(A/I, E) = 0$ ; c). *$E/IE$ is a flat $A/I$-module and $\gamma_{A,I,E}$ is an isomorphism* ; d). *$(A/I^n)\otimes_A E$ is a flat $A/I^n$-module for any $n \in \mathbf{N}$.*

*Proof*. The implications a) $\Rightarrow$ b) $\Rightarrow$ c) $\Rightarrow$ d) result from Theorem 1. To prove d) $\Rightarrow$ a), let us note that for $F$ a finite type submodule of $E$, $F\otimes_A E$ is ideal-separated in the $IB$-adic topology, then $E$ is ideal-separated in the $I$-adic filtration, hence $E$ is a flat $A$-module.

*Remarks*. 1. In the proof, we use the Lemma 1.4 ([3], §2) : „Let $A$ be a ring, $I$ an ideal of finite type of $A$, $E$ a laskerian finite type $A$-module. If $I$ is contained in the Jacobson radical of $A$, then $\bigcap_{n=0}^{\infty} I^n E = 0$".

2. If $B$ is an $A$-algebra, $A$ is a laskerian ring and $B$ is a strong laskerian ring, then the condition of finitness of the ideal $I$ can be cancelled (see [3], § 2, 1.5).

Denote by $\hat{A}$ the $I$-adic completion of the laskerian ring $A$ and assume that $A$ is also a laskerian ring. (Although this statement is not proved, all the laskerian rings which are known have this property).

We can state now some flatness criteria in terms of $\hat{A}$, whose proofs are based upon Theorem 1, Remarks 1 and 2.

**Corollary 1.** *Let $A$ be a laskerian ring, $I$ an ideal of $A$ of finite type, $\hat{A}$ the completion of $\hat{A}$ into $I$-adic topology, $\hat{A}$ laskerian. Then $\hat{A}$ is a flat $A$-module.*

**Corollary 2.** *Let $A$ be a strong laskerian ring, $I$ an ideal of $A$ and $\hat{A}$ the $I$-adic completion of $A$. If $\hat{A}$ is a laskerian ring, then $\hat{A}$ is a flat $A$-module.*

**Corollary 3.** *Let $B$ be an $A$-algebra which is a laskerian ring, $I$ a finite type ideal of $A$ with $IB \subseteq r_B(B)$, $E$ a finite type $B$-module. If $E$ is a flat $A$-module and $E/IE$ is a flat $B/IB$-module, then $E$ is a flat $B$-module.*

*Proof*. Consider the diagram :

$$\begin{tikzcd}
\mathrm{gr}_I(A)\otimes_{A/I}(E/IE) \arrow[rr, "\gamma_{A,I,E}"] \arrow[dr, "\gamma_{A,IB,B}\otimes_{B/IB}(E/IE)"'] & & \mathrm{gr}_I(E) \\
& \mathrm{gr}_I(B)\otimes_{B/IB}(E/IE) \arrow[ur, "\gamma_{B,IB,E}"'] &
\end{tikzcd}$$

All the morphisms in the diagram are surjective by construction or because the functor $-\otimes_{B/IB}E/IE$ is exact, and, since $\gamma_{A,I,E}$ is bijective ($E$ is a flat $A$-module), the other two morphisms are bijective. Now $B$ is a laskerian ring, $IB$ is a finite type ideal of $B$, $E$ is ideal-separated in $IB$-adic topology, since $E$ is a laskerian $B$-module, $IB$ is contained in the Jacobson radical of $B$. By using Theorem 1, we obtain that $E$ is a flat $B$-module.

**Corollary 4.** *Let $A \xrightarrow{u} B \xrightarrow{v} C$ be an exact sequence of local morphisms and laskerian rings, $u$ being a flat morphism. Let $E$ be a finite type $C$-module and $I$ a finite type ideal of $A$. The following statements are equivalent : a). $E$ is a flat $B$-module ; b) $E$ is a flat $A$-module and $E/IE$ is a flat $B/IB$-module.*

To prove it, we apply Corollary 3 and Theorem 2 (for $v : B \to C$, the ideal $IB$ and the $C$-module $E$).

**Proposition.** *Let $v : A \to B$ be a morphism of laskerian rings and let $I$ be a finite type ideal of $A$ such that $IB \subseteq r_B(B)$. Let $u : E' \to E$ be a morphism of finite type $B$-modules, $E$ being a flat $A$-module. The following statements are equivalent : a) $u$ is injective and $E''=$*Coker *$u$ is a flat $A$-module ; b) $u' := A/I\otimes_A u : A/I\otimes_A E' \to A/I\otimes_A E$ is injective and $E''/IE''$ is a flat $A/I$-module.*

*Proof*. a) $\Rightarrow$ b) : By applying the tensor product by $A/I$ to the exact sequence $0 \to E' \xrightarrow{u} E \to E'' \to 0$, we obtain the exact sequence $0 = \mathrm{Tor}_1^A(A/I, E'') \to A/I\otimes_A E' \to A/I\otimes_A E \to A/I\otimes_A E'' \to 0$, since $E''$ is flat. Therefore $u'$ is injective, and, by applying Theorem 2, it follows that $E''/IE''$ is a flat $A/I$-module.

b) $\Rightarrow$ a) : First, by using Theorem 2, we obtain that $E''$ is a flat $A$-module. Then, the exact sequence $0 \to \mathrm{Im}\, u \to E \to E''$ gives us that $\mathrm{Im}\, u$ is a flat $A$-module. The exact sequence $0 = \mathrm{Tor}_1^A(A/I, \mathrm{Im}\, u) \to A/I\otimes_A \mathrm{Ker}\, u \to$ $\to A/I\otimes_A E' \xrightarrow{u} A/I\otimes_A \mathrm{Im}\, u \to 0$ gives that $v$ is an isomorphism. $I$ being a finite type ideal of $A$, we can apply the Nakayama lemma to $(\mathrm{Ker}\, u/I\,\mathrm{Ker}\, u) = 0$, and we obtain $\mathrm{Ker}\, u = 0$, hence $u$ is injective.

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